Robustification of a Nonlinear Dynamical System with a Stability Index and a Matrix Inequality

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Abstract: A problem in the field of nonlinear dynamics is considered with a technique of control theory. In particular, for preventing a nonlinear dynamical system from making undesirable bifurcation, it is proposed to control its parameter with a stability index and a matrix inequality. A basic idea is to update the parameter of the system so as to minimize the stability index. Since the stability index is not differentiable as a function of the parameter, its minimization is carried out through equivalent transformation to a smooth minimization problem, where a matrix inequality is used. The proposed method is applied to a simple dynamical system and shown to be effective. Its generalization is also considered for driving a system to avoid chaos.

Key Words: stability index, matrix inequality, penalty function method, local expansion rate.

1. Introduction

Consider a nonlinear dynamical system dependent on some parameter. As the parameter value varies, the system often makes bifurcation and changes its behavior qualitatively, which is a subject of bifurcation analysis. On the other hand, from an engineering point of view, it is important to keep the system from making such qualitative change. For example, an alternating pulse of the heart is claimed to be a consequence of bifurcation. In particular, they noticed the stability index at a fixed point. Indeed, Kitajima et al. [2] did not minimize the stability index directly but a function closely related to it. They also needed an assumption that the eigenvalue of maximum magnitude is real.

In this paper, we try more direct approach to this problem with a technique from control theory, i.e., a matrix inequality, and reduce the problem to a smooth minimization problem. The resulting problem can be solved with the penalty method and gives an update rule of the parameter, which resembles that of Kitajima et al. [2]. The obtained update rule is tested with a simple dynamical system to show its efficacy. Finally, generalization of this approach is considered for keeping the system from falling into chaos. The results of this paper were partly presented at conferences [3],[4] and in a book [5].

The following notation is used. The transpose of a matrix or a vector is denoted by T. The trace of a matrix is expressed by tr and a diagonal matrix is expressed by diag. For a square matrix D, the symbol ρ(D) denotes the maximum absolute value over the eigenvalues of D. For a symmetric matrix T, the matrix inequalities $T > O$ and $T ≥ O$ stand for positive definiteness (all the eigenvalues are positive) and positive semidefiniteness (all the eigenvalues are nonnegative) of T, respectively.

2. Considered Problem

Consider a discrete-time dynamical system

$$z(k + 1) = f(z(k), λ) \quad (k = 0, 1, 2, \ldots)$$

having a state $z \in \mathbb{R}^n$ and a parameter $λ \in \mathbb{R}^m$. Here, $f(z, λ)$ is some smooth function. If the value of $λ$ varies due to external noise and/or internal fluctuation, the system may reach undesirable bifurcation. Kitajima et al. [2] proposed mechanism to automatically control the parameter value for suppression of such bifurcation. In particular, they noticed the stability index at a fixed point.

Suppose that a fixed point of this system, that is, a state $z_λ$ satisfying $z_λ = f(z_λ, λ)$, is obtained for any $λ$ in some nonempty open set $\Lambda \subseteq \mathbb{R}^m$ and that it is smooth as a function of $λ$. Then, for the Jacobian of $f(z, λ)$ at $z = z_λ$, that is,

$$D(λ) := \frac{∂}{∂z} f(z, λ) \bigg|_{z=z_λ},$$

the maximum absolute value over its eigenvalues, i.e., $ρ(D(λ))$, is called the stability index at the fixed point $z_λ$. Indeed, it is smaller than unity if and only if the fixed point $z_λ$ is asymptotically stable. Kitajima et al. [2] considered the following problem:

$$S : \text{minimize } ρ(D(λ)) \text{ in } λ$$

subject to $λ ∈ Λ$. 

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If the parameter $\lambda$ is updated so as to converge to a local minimum solution of the problem, the dynamical system is driven to increase its stability. Hence, even if the system is subject to unexpected change of the parameter, it is expected to recover its stability in due course.

**Remark.** We consider in this paper minimization of $\rho(D(\lambda))$ to attain the optimal stability. If the objective is only the stability of the system, it is possible to stop updating $\lambda$ as soon as $\rho(D(\lambda)) < 1$ is satisfied.

3. **Proposed Method**

Although the matrix $D(\lambda)$ is smooth with respect to $\lambda$, the stability index $\rho(D(\lambda))$ is not differentiable in general with respect to $\lambda$ and hence its minimization needs some special technique. Kitajima et al. [2] did not minimize $\rho(D(\lambda))$ directly but minimized a smooth function closely related to it.

In this paper, we use the following result well-known in control theory [6, Section 4.1.2].

**Proposition.** For a square matrix $D$ and a positive number $\rho$, there holds $\rho(D) < \rho$ if and only if there exists a symmetric matrix $X$ that satisfies

$$
\begin{bmatrix}
\rho^2 X & D^T X \\
XD & X
\end{bmatrix} > O.
$$

Use this result and rewrite $\rho^2$ as $\alpha$ to have the following minimization problem equivalent to $S$:

$S'$: minimize $\alpha$ in $(\alpha, \lambda, X)$

subject to $\lambda \in \Lambda, \begin{bmatrix}
\alpha X & D(\lambda)^TX \\
XD(\lambda) & X
\end{bmatrix} > O.$

In this problem, both objective function and constraints are smooth functions of the optimization variables $(\alpha, \lambda, X)$. Hence, a standard method can be used for the optimization.

**Remark.** A different approach is possible to the minimization of $\rho(D(\lambda))$ such as application of nonsmooth minimization. An advantage of the present approach is that we can convert a nonsmooth minimization problem to a smooth one with a simple technique and can avoid all the delicate issues in nonsmooth minimization [7].

Although a minimization problem with a matrix inequality constraint is common in the field of control theory [6, 8], its objective function and constraint are usually affine there. Our problem $S'$ has the matrix inequality constraint nonlinear in $(\alpha, \lambda, X)$ and thus cannot be solved with the usual technique in control theory. In this paper, we apply the penalty function method of Kočvara–Stingl [9] to solve $S'$. Its key idea is to construct a penalty function that has a small value when the matrix inequality constraint is satisfied and a large value when it is not satisfied. Adding this penalty function to the objective function, we can perform minimization without considering the matrix inequality constraint explicitly.

The desired penalty function can be constructed as follows. First, we define a function $\phi_p(t)$ having a positive scheduling parameter $p$ by

$$
\phi_p(t) := p \left(1 - \frac{1}{t/p + 1} - 1\right).
$$

For $t \geq 0$, the function $\phi_p(t)$ approaches zero as $p \downarrow 0$; for $t < 0$, it blows up to infinity as $p \downarrow -t$. On the other hand, let $T$ denote the left-hand side matrix of the matrix inequality constraint of $S'$, which is a symmetric matrix of order $2n$.

Suppose that this $T$ is diagonalized by some orthogonal matrix $Q$ as $T = Q \text{diag}(t_1, t_2, \ldots, t_2n)Q^T$. Here, we write the matrix $Q \text{diag}(\phi_p(t_1), \phi_p(t_2), \ldots, \phi_p(t_{2n}))Q^T$ as $\Phi_p(T)$ and consider $\text{tr}(U\Phi_p(T))$ for some arbitrary positive definite matrix $U$. Then this function $\text{tr}(U\Phi_p(T))$ can be used as the desired penalty function. Indeed, for $T \geq 0$, the function $\text{tr}(U\Phi_p(T))$ converges to zero as $p \downarrow 0$; for $T \not\geq 0$, it blows up to infinity as $p \downarrow p_0$, where $p_0$ is some positive number.

Based on the discussion so far, we can expect that the problem $S'$ can be solved by iteration of the following three steps:

1. Update $(\alpha, \lambda, X)$ so as to make

$$
\alpha + \text{tr}(U\Phi_p(\begin{bmatrix}
\alpha X & D(\lambda^T)X \\
XD(\lambda) & X
\end{bmatrix})) \text{ smaller};
$$

2. Update $U$;


More concretely, the update in Step 1 is performed in the steepest descent direction or in the Newton direction of the objective function with an appropriate step size. These directions are computable with the derivatives of $D(\lambda)$. The update in Step 2 is carried out so that the Karush–Kuhn–Tucker condition of $S'$ is satisfied. The update in Step 3 is performed so that $p$ approaches zero. See [9] for the details.

4. **Extension**

In the preceding problems $S$ and $S'$, the availability of a fixed point $z_1$ has been assumed. When this assumption cannot be made, the following problem is considered instead:

$R$ : minimize $\alpha$ in $(z, \alpha, \lambda, X)$

subject to $f(z, \lambda) - z = 0, \lambda \in \Lambda, \begin{bmatrix}
\alpha X & D(\lambda)^T X \\
XD(\lambda) & X
\end{bmatrix} > O.$

In this new problem, $z$ becomes an optimization variable and the equality constraint $f(z, \lambda) - z = 0$ is added.

Just as in the case of $S$, we can obtain an update rule of $(z, \alpha, \lambda, X)$. Namely, we decompose the newly added equality constraint into two inequality constraints $f(z, \lambda) - z \geq 0$ and $-f(z, \lambda) + z \geq 0$, construct the corresponding penalty functions for each, and add them to the objective function. A similar approach is possible also for a periodic point in place of a fixed point.

For a continuous-time dynamical system $\dot{z} = f(z, \lambda)$, the stability index should be defined as the maximum real part over the eigenvalues of the Jacobian $D(\lambda) := (\partial f(z, \lambda)/(\partial z)|_{z=z_1}$ at a fixed point $z_1$. In order to minimize it, the following problem is considered:

minimize $\alpha$ in $(\alpha, X)$

subject to $\lambda \in \Lambda, X > O,$

$$-X(D(\lambda) - \alpha I) - (D(\lambda) - \alpha I)^TX > O.$$

Application of the penalty function method gives an update rule of $(\alpha, \lambda, X)$ as before.
5. Example

Consider a discrete-time dynamical system based on the Kawakami map [10],[11], that is,
\[
\begin{pmatrix}
 x(k+1) \\
 y(k+1)
\end{pmatrix} = \begin{pmatrix}
 ax(k) + y(k) \\
 x(k)^2 + b
\end{pmatrix},
\]
which has the state \( z = (x \ y)^T \) and the parameter \( \lambda = (a \ b)^T \). In the parameter region \( \Lambda = \{ \lambda = (a \ b)^T | (a-1)^2 - 4b > 0 \} \), the system has a fixed point
\[
\begin{align*}
 z_\lambda &= \left( \frac{(a+1 - \sqrt{(a-1)^2 - 4b})}{2}, \frac{(-a+1 + \sqrt{(a-1)^2 - 4b})}{2} \right) \\
 D(\lambda) &= \begin{pmatrix}
 a & 1 \\
 2x & b
\end{pmatrix}
\]

The Jacobian at this fixed point \( z_\lambda \) is denoted by \( D(\lambda) \). The problem \( S' \) is considered in this setting and solved with the method in Section 3. The result is presented in Fig. 1 (a). Here, the horizontal plane stands for the space of the parameter \( \lambda = (a \ b)^T \) and the curved surface is the graph of the stability index \( \rho(D(\lambda)) \). (The graph is drawn only for \( \lambda \in \Lambda \)). On the horizontal plane, the trajectory of the updated parameter \( \lambda \) is shown. It is seen from the figure that the stability index \( \rho(D(\lambda)) \) is not differentiable at some point. Nevertheless, the parameter \( \lambda \) is updated without any problem and reaches the neighborhood of a local minimum \( \lambda = (0 \ 0)^T \). Here, in Step 1 of the method, the steepest descent direction is employed with more priority put on the stability of the convergence rather than on its speed. The step size is chosen according to the Armijo rule. The initial value of the scheduling parameter \( p \) is 200 and is updated by multiplication of 0.9 in Step 3. The additional constraint \( \lambda \in \Lambda \) is guaranteed just by limiting update of the parameter \( \lambda \) in \( \Lambda \).

Next we consider the case where a fixed point is not available and apply the method of Section 3 to the problem \( R \) in Section 4. The result is presented in Fig. 1 (b). Again, the parameter successfully reaches the neighborhood of a local minimum \( \lambda = (0 \ 0)^T \).

6. Avoidance of Chaos

The technique of a matrix inequality can be used for more general purpose. In this section, we use it for minimizing the local expansion rate and preventing the system from falling into chaos.

6.1 Method for Chaos Avoidance

Let the maximum Lyapunov exponent with respect to \( z_0 \) be
\[
\gamma(z_0, \lambda) := \lim_{N \to \infty} \frac{1}{N} \log \rho(D^N(z_0, \lambda)) \\
= \lim_{N \to \infty} \frac{1}{N} \log \rho(D(z_N-1, \lambda) \cdots D(z_1, \lambda)D(z_0, \lambda)),
\]
where \( \{z_0, z_1, \ldots, z_{N-1}\} \) is the trajectory of the dynamical system (1) starting from \( z_0 \). In the special case that \( \{z_0, z_1, \ldots, z_{p-1}\} \) is a periodic orbit of period \( p \), we have
\[
\gamma(z_k, \lambda) = \frac{1}{p} \log \rho(D^p(z_0, \lambda)) \quad (k = 0, 1, \ldots, p - 1).
\]
Since exact computation of the maximum Lyapunov exponent \( \gamma(z_0, \lambda) \) is difficult, we may use instead the finite-time Lyapunov exponent, or the local expansion rate, defined by
\[
\Gamma(N, z_0, \lambda) := \frac{1}{N} \log \rho(D^N(z_0, \lambda))
\]
for some large positive integer \( N \). The local expansion rate \( \Gamma(N, z_0, \lambda) \) usually takes a large value when the system shows a chaotic behavior, and thus is often regarded as an index of chaos. If we update the parameter \( \lambda \) so as to minimize the local expansion rate, the system is expected to avoid falling into chaos. Note that minimization of the local expansion rate \( \Gamma(N, z_0, \lambda) \) is equivalent to that of \( \rho(D^N(z_0, \lambda)) \), for which the technique of a matrix inequality can be used. In particular, we consider the following minimization problem:

\[
\begin{align*}
\text{minimize} \; & \alpha \quad \text{in} \; (a, \lambda, X) \\
\text{subject to} \; & \lambda \in \Lambda, \quad \begin{bmatrix}
\alpha X \\
XD\Gamma(z_0, \lambda) \end{bmatrix} > 0.
\end{align*}
\]

6.2 Experimental Result

We try the proposed method with the system based on the Kawakami map given by Equation (2). In the \((a, b)\) parameter space, a grid equally spaced by 0.01 is taken. Each grid point is chosen as an initial parameter \( \lambda_0 \) and is updated according to the method given in Section 6.1. Figure 2 (a) shows the result together with bifurcation parameter sets. The symbols \( G^p \) and \( I^p \) and \( NS^p \) denote tangent, period-doubling and Neimark–Sacker bifurcations for \( p \)-periodic points, respectively. Chaotic behavior is observed in the shaded parameter region. The parameters \( A_{100} \) obtained after 100 updates are indicated by the small blue dots. A typical trajectory is presented by the red solid line with an arrow. The ends of the line correspond to the initial parameter \( \lambda_0 \) and the final parameter \( A_{100} \). The updates are made in the direction of the arrow.

Figure 2 (b) shows an overlapped image of the local expansion rate for attractors and the bifurcation diagram of periodic points. The colored contour plot presents the values of the local expansion rate, as indicated by the color bar. Cold color (blue) expresses a small local expansion rate and then high stability. We see that the small blue dots in Fig. 2 (a) are mainly distributed in the region with a negative local expansion rate in Fig. 2 (b). Thus our method operates the system to avoid chaos.

7. Conclusion

In this paper, robustification of a nonlinear dynamical system is considered along the approach of Kitajima et al. [2] and a technique with a matrix inequality is introduced. Although the stability index considered here is not differentiable in general, its minimization can be formulated into a smooth minimization problem. Since the resulting minimization problem has a nonlinear matrix inequality constraint, we use the penalty function method of Kočvara–Stingl to obtain an update rule of the parameter. The proposed method is applied to a dynamical system based on the Kawakami map and is shown to be efficient. It can be generalized for avoiding chaos.

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