Local Synchronization of Linear Multi-Agent Systems Subject to Input Saturation

Kiyotsugu TAKABA

Abstract: This paper is concerned with local state synchronization of linear agents subject to input saturation over a fixed undirected communication graph. The author first derives a sufficient condition for locally achieving the synchronization via a relative state feedback control law. Based on this analysis result, the author presents a linear matrix inequality (LMI) condition for designing the synchronizing state feedback gain. The present LMI condition is scalable as long as the eigenvalues of the associated graph Laplacian are available, and is efficiently solved by an existing convex programming algorithm.

Key Words: multi-agent system, synchronization, input saturation, LMI.

1. Introduction

For the last decade, multi-agent coordination has been attracting a great attention in the area of systems and control, since such phenomena can be encountered in many applications in physics, biology, robotics, computer science, etc. (see e.g. [1],[2], and the references therein). The feature of multi-agent systems is that coordinative tasks such as synchronization and consensus are achieved by distributed control of individual agents based on their local interactions.

For coordination of linear multi-agent systems, earlier works mainly focused on consensus of simple agents described by single or double integrators [2]. Recently, more attention has been paid to consensus or synchronization of higher-order general linear agents [3]–[7].

On the other hand, most of practical control systems are subject to input constraints or input saturations due to physical constraints or safety reasons. It is thus important to study the coordination of multi-agent systems under input saturation. There have been some related works: Lin, Xiang, and Wei [8] solved the consensus problem for single integrator agents under input saturation, a leader-follower-type cooperative control was studied by Meng, Zhao and Lin [9], and the discrete-time consensus under input saturation, a single linear system subject to input saturation has been well studied in the literature (e.g. [11],[12]). Finally, we will verify the effectiveness of the present method through numerical simulations.

2. Problem Formulation

2.1 Agent Dynamics

Throughout this paper, we consider the homogeneous multi-agent system consisting of $N$ linear agents subject to input saturation. The dynamics of each agent is described by

\[ \dot{x}_i = A x_i + B \sigma_r(u_i), i = 1, \ldots, N, \]

where $x_i : \mathbb{R}^n \to \mathbb{R}^n$ and $u_i : \mathbb{R}_+ \to \mathbb{R}$ are the state and input of the $i$-th agent. We assume that the input $u_i$ is scalar for simplicity of notation. The results presented in this paper can be easily extended to the multi-input case. The memoryless map $\sigma_r : \mathbb{R} \to \mathbb{R}$ denotes the saturation nonlinearity.

Assumption 1 The agent dynamics (1) is stabilizable.

The saturation nonlinearity $\sigma_r$ is defined by

\[ \sigma_r(u) = \begin{cases} u, & \text{if } -\bar{u}_i \leq u \leq \bar{u}_i, \\ \bar{u}_i, & \text{if } u > \bar{u}_i, \\ -\bar{u}_i, & \text{if } u < -\bar{u}_i. \end{cases} \]

\( \bar{u}_i \): positive constant

\[ \bar{u}_i > 0, \]

\[ N \geq 2. \]

\[ R_+ : = \{ u \in \mathbb{R} | u \geq 0 \}. \]
For later discussion, we also define the dead-zone nonlinearity
\( \phi_i : \mathbb{R} \to \mathbb{R} \) by
\[
\phi_i(u) = u - \sigma_i(u).
\]
Then, \( \phi_i \) satisfies
\[
\phi_i(u)(\phi_i(u) - \theta u) \leq 0 \quad \forall u \in [-\mu, \mu]
\]
for a constant \( \theta \in [0, 1) \) (Fig. 1), where
\[
\mu = \min_{i \in V} \frac{\bar{a}_i}{1 - \theta}.
\]

Assumption 2

(i) The topology of the graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) is time-invariant, namely, it is fixed at all times.

(ii) \( \mathcal{G} \) is a connected graph, namely, there always exists at least one undirected path between any two nodes.

The graph Laplacian \( L \in \mathbb{R}^{n \times n} \) associated with the graph \( \mathcal{G} \) is defined by
\[
L = (\ell_{ij}), \quad \ell_{ij} = \begin{cases} |N_i|, & \text{if } i = j, \\ -1, & \text{if } (i, j) \in \mathcal{E}, \\ 0, & \text{otherwise}. \end{cases}
\]

It should be noted that \( L \) is non-negative definite for undirected graphs, and it always has a simple zero eigenvalue with an eigenvector \( \mathbf{1} := [1, 1, \cdots, 1]^\top \in \mathbb{R}^n \). For ease of later discussion, we denote the eigenvalues of \( L \) by \( \lambda_i, i = 1, \ldots, n \) in the ascending order:
\[
0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n.
\]
The graph \( \mathcal{G} \) is a connected graph if and only if \( \lambda_2 > 0 \).

2.3 Problem statement

The state synchronization problem considered in this paper is to find a feedback control law that satisfies
\[
\lim_{t \to \infty} \| x_i(t) - x_j(t) \| = 0 \quad \forall i, j \in \mathcal{V}
\]
for all initial states \( (x_1(0), \ldots, x_N(0)) \in \mathcal{R} \), where \( \mathcal{R} \) is a certain region in \( \mathbb{R}^n \). Such a control law is said to locally synchronize the multi-agent system, when \( \mathcal{R} \) is a proper subset of \( \mathbb{R}^n \).

As is well known, local stabilization of a linear centralized system with input saturation is essential, since global stabilization is not possible when the system contains a exponentially unstable eigenvalue [16]. Along this line, we will consider the local synchronization problem stated above.

One of the typical strategies for distributed coordinative control is the relative state feedback law
\[
u_i = F \sum_{j \in N_i} (x_i - x_j), \quad (5)
\]
where \( F \) is the feedback gain to be designed. In this paper, we will consider the analysis and design of the multi-agent system with this feedback control law subject to input saturation.

3. State Synchronization Problem

3.1 Analysis

We here define
\[
x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}, \quad \Sigma(u) = \begin{bmatrix} \sigma_1(u_1) \\ \sigma_2(u_2) \\ \vdots \\ \sigma_N(u_N) \end{bmatrix}
\]
Then the multi-agent system of (1) and (5) is equivalently rewritten as
\[
\dot{x} = (I \otimes A)x + (I \otimes B)\Sigma(u), \quad (6a)
\]
\[
u = (L \otimes F)x, \quad (6b)
\]
where $\otimes$ is the Kronecker product, $I$ denotes the identity matrix of compatible size, and $L$ is the graph Laplacian associated with the communication graph $\mathcal{G}$. Moreover, we define
\[
\Phi(u) = u - \Sigma(u) = \begin{bmatrix} \varphi_1(u_1) \\ \vdots \\ \varphi_N(u_N) \end{bmatrix},
\]
where $\varphi$ is defined by (2). Then, (6) is equivalent to
\[
\dot{x} = (I \otimes A + L \otimes BF)x - (I \otimes B)w,
\]
\[
u = (L \otimes F)x,
\]
\[
w = \Phi(u).
\]
It also follows from (3) that
\[
w^T(w - \theta u) \leq 0
\]
holds for $\theta \in [0, 1)$, and for $w = \Phi(u)$, $u \in \mathbb{R}^N$ such that $u_i \in [-\mu, \mu]$, $i = 1, \ldots, N$.

Let $\Lambda$ and $U$ be the diagonal matrix and the orthogonal matrix such that
\[
ULU^T = \Lambda, \quad \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N).
\]
We introduce the coordinate transformation
\[
\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_N \end{bmatrix} = (U \otimes I)x.
\]
Since $UU^T = I$ and $LU^T = U^T \Lambda$, (7) reduces to
\[
\dot{\xi} = (I \otimes A + \Lambda \otimes BF)\xi - (U \otimes B)w,
\]
\[
u = (U^T \Lambda \otimes F)\xi,
\]
\[
w = \Phi(u).
\]
Here, we have used the identity
\[
(X \otimes Y)(Z \otimes W) = (XZ) \otimes (YW).
\]
Since $L \Lambda = 0$, the first row of $U$ is $1^T / \sqrt{N}$. It thus follows that $\xi_i = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i$. It is easily seen from (10) that, if $\lim_{t \to \infty} \|\xi_i(t)\| = 0$, $i = 2, \ldots, N$, then
\[
x(t) \to (U^T \otimes I) \begin{bmatrix} \xi_1(t) \\ \vdots \\ 0 \end{bmatrix} = \frac{1}{\sqrt{N}} \otimes \xi_1(t) \quad (t \to \infty).
\]
This implies that the state synchronization is achieved.

With these preparation, a sufficient condition for local state synchronization is stated in the following theorem.

**Theorem 1** Under Assumptions 1 and 2, for a given feedback gain $F$, assume that there exist a positive definite matrix $P$ and a scalar constant $\theta \in [0, 1)$ satisfying
\[
\begin{bmatrix}
(A + \Lambda BF)^T P + P(A + \Lambda BF) \\
\theta \lambda_i F^T - PB \\
\theta \lambda_i F - B^T P
\end{bmatrix} < 0,
\]
\[
P \geq \lambda_i^2 \mu^{-2} F^T F, \quad \mu = \frac{\bar{u}}{1 - \theta},
\]
where $\mu$ is defined in (4). Then, the multi-agent system of (6) achieves the local state synchronization under input saturation.

(Proof) Based on the earlier discussion, we shall prove the theorem by showing the convergence of $\xi_2, \ldots, \xi_N$ for the system (11) under the assumption that $P > 0$ satisfies (12).

We define
\[
V(\xi) = \xi^T P \xi,
\]
\[
P = \text{diag}(0, P, \ldots, P), \quad J = \text{diag}(0, I, \ldots, I).
\]
and the convex region
\[
\Xi(P) := \left\{\xi \in \mathbb{R}^N \mid V(\xi) \leq 1\right\}.
\]
Note that $V$ is nonnegative definite, and its derivative along the trajectory of the system (11) is given by
\[
\dot{V}(\xi) = 2\xi^T P [(I \otimes A + \Lambda \otimes BF)\xi - (U \otimes B)w].
\]

Claim 1: Calculation of $\dot{V}(\xi)$

Since (12a) is satisfied, there exists a scalar constant $\epsilon > 0$ such that
\[
\begin{bmatrix}
(A + \lambda_i BF)^T P + P(A + \lambda_i BF) + \epsilon I \\
\theta \lambda_i F^T - PB \\
\theta \lambda_i F - B^T P
\end{bmatrix} \leq 0,
\]
\[
\epsilon > 0, \quad i = 2, \ldots, N
\]
Stacking and re-arranging these inequalities yields
\[
\begin{bmatrix}
(I \otimes A + \lambda \otimes BF)^T P + P(I \otimes A + \lambda \otimes BF) + \epsilon J \\
\theta (\lambda \otimes F) - (I \otimes B)^T P
\end{bmatrix} \leq 0
\]
We apply the congruence transformation with $\text{diag}(I, (U \otimes I))$ to the above inequality to obtain
\[
\begin{bmatrix}
(I \otimes A + \lambda \otimes BF)^T P + P(I \otimes A + \lambda \otimes BF) + \epsilon J \\
\theta U^T \Lambda \otimes F - (U \otimes B)^T P
\end{bmatrix} \leq 0
\]
Pre-multiplying the above inequality with $[\xi^T \ w^T]$ and post-multiplying with $[\xi \ w]$ yields
\[
\xi^T P [(I \otimes A + \lambda \otimes BF)\xi - (U \otimes B)w] + [(I \otimes A + \lambda \otimes BF)\xi - (U \otimes B)w]^T P \xi + \theta \xi^T (U^T \Lambda \otimes F) w + \epsilon \xi^T J \xi - 2w^T w \leq 0.
\]
Thus, we have
\[
\dot{V}(\xi) + \epsilon \sum_{i=2}^{N} ||\xi_i||^2 \leq 2w^T (w - \theta u)
\]
with $w = \Phi(u)$, $u = (U^T \Lambda \otimes F)\xi$.

Claim 2: $\xi \in \Xi(P)$ implies $u_i \in [-\mu, \mu]$, $i = 1, \ldots, N$.

Assume that $\xi$ belongs to $\Xi(P)$. Noting the orthogonality of $U$, we stack (12b) up from $i = 2$ to $N$ to get
\[
P \geq \mu^{-2} \Lambda \otimes (F^T F)
\]
\[

(16)
\]

(15)
Pre-multiplying this by $\xi^T$ and post-multiplying by $\xi$ yield

$$\mu^2 \|u\|^2 \leq \xi^T P \xi,$$

where $u$ is given by (11b). Hence, we have

$$u_i \in [-\mu, \mu], \ i = 1, 2, \ldots, N, \ \forall \xi \in \Xi(P). \quad (17)$$

**Claim 3:** $\Xi(P)$ is positively invariant.

It follows from (3), and (17) that $w^T (w - \theta u) \leq 0$ holds when $\xi \in \Xi(P)$ and $u = (U^T \Lambda \otimes F) \xi$. Thus, we obtain from (15) that

$$\dot{V}(\xi) \leq -\epsilon \sum_{i=2}^{N} \|\xi_i\|^2 \leq 0 \quad (18)$$

holds for $\xi \in \Xi(P)$. This implies that $\Xi(P)$ is a positively invariant set, because $V(\xi)$ is non-increasing along the state trajectory starting at $\xi(0) \in \Xi(P)$.

**Claim 4:** Convergence of $\xi_2, \ldots, \xi_N$

By La Salle’s invariance principle, the trajectory of $\xi$ starting from $\Xi(P)$ converges to the largest invariant set contained in $\{\xi \in \mathbb{R}^{nN} | V(\xi) = 0 \} \subset \Xi(P)$. It then follows from (18) that $\xi_2, \ldots, \xi_N$ converge to 0 as $t$ goes to infinity.

Consequently, we conclude that the state synchronization is locally achieved by the relative feedback control law (5).

The matrix inequalities in (12) are affine in $\lambda_i$. In addition, since $\lambda^i's$ are arranged in the ascending order, $\lambda_1, \ldots, \lambda_{N-1}$ can be represented as convex combinations of $\lambda_2$ and $\lambda_N$. Thus, we can reduce the number of matrix inequalities in Theorem 1, and the following result is more suitable for large-scale networks.

**Corollary 1** Under Assumptions 1 and 2, for a given feedback gain $F$, assume that there exist a positive definite matrix $P$ and a positive constant $\theta \in [0, 1]$ satisfying

$$\begin{bmatrix}
(A + \lambda_i BF)^T P + P(A + \lambda_i BF) & 0 \\
\theta \lambda_i F - B^T P & -2I
\end{bmatrix} < 0, \ i = 2, \ldots, N \quad (19a)
$$

$$P \geq \frac{\lambda^2}{\alpha^2} F^T F, \quad (19b)$$

where $\mu$ is defined in (4). Then, the multi-agent system of (6) locally achieves the state synchronization under input saturation.

**Remark 1** The synchronization conditions in Theorem 1 and Corollary 1 are linear matrix inequalities (LMIs) in $P$ when $F$ and $\theta$ are fixed. Hence, these conditions can be efficiently checked by using convex programming algorithms [17].

**Remark 2** In comparison with the previous works on local stabilization of linear centralized systems subject to input saturation (e.g. [18]), the matrix inequalities in Theorem 1 and Corollary 1 imply that the linear saturating system

$$\dot{\bar{u}} = \lambda_i \bar{F} \bar{\xi}, \ i = 2, \ldots, N.$$

is locally stabilized by the state feedback laws

**Remark 3** From (11), the closed-loop multi-agent system without input saturation, i.e. $w = 0$, is given by

$$\dot{\xi} = (I \otimes A + \Lambda \otimes BF) \xi.$$

Hence, in the absence of input saturation, a necessary and sufficient condition for achieving synchronization is that $A + \lambda_i BF, \ i = 2, \ldots, N$ are all Hurwitz stable, i.e. all eigenvalues have negative real parts (see Lemma 1 of [4] for a similar result). The Hurwitz stability of these matrices is guaranteed in Theorem 1 and Corollary 1, since the (1, 1)-block element of (12a), (19a) implies the Lyapunov inequalities

$$(A + \lambda_i BF)^T P + P(A + \lambda_i BF) < 0, \ i = 2, \ldots, N. \quad (20)$$

**Remark 4** It is seen from the proof of Theorem 1 that the convex subset

$$\Omega(P) := \{x \in \mathbb{R}^{nN} | x = (U^T \otimes I) \xi, \ \xi \in \Xi(P)\}$$

is an inner approximation of the region of attraction $\mathcal{R}$ which is described in the problem statement. The subset $\Omega(P)$ is an ellipsoidal hyper-cylinder, where $\xi_i \in \mathbb{R}^n$ belongs to the ellipsoid $\{x \in \mathbb{R}^n | x^T P x \leq 1\}$ for $i = 2, \ldots, N$, and $\xi_1 \in \mathbb{R}^n$ is arbitrary.

**Remark 5** The LMI conditions presented in this paper can be made feasible by appropriate choice of $\theta \in [0, 1]$ at the price of conservativeness. For example, by taking $\theta = 0$ and the Schur complement formula, the LMI (19) in Corollary 1 reduces to

$$(A + \lambda_i BF)^T P + P(A + \lambda_i BF) + \frac{1}{\alpha^2} PBB^T P < 0, \ i = 2, \ldots, N \quad (21a)$$

$$P \geq \frac{\lambda^2}{\alpha^2} F^T F. \quad (21b)$$

Let $P$ be given by $P := \alpha P_0$, where $P_0$ is a positive definite solution of (20), and $\alpha$ is a positive constant. Then, it is easily verified from (20) and $P_0 > 0$ that, for sufficiently large $\alpha > 0$, $P = \alpha P_0$ satisfies the LMIs in (21). This observation implies that the LMI (19) with $\theta = 0$ is feasible if the synchronization in the absence of input saturation is achievable.

In this case, the subset $\Omega(P)$ is confined into the linear region, namely, saturation of input signals is suppressed as long as the states of the closed-loop multi-agent system remain inside $\Omega(P)$.

### 3.2 Design of State Feedback Gain

We will present a method to design a synchronizing feedback gain $F$ based on the results of the previous sub-section.

Based on Corollary 1, we wish to find a positive definite matrix $P$ and a feedback gain $F$ satisfying (19). However, since (19) contains a bilinear term between $P$ and $F$, we need to convexify the matrix inequalities in order to effectively solve the problem.

As a usual technique for the convexification, we perform the change of variables of $P$ and $F$ as $X = P^{-1}$ and $Y = FP^{-1}$ [17]. Then, by applying the congruence transformation with $\text{diag}(P^{-1}, I)$ to (19a), and by applying the Schur complement formula to (19b), we end up with the following theorem.

**Theorem 2** Let $\theta \in [0, 1]$ be given. Under Assumptions 1 and 2, assume that there exist a positive definite matrix $X$ and a matrix $Y$ satisfying
\[
\begin{bmatrix}
(AX + \lambda_1BY)^T + (AX + \lambda_1BY) \\
\theta_1\lambda_1Y^T - B \\
\theta_1\lambda_1Y - B^T
\end{bmatrix} < 0, \quad i = 2, N.
\]
\[\begin{bmatrix}
X & \lambda_NY^T \\
\mu^2I
\end{bmatrix} \geq 0.
\]
Then, there exists a feedback gain \( F \) that achieves the local state synchronization under input saturation. One of such feedback gains is given by
\[ F = YX^{-1}. \]

An inner approximation of the region of synchronization \( \mathcal{R} \) is given by
\[ \Omega(X^{-1}) = \{ x \in \mathbb{R}^{n_N} | x = (U^T \otimes I)\xi, \quad \xi \in \Xi(X^{-1}) \}, \]
where \( \Xi \) and \( U \) are defined in (13) and (9).

The matrix inequalities in (22) are LMIs in the variables \( X, Y \). The size of the LMI problem does not depend on the size of the network, \( N \), as long as the Laplacian eigenvalues \( \lambda_2 \) and \( \lambda_N \) are available. Note that, in particular, analytic formulae of those eigenvalues for some typical network topologies are known as in Table 1 [19]. Therefore, we can effectively design the synchronizing feedback gain \( F \) by convex programming.

Table 1 Laplacian eigenvalues for typical graphs.

<table>
<thead>
<tr>
<th>topology</th>
<th>( \lambda_2 )</th>
<th>( \lambda_N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line</td>
<td>( 2(1 - \cos \frac{\pi}{N}) )</td>
<td>( 2(1 + \cos \frac{\pi}{N}) )</td>
</tr>
<tr>
<td>Cycle</td>
<td>( 2(1 - \cos \frac{\pi}{N}) )</td>
<td>( 2(1 + \cos \frac{\pi}{N}) ) (N: even)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 4 ) (N: odd)</td>
</tr>
<tr>
<td>Star</td>
<td>1</td>
<td>( N )</td>
</tr>
<tr>
<td>Complete</td>
<td>( N )</td>
<td>( N )</td>
</tr>
</tbody>
</table>

Recall from Remark 4 that \( \Omega(X^{-1}) \) is an elliptic hypercylinder, and that \( \chi \) is one of the measures of the size of the ellipsoid \( \{ x \in \mathbb{R}^n | x^T X^{-1} x \leq 1 \} \). Hence, a tight inner approximation \( \Omega(X^{-1}) \subset \mathcal{R} \) can be obtained by maximizing \( \chi \). This maximization is formulated as the following convex programming problem with the LMI constraints.

\[
\begin{bmatrix}
\chi \\
\sigma(\chi)
\end{bmatrix} \quad \text{maximize} \quad \text{trace} \ X \quad \text{subject to (22a)} \quad \text{and (22b)}
\]

4. Numerical Examples

Consider the multi-agent system consisting of the linear agents described by the state equation
\[
\begin{bmatrix}
\dot{x}_{i1} \\
\dot{x}_{i2}
\end{bmatrix} = \begin{bmatrix}
0.03 & 0.9 \\
-0.9 & 0.03
\end{bmatrix} \begin{bmatrix}
x_{i1} \\
x_{i2}
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} \sigma(u_i),
\]
where \( \sigma(= \sigma_f) \) is the saturation nonlinearity with the bound \( \tilde{u}_i = 1, \quad i = 1, \ldots, N \). The open-loop dynamics of this system is exponentially unstable, since it has eigenvalues \( 0.03 \pm 0.9j \).

We will consider two types of communication graphs: a line graph and a small-world network.

Example 1: Line graph

We first consider the system of 4 agents over the line graph illustrated in Fig. 3. The Laplacian eigenvalues \( \lambda_2 \) and \( \lambda_N \) are \( \lambda_2 = 2 - \sqrt{2} \) and \( \lambda_N = 2 + \sqrt{2} \) from Table 1.

For \( \theta = 0.8 \), a solution to the convex optimization problem (CP) is given by
\[
X = \begin{bmatrix}
0.420 & -0.017 \\
-0.017 & 0.422
\end{bmatrix}, \quad Y = \begin{bmatrix}
0.000 & -0.931
\end{bmatrix},
\]
and hence
\[
F = YX^{-1} = \begin{bmatrix}
-0.090 & -2.213
\end{bmatrix}.
\]

With this feedback gain, we perform a numerical simulation. The initial states of each agent are randomly chosen as \( (x_{i1}, x_{i2}) \in [-3, 3] \times [-3, 3] \), \( i = 1, \ldots, 4 \). The simulation result is shown in Fig. 4. It is easily seen from the figure that the state synchronization is achieved even when the inputs are saturated.

- **Example 2: Watts-Strogatz network**

  We consider the synchronization over a Watts-Strogatz (WS) network which is a typical model of complex networks [20]. The WS network is constructed by the following procedure.

  **Step 1.** Construct a regular graph with \( N \) nodes each connected to \( k \) neighbors, called \( k \)-ring graph.

  **Step 2.** For each node, take every edge, and rewire it with another node with probability \( p \). The new node is randomly chosen from all possible nodes that avoid self-loops.
Here, we construct two WS networks with $k = 5$ and $k = 2$, and $p = 0.1$ and $N = 20$. Examples of such networks are given in Fig. 5. Notice that the WS network is sparse when $k$ is small, and is dense when $k$ is large.

For the network in Fig. 5 (a) with $k = 5$, the second smallest and the largest eigenvalues of $L$ are given by $\lambda_2 = 4.654$ and $\lambda_{20} = 19.654$. In this case, the solution to the convex optimization problem (CP) with $\theta = 0.8$ is given by

\[
X = \begin{bmatrix} 1.4396 & -0.0526 \\ -0.0526 & 1.4436 \end{bmatrix}, \quad Y = \begin{bmatrix} -0.0001 & -0.1509 \end{bmatrix},
\]

and hence

\[
F = YX^{-1} = \begin{bmatrix} -0.0039 & -0.1047 \end{bmatrix}.
\]

The simulation results with this feedback gain are shown in Fig. 6, where the initial states of the individual agents are randomly chosen as $(x_{i1}, x_{i2}) \in [-3, 3] \times [-3, 3]$, $i = 1, \ldots, 20$, where the synchronization errors are defined by

\[
\tilde{x}_i(t) = x_i(t) - \bar{x}(t), \quad i = 1, \ldots, N, \quad \bar{x}(t) = \frac{1}{N} \sum_{j=1}^{N} x_j(t).
\]

The figures show that the state synchronization is achieved even when the inputs are saturated.

Next, we consider the network in Fig. 5 (b) with $k = 2$. For this network, $\lambda_2$ and $\lambda_{20}$ are given by $\lambda_2 = 0.5659$, $\lambda_{20} = 9.1647$. In this case, the LMI (22) with $\theta = 0.8$ is infeasible. This result suggests that the synchronization under input saturation is difficult to achieve when the network is sparse, i.e. $\lambda_2$ is close to zero.

The above observation can also be justified from Remark 2. The synchronization condition in Theorem 1 requires the stabilization of $\tilde{\dot{x}} = A\tilde{x} + B\sigma(u)$ by the multiple state feedback laws $\bar{u} = \lambda_i F \tilde{x}$, $i = 2, \ldots, N$. If $\lambda_2$ is very close to zero, then the feedback gain $F$ must be large so that the closed-loop system $\tilde{\dot{x}} = A\tilde{x} + B\sigma(\lambda_2 F \tilde{x})$ should be stabilized. However, if $F$ is large, the control input $\bar{u} = \lambda_2 F \tilde{x}$ is likely to be saturated, leading to instability of the corresponding closed-loop system.

5. Conclusion

We have studied the local state synchronization of linear agents subject to input saturation over a fixed undirected communication graph. A sufficient condition for achieving the synchronization via relative state feedback control has been derived. Based on this analysis result, we have presented an LMI condition for designing a state feedback gain which achieves the local state synchronization. The present LMI condition is scalable as long as the Laplacian eigenvalues of the communication graph are available, and is readily solved by an existing convex programming algorithm. The simulation results also demonstrate the effectiveness of the present method.

The LMI conditions presented in this paper are somewhat conservative, because they make use of only the sector boundness of saturation nonlinearities. It remains as a future work to derive less conservative synchronization conditions based on more detailed structures of saturation nonlinearities.

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