Tracking Performance Limitation for 1-DOF Control Systems Using a Set of Attainable Outputs

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Abstract: This paper deals with analysis of fundamental limitation in tracking control problem. The authors have given the tracking performance limitation of two degree of freedom systems for a class of reference signals, explicitly. This obtained result includes a uniform description of reference signals. In this paper, the result is extended to one degree of freedom systems. In the case of unstable plants, the tracking performance for one degree of freedom systems has not been analyzed except for the step reference. The analysis results of this paper clearly separate the contributions of the plant and the reference characteristics. The optimal performances are illustrated by numerical example with the sinusoidal reference.

Key Words: performance limitation, optimal control, unstable zeros, 1-DOF systems.

1. Introduction

In the history of control theory, much attention has been paid to analysis of fundamental limitations in control system design. Recently, many results have been reported [1]–[11]. These papers derive explicit expressions of the achievable optimal \( H_2 \) performance described in terms of plant parameters such as unstable zeros/poles and so on. This kind of analysis results not only enables us to understand the relationship between the parameters and the achievable performance, but also can be used to design a ‘good’ plant when we can change some parts of the plant.

In addition to the optimal \( H_2 \) performance, the tracking control performance has been analyzed in [2]–[6]. In [2], the performance limitations for step reference inputs have been analyzed for control systems with two degree of freedom (2-DOF). In addition to the tracking error, magnitude of control inputs is also dealt with in [3]. The results for 2-DOF systems with trigonometric references are given in [4],[5]. Furthermore, the optimal performance for a fairly general class of reference inputs is revealed in [12], where the class is defined abstractly. The results in [12] clarify the contributions, to performance limitations, of plants and reference inputs separately.

The tracking control performance limitation is also analyzed for 1-DOF systems in [2]. The limitation is characterized by not only unstable zeros but also unstable poles of plants. In other words, the existence of unstable poles makes the performance limitation for 1-DOF systems worse than 2-DOF systems. However, the limitation is analyzed only for the step responses and other reference inputs are not dealt with in [2].

This paper aims to find the tracking control performance limitation for 1-DOF systems, assuming the fairly general class of the reference inputs defined in [12]. We will obtain the performance limitation for 1-DOF systems by extending the results for 2-DOF systems in [12]. We first parameterize the set of the admissible outputs that are produced by the internally stable closed-loop systems corresponding to the given plant and also track to the given reference signal asymptotically. The obtained optimal performance indices separate clearly the contributions of the plant and the reference input. The obtained results thus clarify the essential difference between 1-DOF and 2-DOF systems.

This paper is organized as follows: The problem formulation is defined in Section 2. The set of all attainable outputs is parameterized in Section 3. By using the parameterization, the explicit expressions of the performance limitations are derived in Section 4. The given results are illustrated by using the examples in Section 5.

Notation is standard. We denote the Laplace transform of \( u(t) \) as \( \hat{u}(s) \). \( L_\infty \) is the space of signals defined by

\[
L_\infty = \left\{ e(t) : \sup_t |e(t)| < \infty \right\} .
\]

For signal \( e(t) \), \( \|e\|_2 \) represents the \( L_2 \) norm

\[
\|e\|_2 = \left( \int_0^\infty |e(t)|^2 dt \right)^{1/2}.
\]

On the other hand, for Laplace transform \( \hat{u}(s) \), \( \|\hat{u}\|_2 \) is the \( L_2 \) norm of \( \hat{u}(j\omega) \), i.e.,

\[
\|\hat{u}\|_2 = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{u}(-j\omega)|^2 d\omega \right)^{1/2}.
\]

If \( \hat{u}(s) \) is a rational function that is analytic in the closed right half plane, \( \|\hat{u}\|_2 \) can be called the \( H_2 \) norm. \( S \) is a set of functions that are proper real rational, i.e. proper rational functions with real coefficients, and analytic in the closed right half plane.

2. Problem Formulation

We consider the tracking control problem for an SISO plant given by a real rational transfer function \( P(s) \). We assume that \( P(s) \) has \( \ell_p \) and \( m_p \) number of unstable poles \( p_1, \ldots, p_{\ell_p} \) and unstable zeros \( z_1, \ldots, z_{m_p} \), respectively. The relative degree of
$P(s)$ is $h_p$. We further assume that all the unstable poles and zeros lie in the open right half plane and distinct, i.e.
\[
\begin{align*}
\operatorname{Re}(z_i) > 0 & \quad \forall i, \quad z_i \neq z_j, \quad \forall i \neq j, \\
\operatorname{Re}(p_k) > 0 & \quad \forall k, \quad p_k \neq p_l, \quad \forall k \neq l, \quad (1)
\end{align*}
\]

Moreover, we assume that the reference signal is given by its Laplace transform and belongs to the following set:
\[
\mathcal{R} = \{ \hat{r}(s) \in \mathcal{N} : \mathcal{L}^{-1}[\hat{r}(s)] \in L_{\infty} \}
\]

where $\mathcal{N}$ is the set of strictly proper real rational functions. In [2]–[6], the class of the reference signals are defined in a specific manner. On the other hand, no specific forms for the reference signals are assumed in this paper. In fact, $\mathcal{R}$ is defined abstractly and even neutrally stable poles of reference inputs are not specified explicitly. Set $\mathcal{R}$ defines the fairly general class of reference inputs. For example, any linear combinations of the Laplace transforms of the trigonometric functions, the step function and the decaying exponential functions $e^{-a t}$ where $a > 0$ belong to $\mathcal{R}$.

Assume that $\hat{r}(s) \in \mathcal{R}$ has $m_r$ number of unstable zeros $z_{m_r+1}, \ldots, z_{m_r+m_a}$ and the relative degree of $\hat{r}(s)$ is $h_r$. Moreover, all the unstable zeros of $P(s)$ and $\hat{r}(s)$ are distinct, i.e. (1) holds, even if the unstable zeros of $\hat{r}(s)$ are taken into account. Then, we denote the total number of the unstable zeros and the relative degree as $m_r$ and $h_r$, respectively, i.e.
\[
m_a = m_r + m_r, \quad h_a = h_r + h_r.
\]

Moreover, we assume that there is no unstable pole/zero cancellation between $\hat{r}(s)$ and $P(s)$.

For 1-DOF control systems, we deal with the tracking control problem depicted in Fig. 1. $y(t)$ and $r(t)$ are, respectively, the control output and the reference input such that $\hat{r}(s) \in \mathcal{R}$ holds. $C(s)$ is the feedback controller, while $G(s)$ is the closed-loop system from $r(t)$ to $y(t)$.

The aim of this paper is to analyze the performance limitation of the $L_2$ norm of the transient tracking error. The precise problem formulation is as follows:

**Problem 1** Let $P(s)$ and $\hat{r}(s)$ be given. Find $J = \inf \{ ||e||_2^2 \}$, where $e(t) = y(t) - r(t)$ is the error signal.

Note that $J$ depends on both $P(s)$ and $\hat{r}(s)$. For the specifically defined class of the reference signals, $J$ has been analyzed in [2]–[6]. On the other hand, we will derive $J$ for the more general class $\mathcal{R}$ of reference signals.

3. The Parameterization of Admissible Outputs

To derive the explicit expression of $J$, we first parameterize all the admissible outputs. The admissible outputs are such signals that are produced by the internally stable closed-loop systems and that track the reference input. Moreover, the parameterization will be given based on the closed-loop transfer functions rather than the feedback controllers.

We here characterize the internal stability condition in terms of the closed-loop system transfer function $G(s)$. The derivations based on $G(s)$ do not lose generality. In fact, $G(s)$ in Fig. 1 is written by
\[
G(s) = \frac{P(s)C(s)}{1 + P(s)C(s)}. \quad (3)
\]

It follows that the map from $C(s)$ to $G(s)$ is a one-to-one correspondence and that $C(s)$ can be written in term of $G(s)$ as follows:
\[
C(s) = \frac{G(s)}{P(s)(1 - G(s))}. \quad (4)
\]

All the transfer functions defining the internal stability can be written in terms of $G(s)$ as follows:
\[
\begin{align*}
\frac{P(s)}{1 + P(s)C(s)} &= P(s)(1 - G(s)), \quad (5) \\
\frac{1}{1 + P(s)C(s)} &= 1 - G(s), \quad (6) \\
\frac{C(s)}{1 + P(s)C(s)} &= \frac{G(s)}{P(s)}, \quad (7) \\
\frac{P(s)C(s)}{1 + P(s)C(s)} &= G(s). \quad (8)
\end{align*}
\]

Consequently, the closed-loop system is internally stable, if the following conditions hold:
\[
G(s) \in S, \quad P(s)^{-1}G(s) \in S, \quad (9) \\
P(s)(1 - G(s)) \in S. \quad (10)
\]

Moreover, the output tracking is defined by
\[
\lim_{t \to \infty} (r(t) - y(t)) = 0 \quad (11)
\]

where $y(t)$ is the output produced by
\[
\hat{y}(s) = G(s)\hat{r}(s). \quad (12)
\]

It follows that the set $\mathcal{Y}$ of all the admissible outputs is given by the set of all $\hat{y}(s)$ such that there exists a $G(s)$ satisfying (9), (10), (11) and (12). However, rather than $\mathcal{Y}$, it is technically easier to deal with the set $\mathcal{E}$ of the error signals
\[
\hat{e}(s) = \hat{y}(s) - \hat{r}(s). \quad (13)
\]

Once $\mathcal{E}$ is given, $\mathcal{Y}$ is given explicitly by
\[
\mathcal{Y} = \hat{r}(s) + \mathcal{E}. \quad (14)
\]

Hence, we will find an explicit expression of $\mathcal{E}$.

The set $\mathcal{E}$ is a subset of $\mathcal{E}_c$, where $\mathcal{E}_c$ is the set of all $\hat{e}(s)$ such that (9), (11) and (12). The explicit expression of $\mathcal{E}_c$ has been given in [12] as follows:

**Theorem 1** [12] Let $P(s)$ and $\hat{r}(s)$ be the given plant and the reference input assumed in this paper, respectively. Moreover, let $a > 0$ be an arbitrary given positive real number. Then, the following equivalence holds:
\[
\mathcal{E}_c = U_c(s) + V_c(s)S, \quad (14)
\]
where \(U_s(s) \in \mathcal{S}\) and \(V_s(s) \in \mathcal{S}\) are the real rational functions defined by the following recursion:

\[
U_s(s) = K^{(s_1)}(s), \quad V_s(s) = L^{(s_1)}(s),
\]

\[
K^{(s+1)}(s) = K^{(s)}(s) + \alpha_k L^{(k)}(s),
\]

\[
L^{(k+1)}(s) = \frac{1}{s + a} L^{(k)}(s),
\]

where \(\alpha_k = -\lim_{s \to \infty} (s^k \hat{\nu}(s) + K^{(k)}(s)))\),

\[
K^{(0)}(s) = -\sum_{i=1}^{m_k} H_i(s) \hat{\tau}(z_i),
\]

\[
H_i(s) = \left(\frac{z_i + a}{s + a}\right)^{m_k-1} \prod_{j \neq i} \frac{s - z_j}{s - z_j},
\]

\[
L^{(0)}(s) = \prod_{i=1}^{m_k} \frac{s - z_i}{s + a} L^{(0)}(s).
\]

Theorem 1 gives the explicit parameterization of \(E_s\) as the set of the proper stable real rational functions. It is similar to the KYJB parameterization of internally stabilizing controllers [13]. Although the parameterization is defined by using \(a > 0\), it is independent of choices of \(a > 0\).

The difference between \(E\) and \(E_s\) is caused by (10). When \(P(s)\) is stable, \(E = E_s\) holds, since (9) leads to (10). When \(P(s)\) is unstable, \(E\) is a proper subset of \(E_s\). In other words, \(E\) can be obtained by restricting \(E_s\) by using (10).

Such the restriction can be stated in terms of \(e(s)\) as follows:

**Lemma 1** Let \(P(s)\) and \(\hat{\tau}(s)\) be the given plant and the given reference signal, respectively. Then, \(\hat{e}(s) \in E\) holds, if \(\hat{e}(s) \in E_s\) and the following equations hold:

\[
\hat{e}(p_i) = 0 \quad \forall i = 1, \ldots, t_p.
\]

**proof:** (Necessity) Suppose that \(\hat{e}(s) \in E\) holds, i.e. (9) through (12) hold for some \(G(s)\). Obviously, \(E \subseteq E_s\) yields \(\hat{e}(s) \in E_s\). Since \(G(s)\) can be written by

\[
G(s) = \frac{\hat{\tau}(s)^{\theta} + \hat{\tau}(s) \hat{e}(s)}{\hat{\tau}(s)},
\]

(10) leads to the following condition:

\[
P(s) (1 - G(s)) = -\frac{P(s) \hat{e}(s)}{\hat{\tau}(s)} \in S
\]

Since \(\hat{\tau}(s)\) does not share any unstable poles with \(P(s)\), \(\hat{e}(s)\) must satisfy (22).

(Sufficiency) Suppose that \(\hat{e}(s) \in E_s\) and (22) hold. Obviously, the only possible choice of \(G(s)\) is given by (23). For the choice of \(G(s)\), (9) and (11) hold, since \(\hat{e}(s) \in E_s\) is assumed. (9) further implies \(\hat{\tau}(s) = 1 - G(s) \in S\). Since \(\hat{\tau}(s)\) has no zeros at \(s = p_i\), (22) yields \(1 - G(p_i) = 0\) and thus \(P(s) (1 - G(s)) \in S\) holds. It follows \(\hat{e}(s) \in E\).

As a consequence of Lemma 1, an explicit characterization of \(E\) is given by the following theorem:

**Theorem 2** Let \(P(s)\) and \(\hat{\tau}(s)\) be the given plant and the reference input, respectively. Let \(b > 0\) be an arbitrary given positive real number. Then, the following equivalence holds:

\[
E = U(s) + V(s)S.
\]

The real rational functions \(U(s) \in S\) and \(V(s) \in S\) are given by the following recursive equations:

\[
U(s) = K^{(p)}(s), \quad V(s) = L^{(p)}(s),
\]

where

\[
L^{(k)}(s) = \frac{s - p_{k+1}}{s + b} L^{(k)}(s),
\]

\[
K^{(k)}(s) = K^{(k)}(s) + \beta_k L^{(k)}(s),
\]

or equivalently

\[
L^{(k)}(s) = \varphi_k(s)V(s), \quad \varphi_k(s) = \prod_{j=1}^k \frac{s - p_j}{(s + b)^k},
\]

\[
K^{(k)}(s) = U(s) + \sum_{n=1} \beta_n \varphi_n(s) V(s).
\]

\(U(s)\) and \(V(s)\) are defined in Theorem 1.

**proof:** The proof is given in Appendix.

Theorem 2 gives the explicit parameterization of \(E\) for possibly unstable plants. As is Theorem 1, the parameterization is quite similar to the KYJB parameterization, where the parameterization is independent of choices of \(a > 0\) and \(b > 0\). Note that the parameterization is obtained without any information other than the unstable poles and zeros (including infinite zeros) of \(P\) and \(\hat{\tau}\), and the values \(\hat{\tau}(z_i)\) and \(\lim_{s \to \infty} \hat{\tau}(s)\). Moreover, the parameterization is independent of the neutrally stable poles of \(\hat{\tau}\).

4. **The Performance Limitations**

Now, we consider minimizing the \(L_2\) norm of \(e(t)\). Since the Parseval’s equality implies \(\|e\|^2 = \|\hat{e}\|^2\), the minimization problem is equivalent to

\[
\text{minimize } \|e\|^2 \text{ subject to } \hat{e}(s) \in E
\]

which is further equivalent to

\[
\text{minimize } \|U + VQ\|^2 \text{ subject to } Q(s) \in S.
\]

Note that the equivalence is deduced by Theorem 2. Problem (32) is a standard \(H_2\) model-matching problem, since \(U(s) \in S\) and \(V(s) \in S\) hold, and \(U(s)\) is strictly proper by its construction.

By using the explicit characterization of \(U(s)\) and \(V(s)\) in Theorem 2, the optimal value and its minimizer can be given explicitly in the following theorem:

**Theorem 3** Let \(P(s)\) and \(\hat{\tau}(s)\) be the given plant and the reference input, respectively. Moreover, \(U(s)\) and \(V(s)\) are the real rational functions defined in Theorem 2. Then,

\[
J = \inf_{\hat{e}(s) \in E} \|\hat{e}\|^2, \quad \arg \inf_{\hat{e}(s) \in E} \|\hat{e}\|^2
\]

can be written respectively as follows:

\[
J = \inf_{\hat{e}(s) \in E} \|\hat{e}\|^2, \quad \arg \inf_{\hat{e}(s) \in E} \|\hat{e}\|^2
\]
\[ J = p^* M p, \]  
\[ \hat{e}^{\text{opt}}(s) = - \left( \sum_{n=1}^{m_p} H^{\text{opt}}_n(s) w_i \hat{r}(z_i) \right), \]  
where

\[ \rho = \left[ \sigma_1 \hat{r}(z_1) \cdots \sigma_{m_p} \hat{r}(z_{m_p}) \right]^T, \]  
\[ (M)_{ij} = \frac{w_i q_j q_i w_j}{z_i + z_j}, \]  
\[ \sigma_i = \prod_{n=m_p+1}^{m_i} \theta(z_i, z_n), \]  
\[ w_i = \prod_{n=1}^{m_p} \theta(z_i, p_n), \]  
\[ q_j = (z_i + z_j) \prod_{n=1,j \neq i}^{m_j} \theta(z_i, z_n), \]  
\[ H^{\text{opt}}_i(s) = \frac{q_i}{s + z_i} \prod_{k=1,k \neq i}^{m_i} \theta(s, -\zeta_k). \]  
\[ \theta(s, \zeta) \text{ is the inner function in terms of } s \text{ and defined by} \]

\[ \theta(s, \zeta) = \frac{s + \zeta}{s - \zeta}. \]

**proof:** The proof is given in Appendix.

Theorem 3 gives the explicit forms of the optimal norm and the optimal tracking error in (33) and (34), respectively, for the general class \( R \) of the reference inputs. The obtained performance limitation (33) separates the contributions of the plant and the reference input; \( J \) is given by the quadratic form of the matrix \( M \) and the vector \( \rho \), where \( M \) is a function of the unstable poles and zeros of \( P(s) \), while \( \rho \) is composed of the values of \( \sigma_i \) and \( \hat{r}(s) \) at the unstable zeros \( z_i \). Note that \( \sigma_i \) replaces \( z_i - z_k \) in \( \hat{r}(z_i) \) to \( z_i + z_k \). This separation is an outcome of defining abstractly the class of reference inputs.

The numerator of \( (M)_{ij} \) is given by the product of \( w_i \) and \( q_j \). If an unstable pole and an unstable zero of \( P(s) \) lie closely, the magnitude of \( w_i \) is large and so is \( J \). Similarly, if a pair of unstable zeros of \( P(s) \) lies closely, the magnitude of \( q_j \) is large and, again, so is \( J \). \( M \) thus characterizes the factor that is inherent of \( P(s) \) irrelevant to reference inputs and that produces the performance limitation.

For 2-DOF control systems, performance limitation \( J_2 \) is derived in [12]. In [12], it is shown that the admissible output set for 2-DOF control systems is given by \( \mathcal{E}_s \), and that \( J_2 = \inf_{\tau \in \mathcal{E}_s} \| \tau \|_2 \). In fact, \( J_2 \) given in [12] coincides with \( J \) in Theorem 3 calculated assuming \( \ell_p = 0 \). Moreover, since \( \mathcal{E}_s \supseteq \mathcal{E}_t \) holds, \( J \geq J_2 \) holds.

Note that the optimal controller \( C^{\text{opt}}(s) \) can be calculated by using \( \hat{e}^{\text{opt}}(s) \) as follows:

\[ C^{\text{opt}}(s) = - \frac{\hat{r}(s) + \hat{e}^{\text{opt}}(s)}{P(s)\hat{e}^{\text{opt}}(s)}. \]  

**5. Numerical Example**

This section illustrates the obtained results by using numerical examples.

Suppose \( m_o = m_p = 1 \) and \( \ell_p = 1 \). Theorem 3 yields the following equation:

\[ J = 2z \left( \frac{z + p}{z - p} \right) \hat{r}(z)^2, \]
where \( z > 0 \) and \( p > 0 \) are the unstable zero and pole of \( P \), respectively. On the other hand, \( J_2 \) in [12], which corresponds to the case of \( \ell_p = 0 \), is given by

\[ J_2 = 2z \hat{r}(z)^2. \]

Since \( \left[ \frac{z + p}{z - p} \right] > 1 \) holds, \( J \geq J_2 \) by the factor of \( \left( \frac{z + p}{z - p} \right)^2 \). Note that this factor is irrelevant to choices of \( \hat{r}(s) \).

The optimal outputs for sinusoidal signal \( r(t) = \sin(t) \) are shown in Fig. 2. The thick-dashdotted line is the reference signal. The thick-solid line is the optimal output of stable plant with unstable zero \( z = 1 \). The thin lines are the optimal outputs of unstable plants with \( z = 1 \). The thin-solid and thin-dashdotted outputs show the cases of the plant with \( p = 2 \) and \( p = 10 \), respectively. We can see that the output of \( p = 10 \) is closer to the output of the stable plant than \( p = 2 \). Actually, \( J = \frac{2\hat{r}(1)^2}{p} = 10 \) is closer to \( J_2 = 2\hat{r}(1)^2 \) than \( J = 18\hat{r}(1)^2 \) of \( p = 2 \).

![Fig. 2 Optimal outputs and reference.](image-url)

**6. Conclusion**

The limitation on the tracking performance for the general class of references has been analyzed for the 1-DOF control system. We have first characterized the set of the attainable output signals and parameterized that based on the set of the stable proper real rational functions. By using the parameterization, we have derived the optimal errors and the corresponding norms explicitly for a class of references. The obtained formula clarify how the optimal norm depends on the plant and the reference input.

**References**

Appendix A Proof of Theorem 2 and Theorem 3

proof of Theorem 2: \((\mathcal{E} \supseteq U(s) + V(s)\mathcal{S})\) Let \(\dot{e}(s) \in U(s) + V(s)\mathcal{S}\) be arbitrarily given. Then, there exists a \(Q(s) \in \mathcal{S}\) such that the following equation holds:

\[
\dot{e}(s) = \hat{K}(s) + L(s)Q(s).
\]

Using (29) and (30), \(\dot{e}(s)\) can be written by

\[
\dot{e}(s) = U(s) + V(s)\hat{Q}(s),
\]

\[
\hat{Q}(s) = \sum_{i=1}^{\ell_p} \beta_i \varphi_i(s) + \psi(s)Q(s),
\]

where \(\hat{Q}(s) \in \mathcal{S}\) holds, since \(Q(s) \in \mathcal{S}\) holds and also \(\varphi_i(s) \in \mathcal{S}\) holds for \(i = 1, \ldots, \ell_p\). Consequently, \(\dot{e}(s) \in \mathcal{E}\) holds.

On the other hand, (29) leads to the following conditions:

\[
\forall k = 1, \ldots, \ell_p, \forall i = 1, \ldots, k \quad \hat{L}(s)(p_i) = 0.
\]

We will show by induction that a similar condition holds for \(\hat{K}(s)\). By definition, \(\hat{K}(p_i) = 0\) holds. Suppose that the following equations hold for some \(k < \ell_p\):

\[
\hat{K}(p_i) = 0 \quad \forall i = 1, \ldots, k.
\]

Then, (25) and (1.1) lead to the following equations:

\[
\hat{K}(p_i) = 0 \quad \forall i = 1, \ldots, k + 1
\]

where the case of \(i = k + 1\) is proven by the definition of \(\beta_k\). Hence, by induction, (A.2) holds for any \(k = 1, \ldots, \ell_p\). It follows that \(\hat{K}(s)(p_i) = 0\) and \(\hat{L}(s)(p_i) = 0\) hold for any \(i = 1, \ldots, \ell_p\). As a conclusion, (22) and \(\dot{e}(s) \in \mathcal{E}\), hold, i.e. \(\dot{e}(s) \in \mathcal{E}\) holds.

\((\mathcal{E} \supseteq U(s) + V(s)\mathcal{S})\) Let \(\dot{e}(s) \in \mathcal{E}\) be given arbitrarily. Since \(\dot{e}(s) \in \mathcal{E}\) implies \(\dot{e}(s) \in \mathcal{E}\), there exists a \(Q(s) \in \mathcal{S}\) such that the following equation holds:

\[
\dot{e}(s) = U(s) + V(s)Q(s)
\]

\[
\quad = \hat{K}(s) + \hat{L}(s)Q(s).
\]

Suppose that there exists a \(Q(s) \in \mathcal{S}\) for some \(k < \ell_p\) such that the following condition holds:

\[
\dot{e}(s) = \hat{K}(s) + \hat{L}(s)Q(s).
\]

Since \(\dot{e} \in \mathcal{E}\) is assumed, the following equation must hold:

\[
\dot{e}(p_{k+1}) = \hat{K}(p_{k+1}) + \hat{L}(p_{k+1})Q(s) = 0.
\]

Note that \(L(p_{k+1}) \neq 0\) can be shown, since there are no unstable pole/zero cancellations and all the unstable poles are distinct. Hence, (A.4) yields

\[
Q(s) = \frac{\hat{K}(p_{k+1})}{\hat{L}(p_{k+1})} = \beta_k.
\]

We here define \(Q(s)\) as

\[
Q(s) = \left(\hat{K}(s) - \beta \right) \frac{s + b}{s - p_{k+1}}.
\]

Then, \(Q(s)\) holds for any \(k = 1, \ldots, \ell_p\). In particular, \(Q(s)\) is given by

\[
Q(s) = \left(\hat{K}(s) - \beta \right) \frac{s + b}{s - p_{k+1}}.
\]

It follows that (A.3) holds for any \(k = 1, \ldots, \ell_p\). In particular, (A.3) for \(k = \ell_p\) implies \(\dot{e}(s) \in U(s) + V(s)\mathcal{S}\). The proof is thus completed. \(\square\)

proof of Theorem 3: Let \(\Theta(s)\) be the inner factor of \(V(s)\). Then, \(J\) is given by

\[
J = \|(\Theta^{-1}U)_{\text{anz}}\|_2^2,
\]

where \((\cdot)_{\text{anz}}\) represents the anti-stable part. Since \(L(s)\) is written explicitly by (29), the inner factor \(\Theta(s)\) of \(V(s) = L(s)\) is given by

\[
\Theta(s) = \left(\prod_{i=1}^{\ell_p} \beta(s, -p_i)\right)\Theta(s),
\]

where \(\Theta(s)\) is the inner factor of \(V(s)\). The explicit form of \(\Theta(s)\) is given in [12] as follows:

\[
\Theta(s) = \prod_{i=1}^{m_s} \beta(s, -z_i).
\]

Note that the unstable zeros of \(\Theta(s)\) are \(p_i (i = 1, \ldots, \ell_p)\) and \(z_n (n = 1, \ldots, m_s)\). On the other hand, \(U(s) = \hat{K}(s)\) is written by (30). By construction, \(U(p_i) = \hat{K}(p_i) = 0\) holds for any \(i = 1, \ldots, \ell_p\). Then, none of \(p_i\) is the unstable pole of \(\Theta(s)\). Let the set of unstable zeros of \(\Theta(s)\) be denoted by \(\mathcal{Z}(n = 1, \ldots, m_s)\). Furthermore, \(V_i(z_n) = 0\) holds and the partial-fraction decomposition of \(\Theta(s)_{\text{anz}}\) is given by [12]

\[
\Theta(s)_{\text{anz}} = \sum_{i=1}^{m_s} \frac{q_i z_i}{s - z_i}.
\]

Hence, we obtain

\[
\Theta(s)_{\text{anz}} = \sum_{i=1}^{m_s} \frac{q_i w_i z_i}{s - z_i}.
\]
(A. 9) is given by replacing \( q_i \) in (A. 8) to \( q_i w_i \). Consequently, (33) can be readily obtained by replacing \( q_i \) in the results in [12] to \( q_i w_i \). On the other hand, (34) can be derived in the way similar to Theorem 2 in [12].

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