Inverse Optimal Adaptive Consensus Control of Multi-Agent Systems Based on \( H_{\infty} \) Control Criterion

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Abstract: Design methods of inverse optimal adaptive consensus control of multi-agent systems composed of the first-order and the second-order regression models are presented based on \( H_{\infty} \) control criterion. The proposed control schemes are deduced as solutions of certain \( H_{\infty} \) control problems, where estimation errors of tuning parameters and imperfect knowledge of leaders are regarded as external disturbances to processes. The resulting control systems are shown to be robust to uncertain system parameters and the desirable consensus tracking is achieved asymptotically or approximately via adaptation schemes and \( L_2 \)-gain design parameters.

Key Words: adaptive control, consensus control, multi-agent system, \( H_{\infty} \) control.

1. Introduction

Recently, cooperative control problems of multi-agent systems have attracted much attentions, and many control methodologies have been developed in relation to those fields, such as formation control, task assignment, traffic control, and scheduling et al. (for example, [1]–[11]). Among those, distributed consensus tracking of multi-agent systems with limited communication networks, has been a basic and important topic, and various research results have been reported for various processes and under various conditions such as [12]–[16]. In those research works, adaptive control or sliding mode control strategies were also proposed in order to deal with uncertainties of agents, and stability of control systems was analyzed by utilizing Lyapunov function approaches. Furthermore, robustness properties of the control systems were also investigated. However, so much attention does not have been paid on control performance such as optimal property or transient performance in those works.

The purpose of the present paper is to present design methods of adaptive consensus control of multi-agent systems composed of the first-order and the second-order regression models based on the notion of inverse optimality and \( H_{\infty} \) control criterion [17],[18]. The proposed control schemes are derived as solutions of certain \( H_{\infty} \) control problems, where estimation errors of tuning parameters and imperfect knowledge of leaders are regarded as external disturbances to processes. The resulting control systems are shown to be robust to uncertain system parameters and the desirable consensus tracking is achieved asymptotically or approximately via adaptation schemes and \( L_2 \)-gain design parameters. Several simulation studies also confirm the effectiveness of the proposed methodologies.

2. Multi-Agent System and Information Network

First, mathematical preliminaries on information network graph of multi-agent systems are summarized briefly [14],[15]. As a model of interaction among agents, a weighted undirected graph \( G = (\mathcal{V}, \mathcal{E}, A) \) is considered, where \( \mathcal{V} = \{1, \cdots, N\} \) is a node set, which corresponds to a set of agents, and \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) is an edge set. An edge \((i, j) \in \mathcal{E}\) indicates that the agent \( i \) and \( j \) can communicate with each other. Associated with the edge set \( \mathcal{E} \), a weighted adjacency matrix \( A = [a_{ij}] \in \mathbb{R}^{N \times N} \) is introduced, and the entry \( a_{ij} \) of it is defined by

\[
\begin{align*}
  a_{ij} &= a_{ji} > 0 \iff (i, j) \in \mathcal{E}, \\
  a_{ij} &= a_{ji} = 0 \iff \text{otherwise}.
\end{align*}
\]

A path is a sequence of edges in the form \((i_1, i_2), (i_2, i_3), \cdots\), where \( i_j \in \mathcal{V} \). The undirected graph is called connected, if there is always an undirected path between every pair of distinct nodes. For the adjacency matrix \( A = [a_{ij}] \), the Laplacian matrix \( L = [l_{ij}] \in \mathbb{R}^{N \times N} \) is defined by

\[
l_{ii} = \sum_{j=1}^{N} a_{ij}, \quad l_{ij} = -a_{ij}, \quad (i \neq j).
\]

The Laplacian matrix is known to be symmetric and positive-semidefinite. Furthermore, it should be noted that the Laplacian matrix has a simple 0 eigenvalue with the associated eigenvector \( \mathbf{1} = [1 \cdots 1]^T \), and that all other eigenvalues are positive, if the corresponding undirected graph is connected.

In this manuscript, a consensus control problem of leader-follower type is considered. Let \( x_i \) be a leader which each agent \( i \in \mathcal{V} \) should follow (\( i \) is called a follower). Then, \( a_0 \) is defined such as

\[
a_0 = \begin{cases} 
0 & : \text{leader’s information is available to follower } i, \\
> 0 & : \text{otherwise},
\end{cases}
\]

and from \( a_0 \) and \( L \), the matrix \( M \in \mathbb{R}^{N \times N} \) is defined by
\[ M = L + \text{diag}(a_{10} \cdots a_{N0}). \]  
(2)

\[ \text{M is shown to be symmetric and positive definite, if 1. at least one } a_{i0} \ (1 \leq i \leq N) \text{ is positive, and 2. the graph } G \text{ is connected.} \]  
[15]

1. The graph \( G \) is connected.
2. At least one \( a_{i0} \) (\( 1 \leq i \leq N \)) is positive, that is, the information of the leader \( x_0 (\hat{x}_0, \bar{x}_0) \), is available to at least one follower \( i \).
3. \( x_0, \hat{x}_0, \bar{x}_0 \) are uniformly bounded.

### 3. Inverse Optimal Adaptive Consensus Control for First-Order Models Based on \( H_{\infty} \) Control Criterion

#### 3.1 Problem Statement

A multi-agent system composed of the first-order regression models is considered such that

\[ \dot{x}(t) = \Omega(t) \theta + B_i u_i(t), \quad (i = 1, \cdots, N), \]  
(3)

where \( x_i \in \mathbb{R}^n \) is a state, \( u_i \in \mathbb{R}^n \) is an input, \( \theta \in \mathbb{R}^m \) is an unknown parameter vector, and \( \Omega \in \mathbb{R}^{n \times n} \) is a regressor matrix composed of \( x_i \) and its structure is known. It is assumed that \( \Omega \) is bounded for bounded \( x_i, B_i \in \mathbb{R}^{n \times m} \) is an unknown matrix of the form

\[ B_i = \text{diag}(b_{i1}, \cdots, b_{in}), \]  
(4)

and the sign of \( b_{ij} \) is known a priori. Hereafter, let \( b_{ij} > 0 \) without loss of generality. The communication structure among agents and a leader is prescribed by the information network graph \( G \) with the associated adjacency matrix \( A \), the Laplacian matrix \( L \), and the matrix \( M \). The control objective is to achieve consensus tracking of the leader-follower type such as

\[ x_i \to x_j, \quad (i, j = 1, \cdots, N), \]  
(5)

\[ x_i \to x_0, \quad (i = 1, \cdots, N), \]  
(6)

under the restricted communication structure among agents.

#### 3.2 Control Law and Error Equation

Associated with the information network graph \( G \), the following control law is employed.

\[ u_i(t) = \dot{P}_i(t) - N_i \theta(t) \]

\[ -\alpha \sum_{j = 0}^{N} a_{ij} [x_i(t) - x_j(t)] + n_{i0} \dot{x}_0(t) + v_i(t), \]  
(7)

where \( a_{ij} (1 \leq i \leq N, 0 \leq j \leq N) \) is an entry of the adjacency matrix \( A \) and (1), and \( \alpha > 0 \) is a design parameter. \( \dot{\gamma} \) is denoted as a current estimate of \( \gamma \), and \( P_i \) is defined by

\[ P_i = \text{diag}(p_{i1}, \cdots, p_{in}), \quad p_{ij} = 1/b_{ij}. \]  
(8)

Concerned with \( a_{i0}, n_{i0} \) is defined as follows:

\[ n_{i0} = \begin{cases} 1 & \text{if } a_{i0} > 0, \\
0 & \text{otherwise}. \end{cases} \]  
(9)

Furthermore, \( v_i \) is a stabilizing signal to be determined later based on \( H_{\infty} \) control criterion. A tracking error between the leader \( x_0 \) and the follower \( x_i \) is defined by

\[ \dot{x}_i(t) = x_i(t) - x_0(t), \]  
(10)

and the substitution of (7) and (10) into (3) yields

\[ \dot{x}_i(t) = \Omega_i(t) \theta + B_i u_i(t) - \dot{x}_0(t) \]

\[ = \Omega_i(t) \theta - \dot{\hat{\theta}}(t) + U_{i0}(t) B_i [\dot{\hat{\theta}}(t) - p_i] + B_i v_i(t) \]

\[ + \alpha \left( -\sum_{j = 1}^{N} l_{ij} \dot{x}_j(t) \right) + (n_{i0} - 1) \dot{x}_0(t), \]  
(11)

\[ U_{i0} = \text{diag}(u_{i01}, \cdots, u_{i0n}), \]  
(12)

\[ u_{i0} = [u_{i01}, \cdots, u_{i0n}]^T, \]  
(13)

\[ p_i = [p_{i1}, \cdots, p_{in}]^T. \]  
(14)

From (11), the total representation of the multi-agent system is given in the forms

\[ \dot{x}(t) = \Omega(t) \theta - \dot{\hat{\theta}}(t) + U_{i0}(t) B(p - \hat{p}(t)) \]

\[ -\alpha (M \otimes I) \dot{\gamma} + [(N_0 - 1) \otimes I] \dot{x}_0(t) \]

\[ + B v(t), \]  
(15)

\[ \dot{\gamma} = [\dot{\gamma}_1, \cdots, \dot{\gamma}_n]^T, \]  
(16)

\[ \Omega = \text{block diag}(\Omega_1, \cdots, \Omega_N), \]  
(17)

\[ \theta = [\theta_1^T, \cdots, \theta_n^T]^T, \]  
(18)

\[ U_0 = \text{block diag}(U_{i01}, \cdots, U_{i0n}), \]  
(19)

\[ p = [p_1^T, \cdots, p_n^T]^T, \]  
(20)

\[ N_0 = [n_{10}, \cdots, n_{n0}]^T, \]  
(21)

\[ \mathbf{1} = [1, \cdots, 1]^T \]  
(22)

\[ v = [v_{11}, \cdots, v_{nn}]^T, \]  
(23)

where \( \otimes \) denotes Kronecker product.

#### 3.3 Inverse Optimal Adaptive Consensus Control for First-Order Models Based on \( H_{\infty} \) Control Criterion

A positive function \( W_0 \) is defined by

\[ W_0(t) = \frac{1}{2} \dot{x}(t)^T (M \otimes I) \dot{x}(t) \]

\[ + \frac{1}{2} \left[ \dot{\theta}(t) - \hat{\theta}(t) \right]^T \Gamma_{\theta}^{-1} \left[ \dot{\theta}(t) - \hat{\theta}(t) \right], \]  
(24)

\[ (\Gamma_{\theta} = \Gamma_{\theta}^T > 0, \text{diagonal}), \]  
(25)

\[ b = [b_{11}, \cdots, b_{nn}]^T, \quad \hat{b} = [b_{11}, \cdots, b_{nn}]^T. \]  
(26)

The tuning law of \( b \) is determined such as

\[ \dot{b}(t) = P_r \Gamma_1 V(t)^T (M \otimes I) \dot{x}(t), \]  
(27)

\[ V = \text{block diag}(V_1, \cdots, V_n), \]  
(28)

\[ V_1 = \text{diag}(v_{11}, \cdots, v_{nn}), \]  
(29)

\[ v_i = \left[ v_{i1}, \cdots, v_{in} \right]^T. \]  
(30)
where \( \Pr(\cdot) \) are projection operations in which tuning parameters are constrained to bounded regions deduced from upper-bounds and lower-bounds of each element of \( b \) [19]. Then, the time derivative of \( W_0 \) along its trajectory is given as follows:

\[
W_0(\dot{t}) \leq \dot{x}(t)^T (M \otimes I) \Omega(t)(\dot{\theta} - \dot{\tilde{\theta}}(t)) + \dot{x}(t)^T (M \otimes I) U_0(t) B [\dot{p}(t) - p] - \alpha \dot{x}(t)^T (M \otimes I') \ddot{x}(t) + \dot{x}(t)^T (M \otimes I) \dot{B}(t)v(t) + \ddot{x}(t)^T (M \otimes I) \dot{B}(t)v(t). \tag{31}
\]

From the evaluation of \( W_0 \), the next virtual system is introduced.

\[
\dot{x} = f + g_{11} d_1 + g_{12} d_2 + g_{13} d_3 + g_{2} v, \tag{32}
\]

\[
f = -\alpha (M \otimes I) \ddot{x}, \tag{33}
\]

\[
g_{11} = \Omega, \quad g_{12} = U_0, \quad g_{13} = I, \quad g_{2} = \dot{B}, \tag{34}
\]

\[
d_1 = (\theta - \bar{\theta}), \quad d_2 = B [\dot{p} - p], \tag{35}
\]

\[
d_3 = ((N_0 - 1) \otimes I) \dot{x}_0. \tag{36}
\]

The virtual system is to be stabilized by a control input \( v \) based on \( H_\infty \) control criterion, where \( d_1, d_2, d_3 \) are regarded as external disturbances to the process [17],[18]. For that purpose, the following Hamilton-Jacobi-Isaacs (HJI) equation and its solution \( V_0 \) are introduced.

\[
L V_0 + \frac{1}{4} \left\{ \sum_{i=1}^3 \frac{\|L_{0i} V_0\|^2}{\gamma_i} - (L_{02}, V_0) R^{-1} (L_{02}, V_0) \right\} + q = 0, \tag{37}
\]

\[
V_0 = \frac{1}{2} \ddot{x}^T (M \otimes I) \ddot{x}, \tag{38}
\]

\[
\frac{\partial V_0}{\partial \dot{x}} h, \quad h = f, g_{11}, g_{2}, \tag{39}
\]

where \( q \) and \( R \) are a positive function and a positive definite matrix respectively, and those are derived from HJI equation based on inverse optimality for the given solution \( V_0 \) and the positive constants \( \gamma_1, \gamma_2, \gamma_3 \). The substitution of the solution \( V_0 \) into HJI equation (36) yields

\[
-\alpha \dot{x}^T (M \otimes I)^2 \ddot{x} + \frac{1}{4} \ddot{x}^T (M \otimes I) \left\{ \Omega^T \gamma_1^2 \right\} + \frac{U_0 U_0^T}{\gamma_2^2} + \frac{I}{\gamma_3^2} - \dot{B} R^{-1} \dot{B}^T (M \otimes I) \ddot{x} + q = 0. \tag{40}
\]

Then, \( R \) and \( q \) are obtained such as

\[
R = \left( \frac{\dot{B}^T \Omega \dot{B} - \gamma_1^2}{\gamma_2^2} + \frac{\dot{B}^{-1} U_0 U_0^T \dot{B}^{-T}}{\gamma_2^2} + \frac{\dot{B}^{-1} \dot{B}^T}{\gamma_3^2} + K \right)^{-1}, \tag{41}
\]

\[
q = \alpha \dot{x}^T (M \otimes I)^2 \ddot{x} + \frac{1}{4} \ddot{x}^T (M \otimes I) \dot{B} K \dot{B}^T (M \otimes I) \ddot{x}, \tag{42}
\]

where \( \dot{B} = (\dot{B}^T)^{-1} = (\bar{\dot{B}}^T)^{-1} \), and \( K \) is a diagonal and positive definite matrix (a design parameter). From \( R, \nu \) is derived as a solution of the corresponding \( H_\infty \) control problem as follows:

\[
\nu = -\frac{1}{2} R^{-1} (L_{02}, V_0) = -\frac{1}{2} R^{-1} \dot{B}^T (M \otimes I) \ddot{x}, \tag{43}
\]

where the entries of \( \dot{B} \) in \( R \) and \( v \) are constructed from the elements of \( \dot{\tilde{\theta}} \). Then, the time derivative of \( W_0 \) is evaluated by

\[
\dot{W}_0 \leq -q - v^T R v + \left( \frac{1}{2} R^{-1} \dot{B}^T (M \otimes I) \ddot{x} \right)^T R \cdot v + \frac{1}{2} R^{-1} \dot{B}^T (M \otimes I) \ddot{x} \right) + \gamma_1^2 ||d_1||^2 - \gamma_1^2 \left| d_1 \right| \cdot \frac{\Omega^T (M \otimes I) \ddot{x}}{2 \gamma_1^2} \right|^2 + \gamma_2^2 ||d_2||^2 - \gamma_2^2 \left| d_2 \right| \cdot \frac{U_0^T (M \otimes I) \ddot{x}}{2 \gamma_2^2} \right|^2 + \gamma_3^2 ||d_3||^2 - \gamma_3^2 \left| d_3 \right| \cdot \frac{(M \otimes I) \ddot{x}}{2 \gamma_3^2} \right|^2, \tag{44}
\]

and the next theorem is obtained.

**Theorem 1.** The partial adaptive control system (7), (27), (41) is uniformly bounded for arbitrary bounded design parameters \( \hat{\theta}, \hat{\theta}_b, \) and \( v \) is a sub-optimal control input which minimizes the upper bound on the following cost functional \( J_\nu \).

\[
J_\nu(t) = \sup_{d, \dot{d} \in \mathbb{R}^2 \times \mathbb{R}^1} \left[ \int_0^t \left( q + v^T R v \right) d\tau + W_0(t) \right]
- \sum_{i=1}^3 \gamma_i^2 \int_0^t ||d_i||^2 d\tau \right]. \tag{45}
\]

Also the next inequality holds.

\[
\int_0^t \left( q + v^T R v \right) d\tau + W_0(t)
\leq \sum_{i=1}^3 \gamma_i^2 \int_0^t ||d_i||^2 d\tau + W_0(0). \tag{46}
\]

Theorem 1 denotes the properties of the partial adaptive control system (7), (27), (41), where the tunings of \( \dot{\tilde{\theta}} \) and \( \dot{\tilde{\theta}}_b \) are not necessarily required. Furthermore, the L2-gain property between \( \sqrt{q} + v^T R v \) and \( d_1 \sim d_3 \) is prescribed by the design parameters \( \gamma_1 \sim \gamma_3 \), and it indicates that the boundedness of the control systems is assured for arbitrary bounded system parameters \( \theta, p \) (both time-invariant and time-varying).

Next, the tuning laws of \( \theta \) and \( \dot{p} \) are determined as follows:

\[
\ddot{\tilde{\theta}}(t) = \Pr \left[ \Gamma_2 \Omega(t) (M \otimes I) \ddot{x}(t) \right], \tag{47}
\]

\[
\dot{p}(t) = \Pr \left[ -\Gamma_3 U_0(t) (M \otimes I) \ddot{x}(t) \right], \tag{48}
\]

where \( \Gamma_2 = \Gamma_2^T > 0 \) (diagonal), \( \Gamma_3 = \Gamma_3^T > 0 \) (diagonal). \( \Pr(\cdot) \) are projection operations in which tuning parameters \( \dot{\tilde{\theta}} \) and \( \dot{\tilde{\theta}} \) are constrained to bounded regions deduced from upper-bounds of \( ||\dot{\theta}|| \) and upper-bounds and lower-bounds of each element of \( p \), respectively [19]. The entries of \( \dot{\tilde{\theta}} \) are generated from the elements of \( \dot{\tilde{\theta}} \). A positive function \( W \) is defined by

\[
W(t) = \frac{1}{2} \ddot{x}(t)^T (M \otimes I) \ddot{x}(t)
+ \frac{1}{2} \left( \ddot{\tilde{\theta}}(t) - \bar{\ddot{\theta}}(t) \right)^T \Gamma_1^{-1} \left( \ddot{\tilde{\theta}}(t) - \bar{\ddot{\theta}}(t) \right)
+ \frac{1}{2} \left( \ddot{p}(t) - \bar{\ddot{p}}(t) \right)^T \Gamma_2^{-1} \left( \ddot{p}(t) - \bar{\ddot{p}}(t) \right)
+ \frac{1}{2} \left( \dot{p}(t) - p(t) \right)^T B \Gamma_3^{-1} \left( \dot{p}(t) - p(t) \right). \tag{49}
\]
The time derivative of $W$ along its trajectory is given such as
\[
\dot{W}(t) \leq -\alpha \ddot{x}(t)^T (M \otimes I)^2 \ddot{x}(t) - \frac{1}{4\gamma_3^2} \ddot{x}(t)^T (M \otimes I)^2 \ddot{x}(t) - \frac{1}{2} \ddot{x}(t)^T (M \otimes I) \dot{B}(t)K \dot{B}(t)^T (M \otimes I) \ddot{x}(t) + \gamma_3^2 \| \dot{d}_i(t) \|^2, \tag{48}
\]
and the following theorem is obtained.

**Theorem 2.** The total adaptive control system (7), (27), (41), (45), (46) is uniformly bounded, and if $\ddot{x}_0(t) = 0$ or the information of the leader $x_0$ is available for all followers ($d_i = ((N_0 - 1) \otimes I) x_0 = 0$), then it follows that
\[
\lim_{t \to \infty} \ddot{x}(t) = 0. \tag{49}
\]
Otherwise, when $\ddot{x}_0(t) \neq 0$ and the information of $x_0$ is not available for all followers ($d_i = ((N_0 - 1) \otimes I) x_0 \neq 0$), then the next relation holds.
\[
\lim_{t \to \infty} \sup \frac{1}{T} \int_0^T \| \ddot{x}(t) \|^2 dt \leq \text{const} \cdot \gamma_3^2. \tag{50}
\]

Theorem 2 states that the asymptotic consensus tracking is achieved under a specified condition, and also shows that the approximate consensus tracking with the ratio of $\gamma_3 (> 0)$ is still assured, even if that condition is not satisfied. Furthermore, it should be noted that the adaptive control schemes are constructed via $(M \otimes I) \ddot{x}$ and local informations of each agents, and are implemented in a distributed fashion.

4. **Inverse Optimal Adaptive Consensus Control for Second-Order Model Based on $H_{\infty}$ Control Criterion**

4.1 **Problem Statement**

Next, a multi-agent systems composed of the second-order regression models is considered such that
\[
\ddot{x}_i(t) = \Omega(t) \theta + B_i u_i(t), \quad (i = 1, \ldots, N), \tag{51}
\]
where $x_i$, $u_i$, $\theta$, $\Omega$, $B_i$ are defined similarly to the previous case, and the form of $B_i$ is the same as the former one. $\Omega_i$ is composed of $x_i$ and $\dot{x}_i$, and is bounded for bounded $x_i$ and $\dot{x}_i$. The communication structure among agents is prescribed by the information network graph $\mathcal{G}$ with the associated adjacency matrix $A$, the Laplacian matrix $L$, and the matrix $M$. The control objective is to achieve consensus tracking of the leader-follower type together with velocity tracking such as
\[
x_i \rightarrow x_j, \quad \dot{x}_i \rightarrow \dot{x}_j, \quad (i, j = 1, \ldots, N), \tag{52}
x_i \rightarrow x_0, \quad \dot{x}_i \rightarrow \dot{x}_0, \quad (i = 1, \ldots, N), \tag{53}
\]
under the restricted communication structure among agents.

4.2 **Control Law and Error Equation**

Associated with the information network graph, the following control law is employed.
\[
u_i(t) = \dot{P}_i(t) - \Omega(t) \dot{\theta}(t) - \sum_{j=0}^{N} a_{ij} [x_i(t) - x_j(t)] - \sum_{j=0}^{N} a_{ij} [\dot{x}_i(t) - \dot{x}_j(t)] + n_0 \ddot{x}_0(t) \]
\[
\dot{P}_i(t) = \dot{P}_j(t) + v_i(t), \tag{54}
\]
where the definitions of $a_{ij}$ $(1 \leq i \leq N, 0 \leq j \leq N)$, $\alpha > 0$, $P_i$, $n_0$, $v_i$ are the same as the first-order case. A consensus tracking error $\ddot{x}_i$ is denoted by (10), and the substitution of (54) and (10) into (51) yields
\[
\ddot{x}_i(t) = \Omega(t) \theta + B_i u_i(t) - \ddot{x}_0(t) = \Omega(t) \theta - \hat{\theta}(t) + U_0(t) B_i \hat{p}_i(t) - p_i + B_i v(t) \]
\[
\ddot{x}(t) = \Omega(t) \theta - \hat{\theta}(t) + U_0(t) B_i \hat{p}_i(t) - p_i + B_i v(t), \tag{55}
\]
where $U_0$, $u_0$, $p_i$ are defined similarly to the former case. Then, the total representation of the multi-agent system is given such as
\[
\ddot{x}(t) = \Omega(t) \theta - \hat{\theta}(t) + U_0(t) B_i (p - \hat{p}_i(t)) + [(N_0 - 1) \otimes I] \ddot{x}_0(t) + B_i v(t), \tag{56}
\]
where the definitions of $\dot{x}$, $\Omega$, $\theta$, $U_0$, $B$, $p$, $N$, $\dot{x}_0$, $v$ are the same as the previous ones.

4.3 **Inverse Optimal Adaptive Consensus Control for Second-Order Models Based on $H_{\infty}$ Control Criterion**

For the matrix $M$ and the positive constants $\alpha$, $\gamma$, the matrices $P$ and $Q$ are defined such as
\[
P = \begin{bmatrix} \frac{1}{2} M^2 & \frac{1}{2} M \\ \frac{1}{2} M & \frac{1}{2} M \end{bmatrix}, \tag{57}
\]
\[
Q = \begin{bmatrix} \gamma^2 M^2 & \frac{\gamma}{2} M^2 \\ \frac{\gamma}{2} M^2 & \alpha M^2 - \gamma M \end{bmatrix}. \tag{58}
\]
It can be shown that $P$ and $Q$ are both positive definite, if $\gamma$ satisfies the next condition [15].
\[
0 < \gamma < \min \left\{ \sqrt{\lambda_{\text{min}}(M)}, \frac{4\alpha \lambda_{\text{min}}(M)}{4 + \alpha^2 \lambda_{\text{min}}(M)} \right\}, \tag{59}
\]
where $\lambda_{\text{min}}(M)$ is the minimum eigenvalue of the matrix $M$. Hereafter, it is assumed that $\gamma$ satisfies (59). Utilizing a positive definite matrix $P$, a positive function $W_0$ is defined by
Then, the time derivative of $z(t)$ and the definitions of $\tilde{q}$ similarly to the first-order case, the next theorem is obtained.

\[ W_0(t) = \tilde{z}(t)^T (P \otimes I) \tilde{z}(t) + \frac{1}{2} \left[ \dot{\tilde{b}}(t) - b \right]^T \Gamma_1^{-1} \left[ \dot{\tilde{b}}(t) - b \right], \]

\[ \tilde{z} = [\tilde{x}^T, \tilde{\dot{x}}^T]^T, \]

where $\Gamma_1 = \Gamma_1^T > 0$ (diagonal), and $b$ is defined similarly to the first-order case. The tuning law of $\tilde{b}$ is chosen such as

\[ \dot{\tilde{b}}(t) = Pr \left[ \Gamma_1 V(t)^T (M \otimes I) \tilde{x}(t) \right], \]

\[ \tilde{x} = \dot{\tilde{x}} + \gamma \tilde{x}, \]

and the definitions of $V$, $Pr$ are the same as the previous ones. Then, the time derivative of $W_0$ is given by

\[ W_0(t) \leq \tilde{z}(t)^T (M \otimes I) \Omega(t) [\theta - \dot{\theta}_{0}] + \tilde{z}(t)^T (M \otimes I) U_0(t) [\tilde{p}(t) - p] - \tilde{\dot{z}}(t)^T (Q \otimes I) \tilde{z}(t) + \tilde{z}(t)^T (M \otimes I) [(N_0 - 1) \otimes I] \tilde{x}_0(t) + \tilde{z}(t)^T (M \otimes I) \tilde{B}(t) v(t). \]

From the evaluation of $W_0$, the next virtual system is introduced.

\[ \dot{\tilde{z}} = f + g_{11} d_1 + g_{12} d_2 + g_{13} d_3 + g_2 v, \]

\[ f = \begin{bmatrix} 0 & I \\ - (M \otimes I) & - \alpha (M \otimes I) \end{bmatrix} \tilde{z}, \]

\[ g_{11} = \begin{bmatrix} 0 \\ \Omega \end{bmatrix}, \quad g_{12} = \begin{bmatrix} 0 \\ U_0 \end{bmatrix}, \quad g_{13} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \]

\[ g_2 = \begin{bmatrix} 0 \\ \tilde{B} \end{bmatrix}, \]

\[ d_1 = (\theta - \dot{\theta}_0), \quad d_2 = B(\hat{p} - p), \]

\[ d_3 = [(N_0 - 1) \otimes I] \tilde{x}_0. \]

The virtual system is to be stabilized by a control input $v$ based on $H_{\omega}$ control criterion, where $d_1$, $d_2$, $d_3$ are regarded as external disturbances to the process. Then, by repeating the similar discussions to the first-order case, for $R$ defined by (39) and $q$ defined such as

\[ q = \tilde{z}^T (Q \otimes I) \tilde{z} + \frac{1}{4} \tilde{\dot{z}}^T (M \otimes I) \tilde{B} K \tilde{B}^T (M \otimes I) \tilde{x}, \]

and for $v$ deduced such as

\[ v = - \frac{1}{2} R^{-1} (L \tilde{c}_0 V_0)^T = - \frac{1}{2} R^{-1} \tilde{B}^T (M \otimes I) \tilde{x}, \]

the next theorem is obtained.

**Theorem 3.** The partial adaptive control system (54), (62), (70) is uniformly bounded for arbitrary bounded design parameters $\hat{b}$, $\hat{p}$, and $v$ is a sub-optimal control input which minimizes the upper bound on the cost functional $J$ in (43), where $W_0$ and $q$ are newly defined by (60), (69). Also the inequality (44) holds for the newly defined $W_0$ and $q$.

Next, the tuning laws of $\hat{\theta}$ and $\hat{p}$ are determined as follows:

\[ \dot{\hat{\theta}}(t) = Pr \left[ \Gamma_2 \Omega(t)^T (M \otimes I) \tilde{z}(t) \right], \]

\[ \dot{\hat{p}}(t) = Pr \left[ - \Gamma_1 U_0(t)^T (M \otimes I) \tilde{z}(t) \right], \]

where the definition of $Pr$ is the same as the previous one. Then, similarly to the first-order case, the next theorem is obtained.

**Theorem 4.** The total adaptive control system (54), (62), (70), (71), (72) is uniformly bounded, and if $\tilde{x}_0(t) = 0$ or the information of the leader $\tilde{x}_0$ is available for all followers ($d_3 = [(N_0 - 1) \otimes I] \tilde{x}_0 = 0$), then it follows that

\[ \lim_{t \to \infty} \tilde{x}(t) = 0, \]

\[ \lim_{t \to \infty} \tilde{z}(t) = 0. \]

Otherwise, when $\tilde{x}_0(t) \neq 0$ and the information of $\tilde{x}_0$ is not available for all followers ($d_3 = [(N_0 - 1) \otimes I] \tilde{x}_0 \neq 0$), then the next relation holds.

\[ \lim \sup_{t \to \infty} \frac{1}{T} \int_0^T \| \tilde{x}(t) \|^2 dt \leq \text{const} \cdot \gamma_3^2. \]

**Remark 1.** In the design of adaptive consensus control for second-order systems, $y$ satisfying (59), should be known a priori, and it means that each agent should know the entire network architecture $M$ in advance.

### 4.4 Inverse Optimal Adaptive Consensus Control for Second-Order System with Velocity Tracking Based on $H_{\omega}$ Control Criterion

As a special version of the second-order version, the case $y = 0$ can be also considered. Then, although $P$ remains positive definite, $Q$ becomes positive semidefinite

\[ P = \begin{bmatrix} \frac{1}{2} M^2 & 0 \\ 0 & \frac{1}{2} M \end{bmatrix}, \]

\[ Q = \begin{bmatrix} 0 & 0 \\ 0 & \alpha M^2 \end{bmatrix}. \]

and the consensus tracking of velocity is achieved such that

\[ \dot{x}_i \to \dot{x}_j, \quad (i, j = 1, \ldots, N), \]

\[ \dot{x}_i \to \dot{x}_0, \quad (i = 1, \ldots, N). \]

The adaptive consensus control system with velocity tracking is easily deduced by replacing $\tilde{z}(t)$ by $\dot{x}(t)$, and by utilizing newly defined $P$ and $Q$ in (76) and (77), and the following two theorems are obtained.

**Theorem 5.** In the partial adaptive control system (54), (62), (70) together with $y = 0$ (that is, $\tilde{x}$ is replaced by $\dot{x}$), the velocity tracking error $\tilde{x}$ is uniformly bounded for arbitrary bounded design parameters $\hat{\theta}$, $\hat{p}$, and $v$ is a sub-optimal control input which minimizes the upper bound on the cost functional $J$ defined by (43). Also the inequality (44) holds.

**Theorem 6.** In the total adaptive control system (54), (62), (70), (71), (72) together with $y = 0$ ($\tilde{x}$ is replaced by $\dot{x}$), the velocity tracking error $\tilde{x}$ and the tuning parameters $\hat{\theta}$, $\hat{p}$, $\dot{\theta}$ are uniformly bounded, and if $\tilde{x}_0(t) = 0$ or the information of the leader $\tilde{x}_0$ is available for all followers ($d_3 = [(N_0 - 1) \otimes I] \tilde{x}_0 = 0$), then it follows that

\[ \lim_{t \to \infty} \tilde{x}(t) = 0. \]

Otherwise, that is, when $\tilde{x}_0(t) \neq 0$ and the information of $\tilde{x}_0$ is not available for all followers ($d_3 = [(N_0 - 1) \otimes I] \tilde{x}_0 \neq 0$), then the next relation holds.

\[ \lim \sup_{t \to \infty} \frac{1}{T} \int_0^T \| \tilde{x}(t) \|^2 dt \leq \text{const} \cdot \gamma_3^2. \]
5. Numerical Example

In order to show the effectiveness of the proposed methodology, numerical experimental studies for the second-order regression models are performed.

A multi-agent system composed of the second-order regression models is considered such as

\[ \ddot{x}_i(t) = \theta_i x_i(t) + u_i(t), \quad (i = 1, 2, 3), \]
\[ (x_1(0) = 1, \quad x_2(0) = 0, \quad x_3(0) = -1), \]

where \( x_i \in \mathbb{R}, \ u_i \in \mathbb{R}, \) and \( \theta_i \in \mathbb{R} \) is an unknown parameter.

Associated with the information network (Fig. 1), the adjacency matrix \( A = [a_{ij}] \) and \( a_{10} = 1, \ a_{20} = a_{30} = 0. \)

The control objective is to achieve consensus tracking

\[ x_i \rightarrow x_j, \quad \dot{x}_i \rightarrow \dot{x}_j, \quad \dot{x}_i \rightarrow \dot{x}_0, \quad (i, j = 1, 2, 3), \]

where the virtual leader \( x_0 \) is determined such as

\[ \ddot{x}_0 + 2\dot{x}_0(t) + x_0(t) = \sin t. \]

The design parameters are chosen as follows:

\[ \Gamma = 10I, \ K = 100I, \ \alpha = 1, \ \gamma = 0.4, \ \gamma_i = 0.01. \]

As system parameters, both time-invariant (TI) and time-varying (TV) (or piecewise time-invariant (Fig. 2)) cases are considered such that

\[ \theta_1 = 1, \ \theta_2 = 2, \ \theta_3 = 3, \quad (TI), \]
\[ \theta_1 = f_1(t), \ \theta_2 = 2f_1(t), \ \theta_3 = 3f_1(t), \quad (TV), \]
\[ f_1(t) = \begin{cases} 1 & 0 \leq t < 2.5, \ 0 & 2.5 \leq t \leq 7.5, \end{cases} \]
\[ \cdots, \]
\[ 5 \leq t < 7.5, \ 7.5 \leq t \leq 10, \ \cdots. \]

The simulation results of the proposed design scheme (Theorem 3 and Theorem 4) are shown in Fig. 3 (time-invariant case) and Fig. 4 (time-varying case). For comparison, the adaptive control systems which do not contain the stabilizing signal \( v(t) \), are also shown for both cases; Fig. 5 (time-invariant case) and Fig. 6 (time-varying case).

From those results, it is seen that the proposed \( H_{\infty} \) adaptive control strategies which contain \( v(t) \), achieve better tracking and convergence properties together with robustness to abrupt changes of the system parameters. Those are owing to the disturbance attenuation features to estimation errors of tuning parameters in the proposed \( H_{\infty} \) control scheme.

6. Concluding Remarks

Design methods of adaptive \( H_{\infty} \) consensus control of multi-agent systems composed of the first-order and the second-order regression models have been presented in this paper. Effectiveness of the proposed design schemes especially for abrupt changes of system parameters, was also confirmed by the simulation studies. Those are owing to the disturbance attenuation features to estimation errors of tuning parameters in the proposed \( H_{\infty} \) control scheme. An extension to the consensus tracking problems on directed information networks is to be left in
Fig. 5 Simulation result for time-invariant models without \( v \).

Fig. 6 Simulation result for time-varying models without \( v \).

the future research.

References


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