Convergence Error Analysis of DSM with Dual-Decomposition for the Smart Grid

Yoshiro Fukui, Shiro Yano, and Tadahiro Taniguchi

Abstract: In recent years, demand-side management (DSM) has attracted increasing attention in balancing the demand and supply of electricity for future smart grids. Particularly, many researchers consider DSM with dual-decomposition for which the theoretical properties are based on Lagrangian relaxation. It has been proven that the optimal profile of generation and consumption using DSM with dual-decomposition can be obtained. However, the convergence error and the existing range of the optimal price have not been analyzed sufficiently, nevertheless the success of dual decomposition centers on finding an good solution. In this paper, we consider the expanded electricity grid model based on Atzeni and Samadi’s model. We introduce a day-ahead pricing algorithm, which is an extension of Samadi’s algorithm, and we analyze the error and the range. Finally, we show the main parameters that have an impact on price through this theoretical analysis, that is, the maximum sell and purchase value have an impact and the maximum values of other parameters do not.

Key Words: smart grid, demand-side management, dual-decomposition, day-ahead pricing, convergence error analysis.

1. Introduction

In recent years, demand-side management (DSM) [1] has attracted increasing attention in balancing the demand and supply of electricity for future smart grids [2]. Particularly, many researchers consider DSM with dual-decomposition [3]–[6].

The idea of DSM with dual-decomposition is that demand is balanced with supply through the encouragement of peak clipping, load shifting, and valley filling by adjusting electricity price [1]. Electricity utilities and consumers can reduce their generation and electricity costs by establishing demand-and-supply matching [4]. The theoretical properties of DSM with dual-decomposition are based on Lagrangian relaxation [7]. It has been proven that we can obtain the optimal profile of generation and consumption using DSM with dual-decomposition. However, the convergence error and the existing range of the optimal price are not discussed sufficiently, nevertheless the success of dual-decomposition centers on finding a good solution [8].

In this paper, we consider an expanded electricity grid model based on Atzeni and Samadi’s model [3],[4]. We consider a day-ahead pricing algorithm, which is an extension of Samadi’s algorithm [4], and we analyze the error and the range. Finally, we show the main parameters that have an impact on price through this theoretical analysis.

Fig. 1 Variables on inter renewable energy network (i-Rene).

2. Problem Statement

2.1 Smart Grid Model

This paper considers the inter intelligent renewable energy network, (i-Rene) [9],[10], as an expanded electricity grid model of Atzeni and Samadi’s model [3],[4].

Figure 1 shows a schematic view of i-Rene. Each consumer on i-Rene has a power generator, a battery, and a smart meter. The generator can be photo-voltaic, wind power, gasoline engine, or thermoelectric. The battery can be a Li-ion, lead storage, or electric double-layer capacitor (EDLC). The smart meter controls the shiftable electrical load, generator, and battery and enables the buying and selling of electricity from and to the outside grid via a bi-directional communication network.

In this paper, \( N \in \{1,2,...,N\} \) denotes a set of all consumers and \( i \in N \) denotes one consumer. The intended time cycle for the smart meter operation is divided into \( T \) time slots. The term \( T \in \{1,...,T\} \) denotes the set of all time slots per day. Each consumer can generate, charge/discharge storage, and consume electricity at each time slot.

Each consumer \( i \) is characterized in the eight profiles summarized in Table 1, where \( \ell^{\text{min}}, \ell^{\text{max}}, b_{i}^{\text{max}}, b_{i}^{\text{min}}, m_{i}^{\text{max}}, m_{i}^{\text{min}}, g_{i}^{\text{max}} \) denote the minimum and maximum values of each profile.

The term \( \ell^{\text{min}} \) represents the load from appliances that are kept on throughout the day, such as refrigerators. The term
2.3 Electricity Trading Model

In day-ahead trading in i-Rene, consumers trade the next day’s electricity. All consumers trade electricity based on the next day’s predicted information, for example, an electricity generation profile. To simplify the discussion, we assume that each consumer \( i \in \mathcal{N} \) knows the true information for the next day \( \ell_t^{\ast \ast}, \ell_t^{\ast}, b_t^{\ast \ast}, b_t^{\ast}, m_t^{\ast \ast}, m_t^{\ast}, g_t^{\ast \ast}, g_t^{\ast} \) for \( i \in \mathcal{N}, t \in \mathcal{T} \).

The electricity price differs at every time interval \( t \in \mathcal{T} \). Prices \( p_1, \ldots, p_T \) are shared by all consumers. A \( T \)-tuple of market prices \( p_1, \ldots, p_T \) is said to be a market price \( p = (p_t)_{t \in \mathcal{T}} := (p_1, \ldots, p_T) \).

The market price \( p \) is determined by the market. Each consumer \( i \) bids on the preferred sell and purchase values in the market based on the announced market price \( p \). We define \((m_t^{\ast \ast \text{bid}}, m_t^{\ast \text{bid}})_{t \in \mathcal{T}}\) as the preferred values. The market and consumers negotiate the sell and purchase values and the market price based on multiple announcements and bids.

Note that the transmitting condition (4) with respect to \((m_t^{\ast \ast \text{bid}}, m_t^{\ast \text{bid}})_{t \in \mathcal{T}}\) does not hold in many cases because consumers independently bid on \((m_t^{\ast \ast \text{bid}}, m_t^{\ast \text{bid}})_{t \in \mathcal{T}}\). After multiple announcements and bids, the market determines the market clearing sell and purchase values. We define \((m_t^{\ast \text{clear}}, m_t^{\ast \text{clear}})_{t \in \mathcal{T}}\) as the clearing values, which are determined under the condition (4) and \((m_t^{\ast \ast \text{bid}}, m_t^{\ast \text{bid}})_{t \in \mathcal{T}}\). Of course, the market attempts to match \((m_t^{\ast \text{clear}}, m_t^{\ast \text{clear}})_{t \in \mathcal{T}}\) and \((m_t^{\ast \ast \text{bid}}, m_t^{\ast \text{bid}})_{t \in \mathcal{T}}\), but in many cases, it cannot match perfectly.

Consumers then send (or receive) the money \( p_t \gamma^+ m_t^{\ast \text{clear}} - p_t \gamma^- m_t^{\ast \text{clear}} \) to (from) the market and execute a promise to send electricity \( m_t^{\ast \text{clear}} \). If there is an actual electricity amount that is sent to the market. The equation \( m_t^{\ast \text{clear}} \) means that the consumer set an actual electricity value \( m_t^{\ast \text{clear}} \) as the market clearing value \( m_t^{\ast \text{clear}} \).

### 2.4 Welfare of the User

Each consumer has an electricity generator. The term \( C_i : \mathcal{R} \to \mathcal{R} \) denotes the cost of power generation according to the per-slot energy consumption profile \( \ell_t^i \) for each consumer \( i \in \mathcal{N} \) and time slot \( t \in \mathcal{T} \). We assume that \( C_i \) is \( C^2 \) convex for all \( i \in \mathcal{N} \) where the convexity of \( C_i \) is defined by

\[
C_i(d \ell_t^i + (1 - d)\ell_t^i) \leq d C_i(\ell_t^i) + (1 - d)C_i(\ell_t^i),
\]

for all \( \ell_t^i, \ell_t^i \in \mathcal{R} \) and \( d \in (0, 1) \). For example, British Columbia (BC) Hydro in Canada adopts a convex price model [11],[12].

We assume that consumer behavior can be analyzed using the concept of utility function from microeconomics. The utility function of consumer \( i \) at time interval \( t \) is denoted as \( D_i^\prime : \mathcal{R} \to \mathcal{R} \) according to the per-slot energy consumption profile \( \ell_t^i \). We assume that \( D_i^\prime \) is \( C^2 \) concave for all \( i \in \mathcal{N}, t \in \mathcal{T} \), where the concavity of \( D_i^\prime \) is defined by the convexity of \(-D_i^\prime\). Note that the function \( D_i^\prime \) is estimated by previous studies (i.e.,[13]).

The welfare of the consumer \( i \) is defined as [4]

\[
W_i(x_i, p) := \sum_{t \in \mathcal{T}} W_t^i(x_t^i, p),
\]

\[
W_t^i(x_t^i, p) := D_t^\prime(\ell_t^i) - C_t(\ell_t^i) + p_t^G \gamma^+ m_t^{\ast \text{clear}} - p_t^G \gamma^- m_t^{\ast \text{clear}} + p_t \gamma^+ m_t^{\ast \text{clear}} - p_t \gamma^+ m_t^{\ast \text{clear}}.
\]
\[ x_i := (x_i^0)_{i \in \mathcal{N}} \]
\[ x_i^j := (\ell_i^j, \ell_i^e, b_i^j, b_i^e, m_i^j, m_i^e, g_i^j, g_i^e) \] (8)

We assume that each consumer behaves independently, selfishly, and intelligently. Each consumer tries to maximize their welfare \( W_i(x_i, p) \) without considering the welfare of others. From the negotiation discussed in Section 2.3, each consumer can calculate a preferred schedule and bid on preferred sell and purchase values \((m_i^{+\text{bid}}, m_i^{-\text{bid}})_{i \in T}\) to maximize their welfare according to a given market price \( p \). After negotiation, each consumer can calculate a profile \( x_i \) to maximize their welfare \( W_i(x_i, p) \) with respect to \((m_i^{+\text{clear}}, m_i^{-\text{clear}})_{i \in T}\) for a given market price \( p \).

### 2.5 Social Welfare

From a social fairness perspective [4], we define the objective of i-Rene as,

\[
\text{maximize } \sum_{i \in \mathcal{N}} \sum_{t \in T} D(t_i^+) - C(t_i^+) + p_i^G g_i^+ - p_i^G g_i^+, \tag{10}
\]

where (10) implies increasing utilities, reducing electricity cost, and reducing regional payment to an outside conventional fixed-price grid. The objective function of (10) is called social welfare. Note that the objective of i-Rene and each consumer is not necessarily the same.

### 2.6 Problem Statement

The maximizing problem of social welfare can be stated as the following nonlinear programming problem:

\[
\begin{align*}
\text{minimize } & \sum_{i \in \mathcal{N}} \sum_{t \in T} C(t_i^+) - D(t_i^+) + p_i^G g_i^+ - p_i^G g_i^+, \\
\text{subject to } & x_i \in X_i \text{ for } i \in \mathcal{N}, \\
& \sum_{i \in \mathcal{N}} f_i(x_i) = 0 \text{ for } t \in T, \\
& X_i := \{x_i \in \mathbb{R}^{8T} | x_i \geq 0 \}, \\
& h_i(t_i) \leq 0 \text{ for } t_i \in T, j \in \{1, ..., 16\}, \\
& h_i^T(t_i) = 0 \text{ for } t_i \in T, \tag{11}
\end{align*}
\]

where \( x := (x_1, ..., x_N) \in \mathbb{R}^{8NT} \) and \( x_i \) are defined in (8).

For each \( t \in T, i \in \mathcal{N}, \) constraint functions \( h_i^T : \mathbb{R}^{8T} \rightarrow \mathbb{R} \) are defined as follows.

\[
\begin{align*}
h_i^1(x_i) := & \ell_i^{+\text{min}} - \ell_i^+ , \\
h_i^2(x_i) := & - \ell_i^e, \\
h_i^3(x_i) := & - b_i^j, \\
h_i^4(x_i) := & - b_i^e, \\
h_i^5(x_i) := & m_i^j, \\
h_i^6(x_i) := & m_i^e, \\
h_i^7(x_i) := & g_i^j, \\
h_i^8(x_i) := & g_i^e, \\
h_i^9(x_i) := & \ell_i^{-\text{max}} - \ell_i^-, \\
h_i^{10}(x_i) := & b_i^j - b_i^e, \\
h_i^{11}(x_i) := & m_i^j - m_i^e, \\
h_i^{12}(x_i) := & g_i^j - g_i^e, \\
h_i^{13}(x_i) := & s_i^{+\text{min}} - \sum_{k=1}^t (\eta_k b_i^k - b_i^k), \\
h_i^{14}(x_i) := & s_i^{-\text{min}} - \sum_{k=1}^t (\eta_k b_i^k - b_i^k), \\
h_i^{15}(x_i) := & h_i^j + \sum_{k=1}^t (\eta_k b_i^k - b_i^k) - s_i^{+\text{max}}, \\
h_i^{16}(x_i) := & h_i^e + \sum_{k=1}^t (\eta_k b_i^k - b_i^k) - s_i^{-\text{max}}. \\
\end{align*}
\]

### 3. DSM with Dual-Decomposition

In this section, we introduce DSM with dual-decomposition for the problem (11).

#### 3.1 Dual Problem and Subproblem

To handle the dual-decomposition, we consider the following dual problem of (11) with partial relaxation [14] of the constraint (24).

\[
\begin{align*}
\text{maximize } & g(\lambda), \\
\text{subject to } & \sum_{i \in \mathcal{N}} \inf_{x_i \in X_i} L_i(x_i), \\
& L_i(x_i) := - \sum_{t \in T} W_i(x_i^t, \lambda), \tag{27}
\end{align*}
\]

For each \( \lambda \in \mathbb{R}^T, i \in \mathcal{N}, \) the problem of calculating \( \inf_{x_i \in X_i} L_i(x_i) \) is called a subproblem of the dual problem (25). This subproblem is calculated as

\[
\begin{align*}
\text{maximize } & \sum_{i \in \mathcal{N}} \inf_{x_i \in X_i} W_i(x_i^t, \lambda), \tag{28}
\text{subject to } & x_i \in X_i, \tag{29}
\end{align*}
\]

for each \( \lambda \in \mathbb{R}^T, i \in \mathcal{N}. \)

The subproblem (28) implies that each consumer has solved the problem when \( \lambda \) is announced as a market price \( p \). Additionally, a solution of (28) implies a preferred schedule according to an announced market price \( p = \lambda \). We describe an optimal solution of (28) as

\[
\begin{align*}
x_i^{\text{bid}}(\lambda) := & (x_i^{\text{bid}}(\lambda))_{i \in \mathcal{N}}, \\
x_i^{\text{sel}}(\lambda) := & (x_i^{\text{sel}}(\lambda))_{i \in \mathcal{N}} , \\
x_i^{\text{p}}(\lambda) := & (x_i^{\text{p}}(\lambda))_{i \in \mathcal{N}}, \\
\end{align*}
\]

#### 3.2 Day-Ahead Pricing Algorithm Using Dual-Decomposition

We consider the following algorithm.

\[
\begin{align*}
\lambda^{(k+1)} := & [\lambda^{(k)} + \alpha_k \xi(\lambda^{(k)})]^+ , \\
\lambda^{(1)} := & (p_i^T)_{i \in T}, \\
\xi(\lambda^{(k)}) := & (f(x_i^{\text{bid}}(\lambda^{(k)})))_{i \in \mathcal{N}}, \\
f_i(\lambda^{(k)}) := & \sum_{i \in \mathcal{N}} f_i(x_i^{\text{bid}}(\lambda^{(k)}))
\end{align*}
\]
where $k$ is an iteration number, $\alpha_k > 0$ is the step size, and $[A]_{\mathcal{P}_k}$ denotes the projection into a set $\mathcal{P}_k$ defined by:

$$ [A]_{\mathcal{P}_k} \in \arg \min \{ \lambda - \tilde{\lambda} \} $$

$\mathcal{P}_k := \prod_{t \in T} \{ p_t^G + \epsilon, p_t^G - \epsilon \},$

where $\epsilon > 0$ is a sufficiently small value, and $\| \cdot \|$ denotes a standard Euclidean norm.

Note that it is not always true that $(x_i^b(\tilde{\lambda}))_{i \in \mathcal{N}}$ holds the constraints (12) for iteration number $k > 0$. We consider the following heuristic algorithm to calculate a feasible solution $x_i$ to the primal problem:

$$ x_i \in \arg \max_{m_i^+, m_i^- \in \mathcal{S}^{\text{clear}}} \sum_{t \in T} \left| \sum_{i \in \mathcal{N}} W_i(x_i', \tilde{\lambda}(K)) \right| $$

where $K$ is a given maximum value of the iteration number, and $m_i^+, m_i^- \in \mathcal{S}^{\text{clear}}$ are solutions of the following problem:

$$ \min_{(m_i^+, m_i^-) \in \mathcal{S}^{\text{clear}}} \sum_{i \in \mathcal{N}} \left| m_i^+ - m_i^b(\tilde{\lambda}(K)) \right| $$

subject to

$$ \sum_{i \in \mathcal{N}} (y_i^m - y_i^m) = 0. $$

The calculation (37) represents the decision process of a profile $x_i$ to maximize welfare $W_i(x_i, p)$ with a fixed market price $p$ and fixed sell and purchase values $m_i^+, m_i^- \in \mathcal{S}^{\text{clear}}$, as discussed in Section 2.4. The problem (38) means a decision process of $m_i^{\text{clear}}$ and $m_i^{\text{clear}}$ by the market, as discussed in Section 2.3.

As described in Subsection 2.6, the problem addressed in this paper is analyzing the converging error of the algorithm (31) to (39) for the problem (11) and elucidating the main parameters that have an impact for the error.

4. The Relationship between the Lagrange Multiplier and Price

It is well known that the Lagrange multiplier $\lambda$ implies electricity price $p$ for the problem (11). However, the existing range of the Lagrange multiplier is not analyzed. In this section, we show the existing range as a secondary result of the paper.

4.1 Subresult of the Paper

The following nontrivial theorem is a secondary result of the paper.

Theorem 1. Any optimal solution $\lambda \notin \prod_{t \in T} [p_t^G, p_t^G]$. Then the following cases appear:

(A 1) $t \in T$ exists such that $\lambda_t > p_t^G$. In this case, $g_i^b(\lambda_t) > 0$, $m_i^{+b}(\lambda_t) = 0$ and $m_i^{-b}(\lambda_t) = 0$ for all $i \in \mathcal{N}$ because $x_i^b(\lambda)$ is an optimal solution of (28). According to (41), (27) and (7), the following holds:

$$ Y(\lambda) = \left\{ \begin{array}{l} \max_{\lambda_i \in \mathcal{X}} \sum_{i \in \mathcal{N}} W_i(x_i', \lambda) \Big| x_i' \in \mathcal{X} \Big( \forall i \in \mathcal{N} \Big) \\ \arg \max_{\lambda_i \in \mathcal{X}} \sum_{i \in \mathcal{N}} W_i(x_i', \lambda) \Big| x_i' \in \mathcal{X} \Big( \forall i \in \mathcal{N} \Big) \end{array} \right\}. $$

This means $\lambda = \partial^* g(\lambda) = \arg \max_{\lambda_i \in \mathcal{X}} \sum_{i \in \mathcal{N}} W_i(x_i', \lambda) \Big| x_i' \in \mathcal{X} \Big( \forall i \in \mathcal{N} \Big)$. As in case (A 1), a contradiction appears.

(A 2) $t \in T$ exists such that $\lambda_t < p_t^G$. In case (A 1), a contradiction appears.

4.2 Proof of Theorem 1

Because of difficulties in directly proving the theorem, we prepared two propositions.

Definition 1 (Superdifferential). Let $g(\lambda) : \mathbb{R}^T \to \mathbb{R}$ be a concave function. A set $\partial^* g(\lambda)$ defined as follows is called a superdifferential of $g$:

$$ \partial^* g(\lambda) := \{ \xi \in \mathbb{R}^T | g(\lambda) \leq g(\lambda) + \langle \xi, \lambda - \tilde{\lambda} \rangle, \lambda \in \mathbb{R}^T \}, $$

where $\langle \cdot, \cdot \rangle$ denotes an inner product. An element of $\partial g(\lambda)$ is called a supergradient of $g$.

Proposition 1 (A corollary of Proposition 3.4.4 in [15]). Let $X \subset \mathbb{R}^n$ be compact. Let a function $L : X \times \mathbb{R}^T \to \mathbb{R}; (x, \lambda) \mapsto L(x, \lambda)$ be continuous in $X \times \mathbb{R}^T$ together with its partial derivative $\partial L/\partial \lambda$, and $g(\lambda) := \min_{x \in X} L(x, \lambda)$ is concave. Consider the following set $Y(\lambda)$ for given $\lambda$,

$$ Y(\lambda) := \left\{ \frac{\partial}{\partial \lambda} L(x^*, \lambda) \Big| x^* \in \mathcal{X} \right\}. $$

Then,

$$ \partial^* g(\lambda) = \co Y(\lambda), $$

where $\co$ denotes the convex hull of a set.

Proof. Note that if a function is concave, the Fréchet subderivative of the function agrees with the subderivative [16]. Hence, Proposition 1 holds as a corollary of Proposition 3.2.3 in [15].

Proposition 2. Let $f : \mathbb{R}^T \to \mathbb{R}$ be a concave function. $x$ is a local maximum of $f(x)$ if and only if $0 \notin \partial^* f(x)$.

Proof. The proposition is a special case of proposition B.24(f) in [7].

Now, we can prove Theorem 1.

Proof of Theorem 1. Consider an optimal solution $\lambda \notin \prod_{t \in T} [p_t^G, p_t^G]$. Then the following cases appear:

(A 1) $t \in T$ exists such that $\lambda_t > p_t^G$. In this case, $g_i^b(\lambda_t) > 0$, $m_i^{+b}(\lambda_t) = 0$ and $m_i^{-b}(\lambda_t) = 0$ for all $i \in \mathcal{N}$ because $x_i^b(\lambda)$ is an optimal solution of (28). According to (41), (27) and (7), the following holds:

$$ Y(\lambda) = \left\{ \begin{array}{l} \max_{\lambda_i \in \mathcal{X}} \sum_{i \in \mathcal{N}} W_i(x_i', \lambda) \Big| x_i' \in \mathcal{X} \Big( \forall i \in \mathcal{N} \Big) \\ \arg \max_{\lambda_i \in \mathcal{X}} \sum_{i \in \mathcal{N}} W_i(x_i', \lambda) \Big| x_i' \in \mathcal{X} \Big( \forall i \in \mathcal{N} \Big) \end{array} \right\}. $$

This means $\lambda = \partial^* g(\lambda) = \arg \max_{\lambda_i \in \mathcal{X}} \sum_{i \in \mathcal{N}} W_i(x_i', \lambda) \Big| x_i' \in \mathcal{X} \Big( \forall i \in \mathcal{N} \Big)$. As in case (A 1), a contradiction appears.

(A 2) $t \in T$ exists such that $\lambda_t < p_t^G$. In case (A 1), a contradiction appears.
5. Main Result

The following theorem is the main result of this paper:

**Theorem 2.** We consider problem (11) satisfying

\[ \ell^*_i \leq \ell^*_t + g^*_i \quad (\forall t \in T, i \in N), \]

(44)

The iteration (31) then converges to an optimal solution in the following sense: for each iteration number \( k \),

\[ g^* - g^{(k)}_{\text{best}} \leq \frac{kt e^2 + R^2 + G^2 \sum_{i=1}^k \alpha_i}{2}, \quad (45) \]

\[ g^* := \sup_{\lambda \in \Lambda^T} g(\lambda), \]

(46)

\[ g^{(k)}_{\text{best}} := \max_{k=1,...,K} g(\hat{\lambda}_k), \]

(47)

where \( \alpha_k > 0 \) is a step size defined in (31) and \( G \) and \( R \) are constants satisfying

\[ \| \xi(\lambda) \| \leq G \quad (\forall \lambda \in \mathbb{R}^T), \]

(48)

\[ \| \lambda^{(1)} - \lambda^* \| \leq R \quad (\exists \lambda \in \arg\max_{\lambda \in \mathbb{R}^T} g(\lambda)). \]

(49)

The condition (44) means that each consumer \( i \) can obtain electricity \( \ell^*_i \) from a power generator and an outside traditional grid to satisfy the condition \( \ell^*_i \leq [\ell^*_t + g^*_i, \infty) \) in Table 1. The following is a corollary that simplifies Theorem 2.

**Corollary 1.** Consider the problem (11) satisfying (44), \( g^* \) in (46) and \( g^{(k)}_{\text{best}} \) in (47). If \( \alpha_k = \alpha (k = 1, ..., K) \), then for each step \( s \) the following holds:

\[ g^* - g^{(k)}_{\text{best}} \leq \frac{R^2 + 2 e^2 + G^2 \alpha}{2}, \]

(50)

\[ G = \sqrt{T} \sum_{i \in N} (m^i_{\text{max}} + m^i_{\text{max}}), \]

(51)

\[ R = \sqrt{\sum_{i \in T} (p_{i_G} - p_{i_G})^2}, \]

(52)

where \( \alpha > 0 \) is a constant step size.

5.1 Proof of Theorem 2

Because of difficulties in directly proving the theorem, we prepared the following five lemmas.

**Definition 2 (Admissible set).** Consider problem (11). A set \( C \) defined as follows is called an admissible set:

\[ C := \{ x \in \mathbb{R}^{KN} | \xi_i \in X_i (\forall i \in N), \sum_{i \in N} f^i(x_i) = 0 (\forall t \in T) \}. \]

(53)

**Lemma 1.** If \( \ell^*_i < \ell^*_t + g^*_i \) for all \( t \), an optimal solution for the problem (11) exists.

**Proof.** Obviously, the admissible set \( C \) is not empty. Rather, it is closed because the constraint functions (14) to (23) are continuous. \( C \) is bounded by (14) to (20), (23). By the Weierstrass extreme value theorem [17], the lemma was proven.

**Lemma 2.** The problem (11) holds strong duality. Moreover, an optimal solution \( \lambda^* \in \arg\max_{\lambda \in \mathbb{R}^T} g(\lambda) \) exists for the dual problem (25).

**Proof.** The optimal value of the problem (11) is finite by Lemma 1. The admissible set \( C \) is convex because the constraint functions (14) to (23) are convex. Obviously, the objective function defined in (11) is convex. According to Proposition 5.3.2 in [7], the lemma was proven.

**Lemma 3.** For each \( i \in N \), the subproblem (28) has an optimal solution \( \lambda^*_i \).

**Proof.** We can obtain the lemma in the same manner as Lemma 1.

**Lemma 4.** Consider the problem (11). A mapping \( \xi : \mathbb{R}^T \rightarrow \mathbb{R}^T \) defined in (33) to (34) is a supergradient of the dual function \( g(\lambda) \).

**Proof.** Let \( L(x, \lambda) := \sum_{i \in N} L_i(x) \). By the definition of \( L_i(x) \) (27), \( g(\lambda) = \inf_{x \in \mathcal{X}} L(x, \lambda) \) and \( \partial L(x, \lambda) / \partial \lambda = (f_i(x))_{t \in T} \). According to Proposition 1, the lemma was proven.

Now, we can prove Theorem 2.

**Proof of Theorem 2.** An optimal value \( \lambda^* \) exists, and an optimal solution \( g^* := g(\lambda^*) \) for the dual problem (25) by Lemma 2.

According to the definition of \( \mathcal{P} \) and (11.1) in [7],

\[ \| \lambda^* - \lambda^{(k+1)} \|^2 \]

\[ = \| \lambda^* - [\lambda^{(k)} + \alpha_k \xi(\lambda^{(k)})] \|^2 \]

\[ \leq \| \lambda^* - \lambda^{(k)} - \alpha_k \xi(\lambda^{(k)}) \|^2 + T e^2 \]

(54)

(55)

According to Lemma 4, we can apply the same discussion of the proof in [18]. We have

\[ g^* - g^{(k)}_{\text{best}} \leq \frac{k t e^2 + \| \lambda^* - \lambda^{(1)} \|^2 + \sum_{i=1}^k \alpha_i^2 \| \xi(\lambda^{(1)}) \|^2}{2 \sum_{i=1}^k \alpha_i}. \]

(56)

According to the boundedness of the set \( C \), \( G \) and \( R \) satisfying (48) and (49) exist, and the theorem was proven.

5.2 Proof of Corollary 1

Because of difficulties in directly proving the corollary, we prepared the following two lemmas.

**Lemma 5.** \( G \) defined by (51) satisfies the condition (48) of \( G \).

**Proof.** According to the definition of \( \xi(\lambda) \) (33) to (34),

\[ \xi(\lambda) \in \prod_{i \in T} \left[ \sum_{i \in N} m^i_{\text{max}} \sum_{i \in N} m^i_{\text{max}} \right]. \]

(57)

Hence, the lemma holds.
Lemma 6. Consider $R$ defined by (52) and $X^{(1)}$ defined by (32). Then the condition (49) of $R$ holds.

Proof. According to Lemma 1 and the definition of $X^{(1)}$, the lemma was proven.

Now, we can prove Corollary 1.

Proof of Corollary 1. According to Lemma 5 and Lemma 6, $G$ and $R$ defined by (51) and (52) satisfy (48) and (49). According to Theorem 2, the lemma was proven.

6. Simulation

This section shows a simulation result of the algorithm (31) to (39) for the problem (11).

6.1 Experimental Condition

In this simulation, we set $N = 10$ consumers and the number of time intervals as $T = 48$.

We considered the following utility functions [4],[13]:

$$D_j^r(t) := \begin{cases} \omega_j' t - \frac{\theta_j'}{2} t^2 & (0 \leq t \leq \omega_j') \\ (\omega_j')^2 / 2\theta_j' & (\omega_j' < t) \end{cases}$$

(58)

where $\omega_j'$ and $\theta_j'$ are constants. We set $\omega_j' = 10$ and $\theta_j' = 7$ for all $r \in T, i \in N$.

We considered generators with no day-ahead running cost, such as photo-voltaic and wind force power. The generation costs $C_i$ are defined by

$$C_i(t^r) := 0 \quad (\forall t^r_i \in [0, \ell_i^r, \text{max}]).$$

(59)

We used Higashi-Ohmi city’s photo-voltaic power generation data from autumn 2010 [19] as the maximum value of the per-slot energy production profile $\ell_i^r, \text{max}$. We set the top five power generation amounts to $\ell_i^r, \text{max}$ for $i = 1, ..., 5$. In the same manner, we set the five lowest power generation amounts to $\ell_i^r, \text{max}$ for $i = 6, ..., 10$. Figure 2 shows $\ell_i^r, \text{max}$ for each consumer $i$.

We set $s_{\text{init}} = 0, s_{\text{max}} = 4, b_i^r, \text{max} = 0.5, b_i^r, \text{max} = 0.5$, and $\eta_t = 0.9$ for all $t \in T$. We set the efficiency of transmitting electricity inside i-Rene to $\gamma = 0.98$, $\gamma = 0.98$. We set the maximum values to $b_i^r, \text{max} = 0, b_i^r, \text{max} = 0, m_i^r, \text{max} = 4$, and $m_i^r, \text{max} = 4$ for all $i \in N, t \in T$. We set the prices of the conventional grid to $p_i^G = 15$ and $p_i^G = 0$ for all $t \in T$. We set the parameters of the subgradient method to $K = 300, \epsilon = 0.01$, and $\alpha_k = 0.05$ for all $k \leq K$.

6.2 Result

Figure 3 shows the convergence error between $g^{(k)}$ and the true optimal value $g^*$. We confirm that the error approaches zero as Theorem 2 claims.

According to the simulation condition, the right hand side of (50) is calculated as follows:

$$g^* - g^{(k)\text{best}} \leq \frac{1.08 \times 10^5}{k} + 7.68 \times 10^3.$$ (60)

We confirm that the error upper bound obtained by Corollary 1 is too large to understand the error of the simulation result. The main factor of the error is $(Te^2 + G^2\alpha)/2$, which is the second term of the right side of (50).

If we know the range of $\xi (\ell)$, we can calculate a precise error upper bound by Theorem 2. According to the simulation result, the following calculation holds:

$$\max_{k \geq K} \|g(U^{(k)})\| = 3.73 \times 10^1.$$ (61)

This value satisfies (45) which is the condition of $G$. If we know this value before the simulation, the right hand side of (45) with $G = 3.73 \times 10^1$ is calculated as follows:

$$g^* - g^{(k)\text{best}} \leq \frac{1.08 \times 10^5}{k} + 3.48 \times 10^1.$$ (62)

Obviously, (62) is a good upper bound of the convergence error.

These results show that calculating a small $G$ satisfying (45) is important to obtain a precise error upper bound using Theorem 2.

7. Discussion

Namerikawa et al. analyze the stabilizing condition of a price $\lambda$ [6]. In contrast, we analyze a convergence error using DSM with dual decomposition, which is reinforcement of Namerikawa’s research. Additionally, in this section, we discuss the main parameters that have an impact on the price.

According to Corollary 1, we can evaluate the convergence error between the true optimal value $g^*$ and the iterative solution of the dual problem $g^{(k)\text{best}}$ as the right hand side of (45). From the conditions (51) and (52), $G$ and $R$ depend on the maximum sell and purchase values of the market profiles $m_i^r, \text{max}, m_i^r, \text{max}$, and $p_i^G, \text{max}$. The precision of the algorithm depends on $m_i^r, \text{max}, m_i^r, \text{max}$, and $p_i^G, \text{max}$ because the effect of $p_i^G$ vanishes with an increasing number of iteration $k$.

We conclude from the above discussion that the maximum values $m_i^r, \text{max}$ and $m_i^r, \text{max}$ should be small and the other maximum values $\ell_i^r, \text{max} = 4, b_i^r, \text{max} = 4, b_i^r, \text{max} = 4, b_i^r, \text{max} = 4$ and $g^{(k)\text{max}}$ can be large for i-Rene.
8. Conclusion

In this paper, we consider demand-side management (DSM) with dual-decomposition for the smart grid, i-Rene, which is an expanded electricity grid model of Atzeni and Sadi’s model. We analyzed a relationship of the Lagrange multiplier and price and converging error of the DSM with dual-decomposition.

We proved two facts: a market price $p = \lambda$ is within the limits of the electricity price from an outside conventional fixed-price grid to the outside grid; the upper bound of the convergence error is described as (45) by Theorem 2, which is the main theorem of the paper. The simulation result shows that calculating an error upper bound for Theorem 2.

Based on the results, we discuss the main parameters that have an impact on the price, that is, the maximum self/purchase values in the market $m_i^{\text{max}}$ and $m_i^{\text{max}}$ should be small, and the other maximum values $b_i^{\text{max}}, b_i^{\text{max}}, b_i^{\text{max}}$ and $g_i^{\text{max}}$ can be large for i-Rene. Electricity should be traded between consumers that have similar $m_i^{\text{max}}$ and $m_i^{\text{max}}$ values to decrease the error upper bound and identify an good solution for the social welfare maximizing problem for the future smart grid.

Acknowledgments

This research was partially supported by Regional Innovation Strategy Support Program and Super Cluster Program (Sekisui Shiga Area) funded by the Ministry of Education, Culture, Sports, Science and Technology, Japan. This work was supported by JSPS KAKENHI Grant Number 26870711.

References