Bounded Stability of Nonlinear Stochastic Systems

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Abstract: This paper is concerned with stochastic bounded stability for a general class of nonlinear stochastic systems. The bounded stability concept considered here is such that for a given bounded region and realization probability, sample paths remain bounded in the assigned region with the assigned probability. The authors provide a sufficient condition for the proposed bounded stability to be satisfied based on a Lyapunov-like function. The proposed result is applicable to a system with non-vanishing noise at a target point, which the conventional stochastic stability concepts do not deal with.

Key Words: stochastic stability, bounded stability, nonlinear stochastic systems.

1. Introduction

Stochastic systems are dynamical systems having probabilistic uncertainties, and thus are utilized for modeling the actual systems in various fields such as engineering, applied chemistry, economics, biophysics and so on. For those systems, guaranteeing the boundedness of sample paths is a not only theoretically but also practically important problem.

As popular concepts for the boundedness of stochastic systems, stochastic input-to-state stability (SISS) and noise-to-state stability (NSS) are well known. Those are motivated by the deterministic input-to-state stability (ISS) concept [1]. SISS guarantees that the boundedness of sample paths with an arbitrary probability against an essentially bounded unknown external input. The SISS concept in [2] was developed from the concept of \(\gamma\)-input-to-state stability in [3]. While both results in [2],[3] only consider the deterministic external inputs, the authors in [4] extend the SISS concept to deal with stochastic external inputs as well. Besides, the NSS concept proposed in [5] guarantees that the boundedness of sample paths with an arbitrary probability against an essentially bounded unknown noise covariance. It is shown in [2] that NSS can be a special case of the SISS concept in [2],[4]. In addition to those concepts, sufficient conditions for SISS and NSS based on SISS and NSS-Lyapunov functions are also provided, which are useful for analysis and controller design.

Although possessing the SISS property is favorable, it can be practically difficult for general nonlinear stochastic systems, since SISS is a global property and its sufficient condition is often strict. When the external input is equivalent to zero, SISS is reduced to globally asymptotic stability in probability [6],[7], which is even strict to be endowed. Moreover, since the bound of sample paths depends on the essential supremum of the external input, which is unknown in general, it is not assignable. On the contrary, in this paper, we consider stochastic bounded stability as a local property. We suppose the bounded region and realization probability as design parameters, and guarantee that sample paths remain bounded in the given region with given probability. We provide a sufficient condition for the proposed bounded stability to be satisfied based on a Lyapunov-like function. The proposed condition on this function is required only on the assigned bounded region. Therefore, the bounded stability concept considered here is a local property, and its condition is easier to be realized than satisfying global stability criteria in many cases. Beside, the proposed result is applicable to a system with non-vanishing noise at a target point. Although the conventional stochastic stability concepts, e.g., [6]–[9] are not applicable in this case, since the target point is not an equilibrium point, the proposed method guarantees the boundedness of a sample path around the point instead.

In the authors’ previous paper [10] concentrates on a particular class of nonlinear stochastic systems, which is described as a stochastic port-Hamiltonian system [11] including fully/under-actuated mechanical systems in the presence of noise, and has shown a sufficient condition for bounded stability. They have also provided a construction method of a Lyapunov-like function satisfying the condition by using stochastic Hamiltonian structure. Compared to [10], the main result in this paper has the following advantages: first, this paper can deal with more general class of nonlinear stochastic systems. Second, this paper provides more rigorous analysis. Since the literature [10] performed analysis based only on Martingales, it only considers a region where the sign of the expected time variation of the Lyapunov-like function is not positive. Thus, the previous stability result in [10] has to exclude the case where the initial state of the system is on such a region. Meanwhile, this paper equips the sequence of stopping times in addition to the Martingale analysis. The analysis method with the sequence of stopping times used in this paper is partially based on [4]. It enables us to deal with the region where the expected time variation of the Lyapunov-like function can be positive. The advantage of the proposed method that both a bounded region and its realization probability can be assignable for general nonlinear stochastic systems, is significant and useful for many practical control problems.

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2. Preliminaries

We consider a nonlinear stochastic system described by the following Itô stochastic differential equation:

\[ dx = f(x) dt + h(x) dw, \quad x(0) = x^0, \]  

(1)

where the state is denoted by \( x(t) \in \mathbb{R}^n \), and \( w(t) \in \mathbb{R}^r \) denotes a standard Wiener process defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Here, \( \Omega \) is a sample space, \( \mathcal{F} \) is the sigma algebra of the observable random events and \( \mathbb{P} \) is a probability measure on \( \Omega \). A filtration \( \mathcal{F}_t \) represents an increasing family of \( \sigma \)-algebras with \( \mathcal{F}_t \subset \mathcal{F} \), \( \forall t \in [0, \infty) \). We suppose that \( \mathcal{F}_t \) is generated by the Wiener process and is right-continuous and complete. The functions \( f : \mathbb{R}^n \to \mathbb{R}^n \) and \( h : \mathbb{R}^n \to \mathbb{R}^{n \times r} \) are supposed to be sufficiently differentiable. Without loss of generality, we consider the boundedness of sample paths around the origin. However, we deal with non-vanishing noise. Thus, \( h(x) \) does not necessarily vanish at the origin. According to [12], it is also assumed that the linear growth condition and local Lipschitz condition are satisfied for existence and uniqueness of a global solution to the system (1). That is, there exists a positive constant \( k_{L(x)} \) such that

\[ \|f(x)\|^2 + \|h(x)\|^2 \leq k_{L(x)}(1 + \|x\|^2), \quad \forall x \in \mathbb{R}^n, \]

and for each compact subset \( X \subset \mathbb{R}^n \), there exists a positive constant \( k_{L(X)} \) such that

\[ \|f(x) - f(y)\| + \|h(x) - h(y)\| \leq k_{L(X)} \|x - y\|, \quad \forall x, y \in X \subset \mathbb{R}^n. \]

In this paper, we define the norm of a matrix \( A = \|A\| := \sqrt{\lambda_{\max}(A^T A)} \). Here, \( \lambda_{\max}(\cdot) \) and \( \lambda_{\min}(\cdot) \) represent the minimum and maximum eigenvalues of \((\cdot)\), respectively. We use the Euclidean norm for vectors.

Definition 1 For a \( C^2 \) function \( V(x) \), the infinitesimal operator \( \mathcal{L}(V(x)) \) associated with the system (1) is defined as

\[ \mathcal{L}(V(x)) := \frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2} \text{tr} \left( \frac{\partial^2 V(x)}{\partial x^2} h(x) h(x)^T \right), \]

(2)

where \( \text{tr}(\cdot) \) represents the trace of the argument.

Then, Itô’s formula [14] gives the expectation of the time variation of the function \( V(x) \) along a sample path \( x(t) \) as

\[ E^\mathbb{P}[V(x(t)) | x(0) = x^0] = V(x_0) + E^\mathbb{P} \left( \int_0^t \mathcal{L}(V(x(s))) ds \bigg| x(0) = x^0 \right), \]

(3)

where \( E^\mathbb{P}[\cdot] \) denotes the expectation with respect to the probability measure \( \mathbb{P} \).

Definition 2 Define \( t \cap s := \min(t, s) \) and \( t \cup s := \max(t, s) \). Suppose that \( \tau_D \) is the first time of exit of the process \( x(s) \) from an open set \( D \), i.e., \( \tau_D := \inf \{ t \geq 0 \mid x(t) \not\in D \} \). Then, the stopped process \( x_{\tau_D} \) is defined as

\[ x_{\tau_D}(t) := x(t \wedge \tau_D) = \begin{cases} x(t) & t < \tau_D, \\ x(0) & t \geq \tau_D. \end{cases} \]

In this paper, we use the following notations for representing regions: for any positive \( \delta, \nu \in \mathbb{R} \),

\[ D(\delta) := \{ x \in \mathbb{R}^n \mid \|x\| < \delta \}, \]

\[ \bar{D}(\delta) := \{ x \in \mathbb{R}^n \mid \|x\| \leq \delta \}, \]

\[ D_\nu(v) := \{ x \in \mathbb{R}^n \mid V(x) < \nu \}, \]

\[ \bar{D}_\nu(v) := \{ x \in \mathbb{R}^n \mid V(x) \leq \nu \}. \]

Next, we introduce the notion of \((Q_0, Q_1, \rho)\)-stability due to [6] in order to consider the stochastic bounded stability.

Definition 3 ([6]) The systems is \((Q_0, Q_1, \rho)\)-stable if and only if for any initial condition \( x(0) \in Q_0 \subset \mathbb{R}^n \), the probability with respect to a sample path \( x(t) \) satisfies

\[ \mathbb{P}\{x(t) \in Q_1 \subset \mathbb{R}^n, \quad \text{for } 0 \leq t < \infty \} \geq \rho. \]

Here, \( Q_0 \) and \( Q_1 \) are named initial and bounded regions, respectively.

Since we consider the boundedness of sample paths around the origin, we equip the initial region parameter \( \delta_0 \in \mathbb{R} \), \( \delta_0 \geq 0 \) and the bounded region parameter \( \delta_1 \in \mathbb{R} \), \( \delta_1 > 0 \), and parameterize these regions as follows:

\[ Q_0 = \{ x \in \mathbb{R}^n \mid x \in \bar{D}(\delta_0) \}, \]

\[ Q_1 = \{ x \in \mathbb{R}^n \mid x \in \bar{D}(\delta_1) \}. \]

(4)

3. Main Result

We provide the main theorem on stochastic bounded stability of the system (1), which gives a sufficient condition for the state to remain bounded in probability for given a bounded region parameter and realization probability. In the proof of the main theorem, the analysis method with the sequence of stopping times is partially based on [4].

Theorem 1 Consider the system (1). For given bounded region parameter \( \delta_1 \in \mathbb{R} \), \( \delta_1 > 0 \) and realization probability \( \rho \in \mathbb{R} \), \( 0 < \rho < 1 \), there exists a \( C^2 \) function \( V(x) \) satisfying the following conditions:

- there exist class \( K \) functions\(^1\) \( \alpha_m(||x||) \) and \( \alpha_M(||x||) \), which are proper on \( \bar{D}(\delta_1) \), such that
  \[ \alpha_m(||x||) \leq V(x) \leq \alpha_M(||x||) \]
  \[ \forall x \in \bar{D}(\delta_1) \]
  \[ \text{(5)} \]
  holds; and
- for some \( \delta_\rho \in \mathbb{R} \), \( 0 \leq \delta_\rho < \delta_1 \),
  \[ \mathcal{L}(V(x)) \leq 0, \quad \forall x \in \bar{D}(\delta_1) \setminus D(\delta_\rho) \]
  \[ \text{(6)} \]
  holds.

Then, if the following condition holds:

\[ \alpha_M(\delta_0) + \alpha_m(\delta_0) \leq (1 - \rho) \alpha_m(\delta_1), \]

(7)

the systems is \((Q_0, Q_1, \rho)\)-stable, where the initial and bounded regions \( Q_0 \) and \( Q_1 \) are given by Eq. (4), respectively. That is, the following probability inequality is achieved:

\[ \mathbb{P}\left\{ \sup_{t < \infty} \|x(t)\| < \delta_1 \right\} \geq \rho. \]

\[ \text{(8)} \]

\(^1\) A continuous function \( \alpha : [0, \infty) \to [0, \infty) \) is said to belong to class \( K \) if it is strictly increasing and \( \alpha(0) = 0 \).
Proof. We consider a stopped process $x_{Q^V}$, where an open set $Q^V \subset D(\delta_i)$ is defined as

$$Q^V := D_V(\alpha_M(\delta_i)).$$

We define a sequence of stopping times $\{\tau_i\}_{i=1,2,\ldots}$ as

$$\tau_0 = 0,$$

$$\tau_1 = \begin{cases} \inf \{-\tau_0 | x_{Q^V}(t) \in D_V(\alpha_M(\delta_i))\}, & \text{if } \{t > \tau_0 | x_{Q^V}(t) \in D_V(\alpha_M(\delta_i))\} \neq \emptyset, \\ \infty, & \text{otherwise}, \end{cases}$$

$$\tau_2 = \begin{cases} \inf \{-\tau_{i-1} | x_{Q^V}(t) \in Q^V \setminus D_V(\alpha_M(\delta_i))\}, & \text{if } \{t > \tau_{i-1} | x_{Q^V}(t) \in Q^V \setminus D_V(\alpha_M(\delta_i))\} \neq \emptyset, \\ \infty, & \text{otherwise}, \end{cases}$$

$$\tau_{2i+1} = \begin{cases} \inf \{-\tau_{2i} | x_{Q^V}(t) \in D_V(\alpha_M(\delta_i))\}, & \text{if } \{t > \tau_{2i} | x_{Q^V}(t) \in D_V(\alpha_M(\delta_i))\} \neq \emptyset, \\ \infty, & \text{otherwise}. \end{cases}$$

Figure 1 may help to describe the stopping times.

![Illustration of the stopping times $\{\tau_i\}$](image)

We consider the following two cases separately with respect to the initial condition: $\alpha_M(\delta_i) \leq V(x^0)$, and otherwise, respectively.

**Case 1: $x^0$ satisfies $\alpha_M(\delta_i) \leq V(x^0)$**

Since $x_{Q^V}(t) \in Q^V \setminus D_V(\alpha_M(\delta_i))$ for any $t \in [0, \tau_1]$, $L(V(x_{Q^V}(t))) \leq 0$ holds for any $t \in [0, \tau_1]$ from the condition (6). It implies that

$$E^P\{V(x_{Q^V}(t \wedge \tau_1))\} \leq V(x_{Q^V}(0)) \leq \alpha_M(\delta_0)$$

from Eq. (3). According to [4], for any $t \in [\tau_{2i}, \tau_{2i+1}]$, $(i = 1, 2, \ldots)$, we have

$$V(x_{Q^V}(t \wedge \tau_2)) \leq V(x_{Q^V}(\tau_{2i+1})) + \int_{\tau_{2i}}^{\tau_{2i+1}} L(V) \, dx + \int_{\tau_{2i}}^{\tau_{2i+1}} \frac{\partial V}{\partial x} \, dw \quad \text{a.s.}$$

Since $x_{Q^V}(t) \in Q^V \setminus D(\delta_i)$ for any $t \in [\tau_{2i}, \tau_{2i+1}]$, $L(V(x_{Q^V}(t))) \leq 0$ holds, and this fact with Eq. (3) yields

$$E^P\{V(x_{Q^V}(t \wedge \tau_{2i+1}))\} \leq E^P\{V(x_{Q^V}(\tau_{2i+1}))\} \leq E^P\{V(x_{Q^V}(\tau_2))\}. \quad (9)$$

Here, note that

$$\bigcup_{i=0}^{\infty} (\bar{\tau}_{2i+1}, \bar{\tau}_{2i+2}) \bigcup \bigcup_{i=0}^{\infty} (\bar{\tau}_{2i+1}, \bar{\tau}_{2i+2}) = (\tau_1, \infty),$$

$$\bigcup_{i=0}^{\infty} (\bar{\tau}_{2i+1}, \bar{\tau}_{2i+2}) \bigcup \bigcup_{i=0}^{\infty} (\bar{\tau}_{2i+1}, \bar{\tau}_{2i+2}) = (\tau_1, \infty),$$

hold. Thus, we have

$$E^P\{V(x_{Q^V}(t \wedge \tau_1))\} = E^P\{V(x_{Q^V}(t \wedge \tau_1))I_{[\tau_1, \tau_2]}\}$$

$$+ E^P\{V(x_{Q^V}(t \wedge \tau_1))I_{[\tau_2, \infty]}\}$$

$$= E^P\{V(x_{Q^V}(\tau_1))I_{[\tau_1, \tau_2]}\} + \sum_{i=1}^{\infty} E^P\{V(x_{Q^V}(\tau))I_{[\tau_{2i}, \tau_{2i+2}]i}\}, \quad (10)$$

where $I_{[\cdot]}$ denotes the indicator function [13]. Consider the first term in the last equality in Eq. (10). Since $x_{Q^V}(\tau_1)$ lies on the boundary of the set $D_V(\alpha_M(\delta_i))$, the first term is given by

$$E^P\{V(x_{Q^V}(\tau_1))I_{[\tau_1, \tau_2]}\} = P(t \in [0, \tau_1]) \alpha_M(\delta_0). \quad (11)$$

Consider the second term in the last equality in Eq. (10). Since $x_{Q^V}(t) \in D_V(\alpha_M(\delta_i))$ for any $t \in (\tau_{2i+1}, \tau_{2i+2})$, it is estimated as

$$\sum_{i=0}^{\infty} E^P\{V(x_{Q^V}(\tau))I_{[\tau_{2i}, \tau_{2i+2}]i}\} \leq \sum_{i=0}^{\infty} P(t \in (\tau_{2i+1}, \tau_{2i+2})i) \alpha_M(\delta_0). \quad (12)$$

Considering the last term in the last equality in Eq. (10), we have

$$E^P\{V(x_{Q^V}(t))I_{[\tau_{2i}, \tau_{2i+2}]i}\}$$

$$= E^P\{V(x_{Q^V}(t \wedge \tau_2))I_{[\tau_{2i}, \tau_{2i+2}]i}\}$$

$$= E^P\{V(x_{Q^V}(t \wedge \tau_2))I_{[\tau_{2i}, \tau_{2i+2}]i}\}$$

$$- E^P\{V(x_{Q^V}(t \wedge \tau_{2i+1}))I_{[\tau_{2i}, \tau_{2i+2}]i}\}$$

$$= E^P\{V(x_{Q^V}(t \wedge \tau_{2i}))I_{[\tau_{2i}, \tau_{2i+2}]i}\} - E^P\{V(x_{Q^V}(\tau_{2i}))I_{[\tau_{2i}, \tau_{2i+2}]i}\}$$

$$= E^P\{V(x_{Q^V}(t \wedge \tau_{2i}))I_{[\tau_{2i}, \tau_{2i+2}]i}\} - \alpha_M(\delta_i)P(t < \tau_{2i} \text{ for } t > \tau_{2i+1})$$

$$= E^P\{V(x_{Q^V}(t \wedge \tau_{2i}))I_{[\tau_{2i}, \tau_{2i+2}]i}\}$$

$$- \alpha_M(\delta_i)(1 - P(t \in [\tau_2, \tau_{2i+1}]i)) \leq E^P\{V(x_{Q^V}(\tau_{2i}))\} - \alpha_M(\delta_i)(1 - P(t \in [\tau_2, \tau_{2i+1}]i))$$

$$= P(t \in [\tau_2, \tau_{2i+1}]i) \alpha_M(\delta_0). \quad (13)$$

Here, in the fourth equality and the last inequality, the relation $V(x_{Q^V}(\tau_{2i+2})) = V(x_{Q^V}(\tau_{2i+1})) = \alpha_M(\delta_i)$ is utilized, since both $x_{Q^V}(\tau_{2i+2})$ and $x_{Q^V}(\tau_{2i+1})$ lie on the boundary of $D_V(\alpha_M(\delta_i))$. The sixth inequality follows from Eq. (9). From Eq. (13), the last term in the last equality in Eq. (10) is estimated as

$$\sum_{i=0}^{\infty} E^P\{V(x_{Q^V}(t))I_{[\tau_{2i}, \tau_{2i+2}]i}\} \leq \sum_{i=0}^{\infty} P(t \in [\tau_2, \tau_{2i+1}]i) \alpha_M(\delta_i). \quad (14)$$
By substituting Eqs. (11), (12) and (14) into Eq. (10), we have
\[ E^P \{ V(x_Q(t \cup \bar{\tau})) \} \leq \alpha_M(\delta_1). \tag{15} \]

Since \( V(x(t)) = \alpha_m(\delta_1) \) for some \( t \) implies that the state \( x \) reaches the boundary of the region \( Q_V \), we have
\[ P \left\{ \sup_{0 \leq t \leq \bar{\tau}} V(x(t)) \geq \alpha_m(\delta_1) \right\} = P \left\{ V(x_Q(\bar{\tau})) = \alpha_m(\delta_1) \right\}, \quad \forall \bar{\tau} \geq 0. \tag{16} \]

Since \( V(x) \) is positive definite, the following inequality holds due to Chebyshev’s inequality:
\[ E^P \{ V(x_Q(\bar{\tau})) \} \geq \alpha_m(\delta_1) \sigma^P \{ V(x_Q(\bar{\tau})) = \alpha_m(\delta_1) \}, \quad \forall \bar{\tau} \geq 0. \tag{17} \]

It follows from Eqs. (16) and (17) that
\[ P \left\{ \sup_{0 \leq t \leq \bar{\tau}} V(x(t)) < \alpha_m(\delta_1) \right\} \geq 1 - \frac{\sigma_m(\delta_0) + \alpha_m(\delta_2)}{\alpha_m(\delta_1)}. \tag{18} \]

Now, from Eq. (15) we obtain the following estimation:
\[ E^P \{ V(x_Q(\bar{\tau})) \} = E^P \{ V(x_Q(\bar{\tau} \cup \bar{\tau})) \} \leq \sigma_m(\delta_0) + \alpha_m(\delta_2). \tag{19} \]

By substituting Eq. (19) into Eq. (18), and letting \( \bar{\tau} \to \infty \), we have
\[ P \left\{ \sup_{0 \leq t < \infty} V(x(t)) < \alpha_m(\delta_1) \right\} \geq 1 - \frac{\sigma_m(\delta_0) + \alpha_m(\delta_2)}{\alpha_m(\delta_1)} \geq \rho. \tag{20} \]

Then, the condition (7) yields
\[ P \left\{ \sup_{0 \leq t < \infty} V(x(t)) < \alpha_m(\delta_1) \right\} \geq \rho. \tag{21} \]

Since, from Eq. (5), \( V(x(t)) < \alpha_m(\delta_1) \) is a sufficient condition for \( \| x(t) \| < \delta_1 \) (Fig. 2 may help to understand the claim), Eq. (20) implies that the asserted probability inequality (8) holds in the Case 1.

**Case 2**: \( x^b \) satisfies \( V(x^b) < \alpha_M(\delta_d) \)

In this case, \( \bar{\tau}_1 = 0 \) holds almost surely. For \( t = 0 \), we have
\[ V(x_Q(0)) \leq \alpha_M(\delta_d). \tag{21} \]

For \( t > 0 \), since the argument in the Case 1 on \( t > \bar{\tau}_1 \) also valid, and \( P\{t \in (\bar{\tau}_1, \infty)\} = P\{t \in (0, \infty)\} = 1 \) holds, Eq. (15) yields
\[ E^P \{ V(x_Q(\bar{\tau})) \} \leq \alpha_M(\delta_d), \quad \forall \bar{\tau} > 0. \tag{22} \]

From Eqs. (21) and (22), we have
\[ E^P \{ V(x_Q(\bar{\tau})) \} \leq \alpha_M(\delta_d), \quad \forall \bar{\tau} \geq 0. \tag{23} \]

By substituting Eq. (23) into Eq. (18), the inequality (20) also holds under the condition (7). Therefore, as the same argument in the Case 1, the asserted probability inequality (8) holds in the Case 2.

Consequently, the argument in both Case 1 and Case 2 prove the assertion of the theorem.

**Remark 1** Compared to our previous work [10], the main result in this paper has the following advantages: first, this paper can deal with a wider class of the plant system. The literature [10] considers fully/partially-actuated mechanical systems with uncertainties, while more general class of nonlinear systems is dealt with in this paper. Second, Theorem 1 provides more rigorous analysis. Since the literature [10] performed analysis based only on Martingales, it only considers the initial region \( Q_0 \) where \( \mathcal{L}(V) \leq 0 \) holds. That is, \( Q_0 \) has the form as \( \mathcal{D}(\delta_0) \setminus \mathcal{D}(\delta_d) \). Meanwhile, this paper equips the sequence of stopping times in addition to the Martingale analysis. It enables us to deal with the initial region where \( \mathcal{L}(V) \) can be positive. On the contrary, one of the advantages of the literature [10] is to provide not only a sufficient condition for bounded stability, but also a construction method of a function \( V(x) \).

### 4. Numerical Example

This section exhibits application of the proposed method. We consider the stochastic bounded stabilization of a DC motor depicted in Fig. 3 in the presence of noise. Here, \( R \) and \( L \) denote the armature resistance and inductance, and \( J \) and \( D \) denote the rotational inertia and damping, respectively. \( \tau \) denotes the load torque, and \( v_c \) denotes the back electromotive force. The state \( x \) is defined as \( x = (\theta, \omega, i)^T \), where \( \theta, \omega \) and \( i \) represent the angular position, angular velocity and current, respectively. The control input \( u \) is defined as \( u = v_{in} \), where \( v_{in} \) represents the input voltage. Then, we have the following nonlinear stochastic differential equation as the dynamics of the DC motor:

\[
\frac{d}{dt} \begin{bmatrix} \theta \\ \omega \\ i \end{bmatrix} = \begin{bmatrix} \frac{\omega}{L} + \frac{k_v(\theta)}{L} \\ -\frac{\omega}{L} - \frac{k_p(\theta)}{L} - \frac{\tau}{J} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \frac{d}{dt} v_c + \begin{bmatrix} 0 \\ 0 \end{bmatrix} dw_2 + \begin{bmatrix} 0 \\ 0 \end{bmatrix} dw_3 \tag{24} \]

The back electromotive force \( v_c \) and the load torque \( \tau \) are respectively given by

![Fig. 3 Equivalent circuit model of a DC motor.](image)
\[ v_c = K(\theta)\omega \]
\[ \tau = K(\theta)i, \] (25)

where \( K(\theta) \) is a function of \( \theta \) such that there exist positive constants \( K_m \) and \( K_M \) satisfying

\[ K_m \leq K(\theta) \leq K_M, \quad \forall \theta \in \mathbb{R}. \] (26)

We assume that \( K(\theta) \) is twice differentiable, \( h_2(x) \in \mathbb{R}^{15n} \) and \( h_3(x) \in \mathbb{R}^{15n} \) denote the noise ports, and \( w_2 \in \mathbb{R}^{1} \) and \( w_3 \in \mathbb{R}^{1} \) denote the standard Wiener processes. We write \( h(x) \in \mathbb{R}^{15(n+2)} \) as

\[ h(x) := \begin{pmatrix} 0 & 0 \\ h_2(x) & 0 \\ h_3(x) & 0 \end{pmatrix}. \] (27)

Note that the system (24) is a nonlinear stochastic system due to \( K(\theta) \) and is of the form (1), and it cannot be dealt with by the authors’ previous work [10], since the system (24) is an electromechanical system. Now, the control objective is that for given bounded region parameter \( \delta_1 \in \mathbb{R}, \delta_1 > 0 \) and realization probability \( \rho \in \mathbb{R}, 0 < \rho < 1 \) for the system (24), we achieve the probability inequality in Eq. (8).

First, based on feedback linearization technique, we apply the following control input:

\[ u = -\frac{L}{K} \left( \frac{D}{J} + \frac{\partial K}{\partial \theta}i - \frac{K^2}{L} \right) \omega + L \left( \frac{D}{J} + R \right) i + JL \frac{\partial \theta}{\partial \theta}i, \] (28)

where \( \bar{u} \) denotes the new control input. The input (28) can be derived by choosing \( \theta \) as an output function for feedback linearization, which has relative degree 3. Then, we equip the following coordinate transformation:

\[ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \Phi(x) = \begin{pmatrix} \Phi^3(x) \\ \Phi^1(x) \\ \Phi^2(x) \end{pmatrix} = \begin{pmatrix} \omega \\ \frac{\partial \Phi^1}{\partial x^1}h_1^T \omega \\ \frac{\partial \Phi^2}{\partial x^2}h_1^T \omega \end{pmatrix}. \] (29)

According to Itô’s formula [13],[14], under the control input in Eq. (28) and the coordinate transformation in Eq. (29), the system (24) is converted into

\[ \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bar{u} dt + \begin{pmatrix} \frac{\partial \Phi^1}{\partial x^1}h_1^T \\ \frac{\partial \Phi^2}{\partial x^2}h_1^T \omega \end{pmatrix} \bar{u} dt + \begin{pmatrix} \frac{\partial \Phi^3}{\partial x^3}h_3^T \\ \frac{\partial \Phi^2}{\partial x^2}h_3^T \omega \end{pmatrix} dw, \] (30)

where \( \bar{u} := (dw_1^T, dw_2^T, dw_3^T)^T \) and

\[ h_3(z) = \begin{pmatrix} 0 \\ h_{31}(z) \\ h_{32}(z) \end{pmatrix} = \begin{pmatrix} 0 & 0 \frac{h_3(\Phi^1(z))}{\nu} \\ h_2(\Phi^1(z)) & 0 \frac{h_3(\Phi^1(z))}{\nu} \end{pmatrix} \begin{pmatrix} 0 \\ h_3(\Phi^1(z)) \end{pmatrix}. \] (32)

Denote the desired poles to be assigned for \( A_c \) in Eq. (31) as \( v_1, v_2 \) and \( v_3 \). The new control input should be chosen as

\[ \bar{u} = -\zeta_1 z_1 - \zeta_2 z_2 - \zeta_3 z_3, \] (33)

where

\[ \zeta_1 = v_1 v_2 v_3, \]
\[ \zeta_2 = v_1 v_2 + v_2 v_3 + v_3 v_1, \]
\[ \zeta_3 = v_1 - v_2 - v_3. \]

By choosing \( v_1, v_2 \) and \( v_3 \) such that \( A_c \) becomes Hurwitz, for any positive matrix \( S \), there exists positive symmetric matrix \( P \) such that the following Lyapunov equation

\[ A_c^T P_c + P_c A_c = -S \] (34)

holds. Then, by equiping a function \( V_c(z) \) as

\[ V_c(z) = z^T P_c z, \] (35)

we have

\[ \mathcal{L}(V_c(z)) = z^T P_c A_c z + z^T A_c^T P_c z + \frac{1}{2} \text{tr} \left( \frac{\partial^2 V_c}{\partial z^2} h_3^T \right) \] (36)

Since the inverse transformation of Eq. (29) is given by

\[ x = \Phi^{-1}(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}, \] (37)

it follows from Eqs. (26) and (37) that

\[ \| x \| \leq \max \left\{ 1, \frac{J}{K_m} \right\} \| z \|. \]

Therefore, if we guarantee that

\[ \| z \| \leq \tilde{\delta}_1 := \min \left\{ 1, \frac{K_m}{J} \right\} \delta_1 \] (38)

with probability more than or equal to \( \rho \), the objective probability inequality (8) is achieved. Now, we suppose that the noise port of the system \( h(x) \) in Eq. (27) is such that there exists positive constant \( k_s \) satisfying

\[ \frac{1}{2} \text{tr} \left[ P_c h_3^T(z) h_3(z)^T \right] \leq k_s (1 + \| z \|^2) \] (39)

with respect to the last term in Eq. (36), which results from the noise effect in Itô calculus. From Eqs. (36) and (39), we have

\[ \mathcal{L}(V_c(z)) \leq - (\lambda_{\max}(S_c) - k_s) \| z \|^2 + k_c. \] (40)

From Eq. (40) we define

\[ \tilde{\delta}_d := \sqrt{\frac{k_c}{\lambda_{\max}(S_c) - k_s}}. \] (41)
In this case, since we can choose functions $\alpha_M$ and $\alpha_v$ in Theorem 1 as $\alpha_M(\|z\|) = \lambda_{max}(P_z)\|z\|^2$ and $\alpha_v(\|z\|) = \lambda_{max}(P_z)\|z\|^2$, respectively. Therefore, Theorem 1 gives a sufficient condition for the control objective. For given bounded region parameter $\delta_1$ and realization probability $\rho$, if we choose $\xi_1$, $\xi_2$, and $\xi_3$ in (33), $S_z$, and $P_z$ in (34), and $\delta_0$ satisfying the following conditions:

\[
\begin{align*}
\lambda_{max}(S_z) - k_z &> 0, \\
\delta_0^2 + \delta_3^2 &\leq (1 - \rho)\lambda_{max}(P_z)\delta_1^2,
\end{align*}
\]

where $\delta_1$ and $\delta_3$ are given by Eqs. (38) and (41), respectively. Then, for the closed-loop system (24) with Eqs. (28) and (33), the probability inequality (8) is achieved, where the initial region parameter $\delta_0$ is given by

\[
\delta_0 = \min \left\{ 1, \frac{K_m}{J} \right\} \delta_1.
\]

Finally, let us show some numerical simulation results. Here, we consider the noise port $h(x)$ in Eq. (27) as $h_1(x) = 0$ and $h_2(x) = (\gamma_1 \omega, \gamma_2 \lambda, \gamma_3)$, where $\gamma_1$, $\gamma_2$, and $\gamma_3$ are positive constants, respectively. Thus, we have

\[
h(x) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \gamma_1 \omega & \gamma_2 \lambda & \gamma_3 \end{bmatrix}.
\]

Then, since Eqs. (32) and (45) yield

\[
h_i(z)h_i(z) = \begin{bmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & K(z_1) \gamma_1^2 / Jz_z + (D_2 \gamma_2 z_2 + \gamma_2 z_2)^2 + K(z_1) \gamma_2^2 / Jz_z \end{bmatrix},
\]

we have

\[
\frac{1}{2} \text{tr} \left[ P_z h_i h_i^\top \right] = \frac{1}{2} \text{tr} \left[ P_{1,3,3} \left( \frac{K(z_1) \gamma_1^2}{J^2} z_z + \frac{D_2 \gamma_2 z_2 + \gamma_2 z_2}{J^2} \right)^2 \right]
\]

where $[P_{1,3,3}]$ represents the $(3, 3)$ element of $P_z$.

The concrete parameters used in the simulation are $R = 1.0 \Omega$, $L = 2.0 \times 10^{-5} \text{H}$, $J = 5.0 \times 10^{-2} \text{kgm}^2$, and $D = 5.0 \times 10^{-5} \text{Nms/rad}$. $K(\theta)$ in Eq. (25) is chosen as $K(\theta) = 0.3 + 0.03 \cos(66\theta)$ Nms/A, which approximates the residual torque ripple after being smoothed by 3 segments commutator. It follows that $K_m = 0.27$ and $K_M = 0.33$. We choose the coefficients of the noise port in Eq. (45) as $\gamma_1 = 0.1$, $\gamma_2 = 0.4$, and $\gamma_3 = 0.1$. We empirically set design parameters as $\nu_1 = \nu_2 = \nu_3 = -1$ and

\[
S_z = \begin{bmatrix} 6 & 2 & 4 \\
2 & 4 & 2 \\
4 & 2 & 6 \end{bmatrix}, \quad P_z = \begin{bmatrix} 0 & 10 & 7 & 3 \\
10 & 3 & 7 & \end{bmatrix}
\]

so that the condition (42) is satisfied. Here, we numerically calculate $k_z$ in Eq. (39) in the range of $-\pi \leq \theta \leq \pi$, $-10 \leq \omega \leq 10$, and $-10 \leq t \leq 10$. The result is $k_z = 0.45$. It follows from Eq. (46) that $\lambda_{max}(S_z) = 2.0$, and $\lambda_{max}(P_z) = 0.88$ and $\lambda_{max}(P_z) = 18$.

We choose the bounded region parameter as $\delta_1 = 10$ and the assigned probability as $\rho = 0.80$. Then, $\delta_1$ is given from Eq. (38) as $\delta_1 = 10$, and $\delta_3$ is given from Eq. (41) as $\delta_3 = 0.54$. According to those parameters, we decide the initial region parameter as $\delta_0 = 0.80$ so that the other condition (43) is satisfied. Then, we have $\delta_0 = 0.80$ from Eq. (44). Consequently, Theorem 1 guarantees that the converted system (31) with Eq. (33) is $(\bar{Q}_0, \bar{Q}_1, \rho)$-stable with $\bar{Q}_0 = \{ x \in \mathbb{R}^3 \mid \|z\| \leq 0.80 \}$ and $\bar{Q}_1 = \{ x \in \mathbb{R}^3 \mid \|z\| < 10 \}$, and also the system (24) with Eqs. (28) and (33) is $(\bar{Q}_0, \bar{Q}_1, \rho)$-stable with $\bar{Q}_0 = \{ x \in \mathbb{R}^3 \mid \|z\| \leq 0.80 \}$ and $\bar{Q}_1 = \{ x \in \mathbb{R}^3 \mid \|z\| < 10 \}$. It implies that $P \sup_{\|x(0)\| \leq \rho} \|x(t)\| < 10 \geq \rho = 0.80$ holds.

We set the initial state as $x(0) = (0.80, 0, 0)^T$, which implies that $x(0) \in \bar{Q}_0$. The simulation is executed on $t \in [0, 10]$ [s]. The simulation results are shown in Figs. 4 to 6. Figure 4 denotes the responses of the state $x = (\theta, \omega, \lambda)^T$. This figure shows that the state is approaching to the origin and fluctuates due to the persistent disturbances. Figure 5 denotes the time history of the function $V(x)$ in Theorem 1 along the closed-loop system, where $V(x)$ is given as $V(x) = V_i(\Phi^{-1}(x))$ from $V_i(x)$ in Eq. (35). This figure shows that the function $V(x)$ decreases to zero and fluctuates around zero. Finally, Figure 6 denotes the time history of the norm of the state $\|x(t)\|$ and the bounded region parameter $\delta_1$, where Theorem 1 guarantees that $\|x(t)\|$ is below $\delta_1 = 10$ for all $t \geq 0$ with probability more than or equal to $\rho = 0.80$. Furthermore, we generate 9 other sample paths $x_i$s under the same controller, and exhibit the maximum envelop curve of the norm of all sample paths in the same figure in dotted line. Although the simulation is executed on a finite time interval, this figure implies that the control objective is achieved.

5. Conclusion

This paper has investigated stochastic bounded stability for a general class of nonlinear stochastic systems. In the proposed framework, the bounded region for sample paths and realization probability are both assignable, and it is guaranteed that
the sample paths remain bounded in the given region with assigned probability. We have provided a sufficient condition for the proposed bounded stability to be satisfied based on a Lyapunov-like function. Finally, numerical simulations have demonstrated the effectiveness of the proposed method.

Since the provided sufficient condition is possibly conservative due to inequality estimates in its derivation, one of the challenges is to decrease the conservativeness.

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References