LMI-Based Lower Bound Analysis of the Best Achievable $H_\infty$ Performance for SISO Systems

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Abstract: In this paper, we study $H_\infty$ performance limitation analysis for continuous-time SISO systems using LMIs. By starting from an LMI that characterizes a necessary and sufficient condition for the existence of desired controllers achieving a prescribed $H_\infty$ performance level, we represent lower bounds of the best $H_\infty$ performance achievable by any LTI controller in terms of the unstable zeros and the unstable poles of a given plant. The transfer functions to be investigated include the sensitivity function $(1 + PK)^{-1}$, the complementary sensitivity function $(1 + PK)^{-1}PK$, and $(1 + PK)^{-1}P$, the first and the second of which are well investigated in the literature. As a main result, we derive lower bounds of the best achievable $H_\infty$ performance with respect to $(1 + PK)^{-1}P$ assuming that the plant has unstable zeros. More precisely, we characterize a lower bound in closed-form by means of the first non-zero coefficient of the Taylor expansion of the plant $P(s)$ around its unstable zero.

Key Words: $H_\infty$ control, performance limitation, LMI, SISO continuous-time systems.

1. Introduction

In advanced control systems design, we introduce a control performance index to evaluate quantitatively the achievement of design objectives. For almost all design problems for which constructive methodologies of optimal controller synthesis are established, we need numerical computation to obtain optimal controllers and the optimal value of the performance index. Such numerical-computation-based approaches, however, do not allow us to draw any definite conclusions on what kinds of plant properties are crucial in achieving the design objectives of interest satisfactorily. To characterize such properties analytically, extensive studies on achievable performance bounds have been carried out under various plant/controller configurations and control performance indices [1]–[7]. As remarkable results [1],[6],[7], for a plant $P$ and a controller $K$, the $H_\infty$ performance limitations with respect to the sensitivity function $(1 + PK)^{-1}$ and the complementary sensitivity function $(1 + PK)^{-1}PK$ have been obtained in terms of the unstable zeros and the unstable poles of the plant $P$.

In the present paper, we propose a novel approach to performance limitation analysis by means of linear matrix inequalities (LMIs). The scope includes the analysis of the best achievable $H_\infty$ performance with respect to $(1 + PK)^{-1}$, $(1 + PK)^{-1}PK$, and $(1 + PK)^{-1}P$, where $P$ is assumed to be SISO. The motivations to explore an LMI-based approach are as follows:

(i) For a given generalized plant, we can represent the existence condition of internally stabilizing controllers satisfying a prescribed $H_\infty$ performance level by an LMI in a unified fashion. The configurations of the plant/controller and the disturbance inputs/performance outputs can be captured by the generalized plant, and these do not affect the resulting “form” of LMIs. Therefore, by using LMIs, it is expected that we can establish a unified approach to performance limitation analysis.

(ii) In the literature on the performance limitation analysis, the main mathematical tool is the complex analysis. In stark contrast, the present LMI-based approach is purely algebraic and hence highly new. By this new approach, it is expected that we can obtain novel results that are beyond the reach of complex-analysis-based approaches.

(iii) LMIs usually work fine only with numerical optimization. Therefore, from a control theoretic viewpoint, it is intriguing if we can obtain analytical results on the achievable $H_\infty$ performance limitations by means of LMIs. In [8],[9], we have shown that the well-known $H_\infty$ norm bounds in model order reduction can be reproduced by LMIs. The current study inherits the basic spirit of [8],[9].

In relation to the statements in (ii), in this paper, we demonstrate that we can indeed obtain novel results for the $H_\infty$ performance limitations of $(||1 + PK)\|_{\infty}P||_{\infty}$. More precisely, assuming that the plant has an unstable zero, we characterize a lower bound of the best achievable $H_\infty$ performance in closed-form by means of the real part of the unstable zero and the first non-zero coefficient of the Taylor expansion of $P(s)$ around the unstable zero. We also show that we can reproduce a part of known results on the sensitivity and the complementary sensitivity functions, by which we validate the soundness of the proposed LMI-based approach.

We use the following notations. We denote by $\mathbb{R}$ and $\mathbb{C}$ the set of real and complex numbers, respectively. We also use $\mathbb{R}_+$ and $\mathbb{C}_+$ $(\mathbb{C}_{++})$ for the set nonnegative real numbers and complex numbers with nonnegative (strictly positive) real parts, respectively. We denote by $\mathbb{S}^n$ the set of real symmetric matrices of size $n$. For $W \in \mathbb{C}^{n\times n}$, we define $\text{He}(W) = W + W^*$. For compactness, we write $(\bullet)XY$ to denote the product $Y^{*}XY$ where $X$ is Hermitian. For a matrix $A \in \mathbb{R}^{n\times n}$ with rank$(A) = r < n$, $A^* \in \mathbb{R}^{(n-r)\times n}$ is a matrix such that $A^*A = 0$ and $A^*A^+ > 0$. The other notations are standard.

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This paper is a refined version of the one presented in [10]. In the present paper we included a complete proof of Lemma 3 which is missing in [10]. As clarified below, Lemma 3 plays an important role in deriving lower bounds of the best achievable \(H_\infty\) performance with respect to \((1+PK)^{-1}P\) particularly when the plant \(P\) has a duplicated (higher degree) unstable zeros.

2. Review of LMI-based \(H_\infty\) Control and Basic Idea of Proposed Approach

2.1 Quick Review of LMI-based \(H_\infty\) Control

Let us consider the SISO LTI plant \(P\) described by

\[
P : \begin{cases}
    \dot{x} = Ax + Bu, \\
y = Cx
\end{cases}
\]

where \(x\) is the state, \(u\) the control input, \(y\) the measured output, and \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times 1}, \ C \in \mathbb{R}^{1 \times n}\). Throughout the paper, we assume that \((A, B)\) is controllable and \((A, C)\) is observable. The transfer function of \(P\) is given by \(P(s) = C(sI - A)^{-1}B\).

In this paper, we consider the output-feedback \(H_\infty\) control problem for a generalized plant \(G\) given below, which is constructed from \(P\) by appropriately introducing a disturbance input \(w\) and a performance output \(z\).

\[
G :
\begin{cases}
    \dot{x} = Ax + B_1w + B_2u, \\
    z = C_1x + D_11w + D_12u, \\
y = C_2x + D_21w.
\end{cases}
\]

In the standard \(H_\infty\) control problem setting, we seek for an output-feedback controller \(K\) of the form

\[
K : \begin{cases}
    \dot{x}_K = A_Kx_K + B_Ky, \\
u = C_Kx_K + D_Ky
\end{cases}
\]

such that the closed-loop system from \(w\) to \(z\), denoted by \(G \star K\), becomes internally stable and \(\|G \star K\|_{\infty} < \gamma\) holds for a prescribed \(H_\infty\) performance level \(\gamma > 0\).

The main question in this paper is to characterize in an analytical form (the lower bounds of) the best achievable \(H_\infty\) performance defined by

\[
\inf_{K \in \mathcal{K}_G} \|G \star K\|_{\infty}.
\]

Here, \(\mathcal{K}_G\) stands for the set of internally stabilizing output feedback controllers of the form (3) for the generalized plant \(G\).

For the best achievable \(H_\infty\) performance analysis, we use the following well known result [11],[12] that provides an LMI-based necessary and sufficient condition for the existence of \(K \in \mathcal{K}_G\) satisfying \(\|G \star K\|_{\infty} < \gamma\).

**Lemma** 1[11],[12] For given \(\gamma > 0\) and the generalized plant \(G\) described by (2), there exists \(K \in \mathcal{K}_G\) of the form (3) such that \(\|G \star K\|_{\infty} < \gamma\) holds if and only if there exist real symmetric matrices \(X\) and \(Y\) such that

\[L(X, Y) := \begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0,\]

\[
\begin{bmatrix} C_1X + D_11B_1^T + \frac{B_2B_1^T}{\gamma} & \frac{B_1D_1^T}{\gamma} \\ D_11B_1^T + \frac{B_2B_1^T}{\gamma} & \frac{D_11D_1^T}{\gamma} - \gammaI \end{bmatrix} < 0,
\]

(5a)

(5b)

2.2 Zeros and Poles

It is shown in [1],[6],[7] that the zeros and poles of the plant of interest are important factors in characterizing \(H_\infty\) performance limitations under feedback control. Since \(P\) in (1) is SISO, the definition of zeros is straightforward. Still, we provide a precise definition for the clarity of the subsequent discussions.

**Definition** 1 The system \(P\) described by (1) is said to have a zero \(\gamma\) of degree \(d\) if

\[P^{(k)}(z) = 0 \quad (k = 0, \ldots, d - 1), \quad P^{(d)}(z) \neq 0\]

where

\[P^{(k)}(s) = (-1)^k k! C(sI - A)^{-(k+1)} B.\]

(6)

(7)

In the above definition, \(P^{(k)}(s)\) is nothing but \(k\)-th derivative of \(P(s)\). This definition of zeros conforms to the standard definition of zeros of analytic functions in the complex analysis [13]. Indeed, Definition 1 basically states that \(z \in \mathbb{C}\) is a zero of degree \(d\) of \(P(s)\) if \(P(s)\) can be represented by the Taylor series expansion around a neighborhood of \(z \in \mathbb{C}\) as in

\[P(s) = (s - z)^d \sum_{k=0}^{\infty} \frac{P^{(k)}(z)}{k!} (s - z)^{k-d}, \quad P^{d}(z) \neq 0.\]

(8)

On the other hand, the poles of \(P\) given by (1) is nothing but the eigenvalues of \(A\). This is due to the minimality assumption on the triplet \((A, B, C)\).

2.3 LMI-based \(H_\infty\) Performance Limitation Analysis: Basic Idea

The goal of this paper is to provide an LMI-based method for the analysis of \(H_\infty\) performance limitations under various generalized plant settings. To this end, the next lemma plays a central role. We believe that this lemma itself is of independent interest from the viewpoint of convex optimization. The proof of this lemma is given in the appendix section, Section A.

**Lemma** 2 For given \(u, v \in \mathbb{C}^n\), let us define \(\gamma^* > 0\) as follows:

\[\gamma^* := \inf_{X \geq 0, Y \geq 0} \sqrt{Y^TY} \text{ subject to}\]

\[L(X, Y) > 0,\]

\[u^TXu - \gamma_1 < 0,\]

\[v^TYv - \gamma_2 < 0.\]

Then, we have \(\gamma^* = \max \{ |u^Tv|, |u^Tv| \}\).

On the basis of Lemmas 1 and 2, we can state as follows the basic idea of our LMI-based approach to the lower bound analysis of the best achievable \(H_\infty\) performance:

(i) By means of the zeros and the poles of plant \(P\), construct complex vectors \(u\) and \(v\) appropriately.

(ii) By using \(u\) and \(v\), in Lemma 1, derive the conditions of the form (9b) and (9c) from (5b) and (5c), respectively.

(iii) By applying Lemma 2, represent (a lower bound of) the best achievable \(H_\infty\) performance in an analytical form as in max \(\{ |u^Tv|, |u^Tv| \}\).
In the next two sections we demonstrate that we can indeed derive lower bounds of the best achievable $H_{\infty}$ performance analytically by the above LMI-based procedure.

3. Analysis of $\|(1 + PK)^{-1}P\|_{\infty}$

Let us consider the closed-loop system shown in Fig. 1. In this figure, the closed-loop system from $w$ to $z$ is given by $M := (1 + PK)^{-1}P$. The reduction of $\|(1 + PK)^{-1}P\|_{\infty}$ is a natural requirement when we have to cope with disturbances at plant input side. From different angle, the reduction of $\|(1 + PK)^{-1}P\|_{\infty}$ is of prime importance when we deal with decentralized stabilization of large-scale interconnected systems, where $P$ corresponds to a subsystem and $K$ a local controller. In this section, we analyze the performance limitation with respect to $\|(1 + PK)^{-1}P\|_{\infty}$.

When we investigate $\|(1 + PK)^{-1}P\|_{\infty}$, the corresponding generalized plant is given essentially by

$$G_M : \begin{cases} \dot{x} = Ax + Bw + Bu, \\ z = Cx, \\ y = Cx. \end{cases}$$

Namely, we can identify $(1 + PK)^{-1}P$ with $G_M \ast K$. Note that, for the generalized plant $G_M$, the LMI (5) reduces to

$$L(X,Y) > 0,$$

$$\begin{bmatrix} B^+ & [AX + XA^T & XC^T] & [B X + T] \\ 0 & CX & -\gamma & 0 \end{bmatrix} < 0,$$

$$\begin{bmatrix} C^T & [YA + A^T Y & YB] & [C^T 1^T] \\ 0 & B^T Y & -\gamma & 0 \end{bmatrix} < 0.$$  \tag{11c}

\[ \text{Fig. 1 Closed-loop system } (1 + PK)^{-1} P. \]

\[ \text{From the definition of } u_i (i = 1, \ldots, d), \text{ we see that the following equalities hold:} \]

$$zu_i^* - u_i^* A = u_{i-1}^* \quad (i = 1, \ldots, d).$$ \tag{13}

In addition, since $z \in \mathbb{C}_+$ is a zero of degree $d$, we see from Definition 1 that

$$u^*_i B = C(zI - A)^{-1}B = 0 \quad (i = 1, \ldots, d).$$ \tag{14}

By repeating similar procedure, we see that for $v_i (i = 0, \ldots, d)$ the next equalities hold:

$$zv_i - Av_i = v_{i-1}, \quad CV_i = 0 \quad (i = 1, \ldots, d).$$ \tag{15}

We second rewrite (11) to facilitate vector manipulations on LMIs. Namely, by Finsler’s Lemma [14], we rewrite (11) as in

$$L(X,Y) > 0,$$

$$\begin{bmatrix} AX + XA^T & XC^T \\ CX & -\gamma \end{bmatrix} - \mu_1 \begin{bmatrix} B^T 0 \end{bmatrix} < 0,$$ \tag{16a}

$$\begin{bmatrix} YA + A^T Y & YB \end{bmatrix} - \mu_2 \begin{bmatrix} C^T 0 \end{bmatrix} < 0.$$ \tag{16c}

where $\mu_1, \mu_2 \in \mathbb{R}$ are variables to be determined. Note that, for given $\gamma > 0$, the LMI (11) holds for some $X, Y$ if and only if the LMI (16) holds for some $X, Y, \mu_1$ and $\mu_2$.

We are now ready to move on to the core of the LMI-based analysis approach. To illustrate the basic idea with minimum complexity of notations, let us first consider the case where $P$ has a zero $z \in \mathbb{C}_+$ of degree one. Then, multiplying (16b) by $[u^*_1]^\ast$ from the left and $[u^*_1]^\ast$ from the right, we have

$$u^*_1 (AX + XA^T - \mu_1 BB^T)u_1$$

$$+ u^*_1 XC^T + CXu_1 < \gamma < 0.$$ \tag{16}

From (14) we have $u^*_i B = 0$ and hence the above inequality reduces to

$$(u^*_i A + C) Xu_1 + u^*_i X (u^*_i A + C)^\ast < \gamma.$$ \tag{18}

Moreover, since $u^*_i A + C = z u^*_i$ from (13), we have

$$(z + z^\ast) u^*_1 X u_1 < \gamma.$$ \tag{19}

By applying similar procedure to (16c), we have

$$(z + z^\ast) v^*_1 Y v_1 < \gamma.$$ \tag{20}

It follows that the LMI (16) holds only if the LMI below holds:

$$L(X,Y) > 0,$$

$$\begin{bmatrix} (z + z^\ast) u_1)^\ast X (z + z^\ast) u_1 \end{bmatrix} < \gamma,$$

$$\begin{bmatrix} (z + z^\ast) v_1)^\ast Y (z + z^\ast) v_1 \end{bmatrix} < \gamma.$$ \tag{21}

We are now on the final stage to characterize a lower bound of $\inf_{x \in \mathbb{R}^p, y} \|(1 + PK)^{-1}P\|_{\infty}$ in an analytical form. To this end, let us define

$$\gamma^* := \inf_{X, Y, \gamma} \gamma \quad \text{subject to (21)}.$$ \tag{22}

Then, since (16) holds only if (21) holds, it is obvious that $\inf_{x \in \mathbb{R}^p, y} \|(1 + PK)^{-1}P\|_{\infty} \geq \gamma^*$. Moreover, from Lemma 2, we see that

$$\gamma^* = \max \{ (z + z^\ast) u^*_1 v_1, (z + z^\ast) u^*_1 v_1 \}$$

$$= \max \{ (z + z^\ast) (C(z - A)^{-1} B) \}$$

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$$= \max \{ (z + z^\ast) (C(z - A)^{-1} B) \}.$$
Lemma 3 is a generalization of (21).

Theorem 2 Suppose the plant P given by (1) has a zero \( z \in \mathbb{C}_s \) of degree one. Then, we have

\[
\inf_{k \in K_{Cu}} \| (1 + PK)^{-1} P \|_{\infty} \geq (z + z^*) |C(zI - A)^{-2} B| \geq 0. \tag{22}
\]

In particular, if \( z = 0 \), we have

\[
\inf_{k \in K_{Cu}} \| (1 + PK)^{-1} P \|_{\infty} \geq (z + z^*) |C(zI - A)^{-2} B| > 0. \tag{23}
\]

The next remarks follow this theorem.

Remark 1 If the plant P has multiple unstable zeros, Theorem 2 yields multiple lower bounds.

Remark 2 From Definition 1, it is important to note that \( |C(zI - A)^{-2} B| > 0 \) if \( z \) is a zero of \( P \) of degree one. We used this fact to ensure the strict positivity in (23). On the other hand, if \( z \) is a zero of degree more than one, we see \( |C(zI - A)^{-2} B| = 0 \) and hence the lower bound in Theorem 2 degenerates to zero. Therefore we need sensible treatments depending upon the degree of zeros. This issue is pursued in the following.

We next move on to the case where \( P \) has an unstable zero of degree \( d(\geq 1) \). In this case, we can obtain the next lemma that is a generalization of (21).

Lemma 3 Suppose the plant P given by (1) has a zero \( z \in \mathbb{C}_s \) of degree \( d \). Then, the LMI (16) holds only if the LMI given below holds:

\[
L(X, Y) > 0, \tag{24a}
\]

\[
(\star) \left[ \frac{d}{\sum_{i=1}^{d} \alpha_{d, i}(z + z^*)^{i-1} u_i} \right] < \gamma, \tag{24b}
\]

\[
(\sqrt{z} + z^* v_i) Y (\sqrt{z} + z^* v_i) < \gamma \tag{24c}
\]

where

\[
\alpha_{d, i} = (-1)^{d-i} i^{d-1} C_{i-1}. \tag{25}
\]

This lemma is crucial for the treatment of a zero \( z \in \mathbb{C}_s \) of degree more than one. As clarified below, once this lemma is established, the lower bound characterization of \( \inf_{K \in K_{Cu}} \| (1 + PK)^{-1} P \|_{\infty} \) is almost straightforward. The proof of this lemma is given in the appendix section, Section B, where the appearance of the particular term \( \sum_{i=1}^{d} \alpha_{d, i}(z + z^*)^{i-1} u_i \) in (24b) is validated by mathematical induction with respect to \( d \).

From Lemmas 2 and 3, we can readily see that, if the plant P has a zero \( z \in \mathbb{C}_s \) of degree \( d \), then

\[
\inf_{k \in K_{Cu}} \| (1 + PK)^{-1} P \|_{\infty} \geq \max \left\{ \left| \sum_{i=1}^{d} \alpha_{d, i}(z + z^*)^{i-1} u_i \right|, \left| \sum_{i=1}^{d} \alpha_{d, i}(z + z^*)^{i-1} v_i \right| \right\}. \tag{26}
\]

Similarly to the case \( d = 1 \), we can confirm the latter term in the above max evaluation is zero, while the former term reduces to

\[
\sum_{i=1}^{d} \alpha_{d, i}(z + z^*)^{i-1} u_i \geq |\alpha_{d, d}(z + z^*)^d u_d| \tag{27}
\]

due to \( u_i^* v_1 = C(zI - A)^{-d+1} B = 0 \) for \( i = 1, \ldots, d - 1 \). To summarize, we arrive at the next theorem.

Theorem 3 Suppose the plant P given by (1) has a zero \( z \in \mathbb{C}_s \) of degree \( d \). Then, we have

\[
\inf_{k \in K_{Cu}} \| (1 + PK)^{-1} P \|_{\infty} \geq (z + z^*)^d |C(zI - A)^{-d+1} B| \geq 0. \tag{26}
\]

In particular, if \( z = 0 \), we have

\[
\inf_{k \in K_{Cu}} \| (1 + PK)^{-1} P \|_{\infty} \geq (z + z^*)^d |C(zI - A)^{-d+1} B| > 0. \tag{27}
\]

It is clear that Theorem 3 reduces to Theorem 2 for the case where \( d = 1 \). Note also that similar observations to Remarks 1 and 2 apply also to Theorem 3. In particular, we emphasize that, if \( z \) is a zero of \( P \) of degree \( d \), then \( |C(zI - A)^{-d+1} B| > 0 \) from Definition 1. We used this fact to ensure the strict positivity in (27). Moreover, from (7) and (8), we see that \( |C(zI - A)^{-d+1} B| \) is nothing but the absolute value of the first non-zero coefficient of the Taylor expansion of \( P(s) \) around \( s = z \). It is interesting that, depending upon the degree of the zero \( z \in \mathbb{C}_s \), the first non-zero coefficient of the Taylor expansion appears in the lower bound characterization.

Remark 3 One may think that (24c) can replaced with a similar form to (24b) and thus another lower bound that is different from (26) can be obtained. This is true. However, such treatment leads us to a lower bound of complicated fashion involving those terms \( u_i^* v_j \) \( (i + j \geq d + 1) \). This is the reason why we adopt the form (24c). Note that, by relying on (24c), we successfully derived (26) that is a natural extension of (22).

### 3.3 Numerical Examples

For a given plant \( P \), we compare the lower bounds given in Theorem 3 with \( \gamma_M^* = \inf_{k \in K_{Cu}} \| (1 + PK)^{-1} P \|_{\infty} \). To generate \( P \), we first represented \( P(s) \) as in \( P(s) = N(s)/D(s) \), and let \( D(s) = s(s^2 + s + 1) \). On the other hand, for \( N(s) \), we tested the following four cases:

1. \( N(s) = (s - 0.1) \),
2. \( N(s) = (s - 0.1)(s - 0.2) \),
3. \( N(s) = (s - 0.1 - 0.1j)(s - 0.1 + 0.1j) \) \( = (s^2 - 0.2s + 0.02) \),
4. \( N(s) = (s - 0.1)^2 \).

Note that the problem to compute \( \gamma_M^* \) is a "singular" \( H_\infty \) control problem since the generalized plant \( G_M \) given in (10) does not satisfy the standard full-rank conditions with respect to \( D_{21} \) and \( D_{22} \) (in the form of (2)). For such singular \( H_\infty \) control problems, it is known since a quite while ago that reliable computation of the best achievable \( H_\infty \) performance is hard [15]. We note that partial theoretical justification for such numerical instability of SDPs (Semidefinite Programming Problems) in singular \( H_\infty \) control problems, even though for state-feedback case, has been shown recently in [16]. To get around such numerical instability in computing the best achievable \( H_\infty \) performance of singular problems, a tailored algorithm is developed by Gahinet and Laub [15] and implemented as hinfric in MATLAB LMI Control Tool Box [17]. We thus used hinfric for the computation of \( \gamma_M^* \) and obtained its estimate denoted by \( \gamma_M^{est} \). The results are shown in Table 1.

From Table 1, we see that there is no gap between the lower
bound in Theorem 3 and the actual best achievable $H_{\infty}$ performance for the case (i), i.e., if the plant has only one unstable zero $z \in \mathbb{R}_+$. This tightness is also observed in our extensive numerical experiments on various plants with only one unstable zero $z \in \mathbb{R}_+$, posing the question of establishing a possible exactness proof. Indeed, very recently, we have succeeded in proving the exactness by a dual LMI approach [18].

On the other hand, for the cases (ii), (iii) and (iv) where the plant has multiple zeros (including complex conjugate ones and duplicated ones), we see from Table 1 that the lower bounds in Theorem 3 are not necessarily tight. This implies that we need further sophistication on the treatments of multiple zeros.

4. Analysis of $\| (1 + PK)^{-1} \|_{\infty}$ and $\| (1 + PK)^{-1} PK \|_{\infty}$

The $H_{\infty}$ performance limitations on the sensitivity function $(1 + PK)^{-1}$ and the complementary sensitivity function $(1 + PK)^{-1}PK$ are well investigated in the literature [6],[7]. In particular, it is shown that the best achievable performances with respect to $\| (1 + PK)^{-1} \|_{\infty}$ and $\| (1 + PK)^{-1}PK \|_{\infty}$ can be represented analytically in terms of the unstable zeros and the unstable poles of the plant. In this section, we reproduce a part of those known results with LMIs, by which we validate the soundness of the proposed LMI-based approach. Similarly to [1],[6],[7], the treatments of these two functions are almost the same, and hence we concentrate our attention on the sensitivity function analysis.

Let us consider the plant $P$ given by (1) and assume that $P$ has an unstable zero $z \in \mathbb{C}_+$ and an unstable pole $p \in \mathbb{C}_+$. Since $(A, B, C)$ is minimal, the pole $p$ is an eigenvalue of $A$ as noted in Subsection 2.2 and hence

$$A \xi = p \xi$$

holds where $\xi \in \mathbb{C}^n \setminus \{0\}$ is an eigenvector corresponding to the eigenvalue $p$. In particular, we select $\xi$ such that

$$C \xi = 1,$$

(29)

which is always possible due to the observability of $(A, C)$.

When we investigate the $H_{\infty}$ control problem for the sensitivity function $(1 + PK)^{-1}$, the generalized plant is given by

$$G_S : \begin{cases} \dot{x} = Ax + Bu, \\ z = Cx + w, \\ y = Cx + w. \end{cases}$$

(30)

Namely, we can identify $(1 + PK)^{-1}$ with $G_S \star K$. For the generalized plant $G_S$, the LMI (5) reduces to

$$L(X, Y) > 0,$$

(31a)

$$\begin{bmatrix} \frac{1}{\gamma} & 0 \\ C^T & 1 \end{bmatrix} \begin{bmatrix} AX + AT^T & X \gamma^T \\ C & 1 \end{bmatrix} \begin{bmatrix} B^T \\ 0 \end{bmatrix} < 0,$$

(31b)

$$\begin{bmatrix} YA + AT^T & \gamma \gamma^T \\ C & 1 \end{bmatrix} \begin{bmatrix} B^T \\ 0 \end{bmatrix} < 0.$$

(31c)

This can be rewritten equivalently as

$$L(X, Y) > 0,$$

(32a)

$$\begin{bmatrix} AX + AT^T & X \gamma^T \\ C & 1 \end{bmatrix} \begin{bmatrix} B^T \\ 0 \end{bmatrix} < 0,$$

(32b)

$$YA + AT^T \gamma - X \gamma^T C < 0$$

(32c)

where $\mu_1 \in \mathbb{R}$ is a variable to be determined. By following similar procedure to (17)–(19), we can obtain from (32b) that

$$(z + z')u_1Xu_1 < \gamma - \frac{1}{\gamma}.$$\hspace{.5cm} (33)

On the other hand, multiplying (32c) by $\xi^*$ from the left and $\xi$ from the right and using (28) and (29), we have

$$(p + p^*) \xi^* Y \xi < \gamma.$$\hspace{.5cm} (34)

It follows from Lemma 2 that

$$\inf_{K \in \mathcal{K}_{gs}} \| (1 + PK)^{-1} \|_{\infty} \geq \gamma^{*},$$

where $\gamma^{*}$ satisfies

$$\sqrt{\gamma^{*^2} - 1} = \sqrt{(z + z^*)(p + p^*)} \max |[u_1^T \xi^*], w_1^T | \xi |].$$\hspace{.5cm} (35)

Here, from (13), we see that $u'_1(zI - A) = C$, and hence $u'_1(zI - A) \xi^* C \xi^* = C \xi^* = 1$. This implies that $(z + p)u'_1 \xi^* = 1$. It follows that $u'_1 \xi^* = 1/(z - p)$. We thus obtain

$$|u_1^T \xi^*|^2 = \frac{1}{(z - p)(z - p^*)}, \quad |w_1^T | \xi |]^2 = \frac{1}{(z - p)(z - p^*)}.$$\hspace{.5cm} (36)

(35) and (36), we readily see

$$\gamma^{*^2} = \max \left\{ \frac{(z + z^*)(p + p^*) + 1, (z + z^*)(p + p^*) + 1} {\frac{(z - p)(z - p^*)}{(z - p)(z - p^*)}} \right\}$$

$$= \max \left\{ \frac{(z + p)(z + p^*)}{(z - p)(z - p^*)}, \frac{(z + p)(z + p^*)}{(z - p)(z - p^*)} \right\},$$

$$= \max \left\{ \frac{(z + p)}{(z - p)}, \frac{(z + p)}{(z - p)} \right\}. \hspace{.5cm} (37)$$

In particular, if $z \in \mathbb{R}_+$ or $p \in \mathbb{R}_+$, we have

$$\gamma^{*} = \frac{|z + p|}{|z - p|}$$

and hence we can conclude that

$$\inf_{K \in \mathcal{K}_{gs}} \| (1 + PK)^{-1} \|_{\infty} \geq \frac{|z + p|}{|z - p|}.$$\hspace{.5cm} (37)

In the case where $P$ has only one unstable zero $z \in \mathbb{R}_+$ and only one unstable pole $p \in \mathbb{R}_+$, it is known that the right-hand side of (37) coincides with the genuine best achievable performance of the sensitivity function [1],[6],[7]. We have shown that this celebrated result can be reproduced by the proposed LMI-based approach. Even though we omit the technical details, we can also reproduce the best achievable performance of the complementary sensitivity function for the case where $P$ has only one unstable zero $z \in \mathbb{R}_+$ and only one unstable pole $p \in \mathbb{R}_+$.

From these observations and from the exactness stated in Subsection 3.3, we deduce that the proposed LMI-based approach is effective especially when the plant has only one unstable zero and/or only one unstable pole.

5. Conclusion

In this paper, we explored an LMI-based approach to $H_{\infty}$ performance limitation analysis for SISO systems. By the pro-
posed LMI-based approach, we derived lower bounds of the best achievable performance with respect to $\| (I + PK)^{-1} P \|_{\infty}$. Interestingly enough, the lower bound is given in terms of the real part of an unstable zero of the plant $P$ and the first non-zero coefficient of the Taylor expansion of $P(s)$ around the unstable zero. Moreover, we demonstrated that known results on the sensitivity and the complementary sensitivity functions can be reproduced partially by the proposed LMI-based approach.

In the future work, it is important to sophisticate the current results for the case where plant has multiple zeros and poles. From more broad perspective, it is also challenging to extend the current results to MIMO systems. The definition of zeros for MIMO systems is rather complicated and hence we need careful treatments. These topics are currently under investigation and partial results have already been obtained in [19].

References


Appendix A: Proofs of Lemma 2

To prove Lemma 2, we need preliminary results given below.

Lemma 4 For given $W \in \mathbb{R}^{m \times n}$, let us define

$$v^* = \sup_{\Omega \in \mathbb{R}^{n \times m}} \text{trace}(W \Omega) \text{ subject to } \|\Omega\|_2 \leq 1.$$

Then, we have $v^* = \sigma_i(W)$ where $\sigma_i(W)$ stands for the $i$-th largest singular value of $W$.

Proof of Lemma 4: We prove the case where $n \leq m$; the proof for the case $n > m$ follows similarly. Let us denote by $W = \Phi \Sigma \Psi$ the singular value decomposition of $W$ where $\Phi \in \mathbb{R}^{m \times n}$, $\Psi \in \mathbb{R}^{n \times m}$ are orthogonal matrices and $\Sigma = [\Sigma_0 \Sigma_{m \times m}] \in \mathbb{R}^{m \times n}$ with $\Sigma_0 = \text{diag}(\sigma_1(W), \ldots, \sigma_n(W)) \in \mathbb{R}^{n \times n}$. Then, we can rewrite (A.1) as

$$v^* = \sup_{\Omega \in \mathbb{R}^{n \times m}} \text{trace}(\Sigma \Omega \Phi) \text{ subject to } \|\Omega\|_2 \leq 1.$$

If we define $\Omega' := \Psi \Phi^T$, we can readily obtain

$$v^* = \sup_{\Omega' \in \mathbb{R}^{n \times m}} \text{trace}(\Omega') \text{ subject to } \|\Omega'\|_2 \leq 1.$$

This clearly shows that $v^* = \sum_{i=1}^n \sigma_i(W)$, which is attained when $\Omega' = \Psi^T [I_n \ 0_{n \times m}] \Phi^T$.

Lemma 5 For given $\Pi \in \mathbb{R}^{2 \times 2}$ with

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix},$$

we have

$$\sigma_1(\Pi) + \sigma_2(\Pi) = \max \left\{ \sqrt{(\Pi_{11} + \Pi_{22})^2 + (\Pi_{12} - \Pi_{21})^2}, \sqrt{(\Pi_{11} - \Pi_{22})^2 + (\Pi_{12} + \Pi_{21})^2} \right\}.$$

Proof of Lemma 5: It is elementary to see that

$$\Pi^T \Pi = \begin{bmatrix} \Pi_{11}^2 + \Pi_{22}^2 & \Pi_{11} \Pi_{12} + \Pi_{12} \Pi_{21} & \Pi_{12} \Pi_{21} + \Pi_{22}^2 \\ \Pi_{11} \Pi_{12} + \Pi_{12} \Pi_{21} & \Pi_{11}^2 + \Pi_{12}^2 & \Pi_{12}^2 + \Pi_{22}^2 \end{bmatrix}.$$

It follows that

$$\sigma_1(\Pi^2) + \sigma_2(\Pi^2) = (\Pi_{11}^2 + \Pi_{12}^2 + \Pi_{21}^2 + \Pi_{22}^2),$$

$$\sigma_1(\Pi^2) \sigma_2(\Pi^2) = (\Pi_{11}^2 + \Pi_{12}^2)(\Pi_{12}^2 + \Pi_{22}^2) - (\Pi_{11} \Pi_{12} + \Pi_{12} \Pi_{21})^2 = (\Pi_{11} \Pi_{22} - \Pi_{12} \Pi_{21})^2.$$

From these equalities we readily obtain
The above nonnegativity constraint implies that if \( \Pi_{12} \geq 0 \), we have
\[
(\sigma_1(\Pi) + \sigma_2(\Pi))^2 = \Pi_{11}^2 + \Pi_{12}^2 + \Pi_{21}^2 + \Pi_{22}^2 + 2\Pi_{11}\Pi_{22} - 2\Pi_{12}\Pi_{21}.
\]
Therefore, the optimization problem (9) is essentially equivalent to the following SDP:
\[
\gamma^* := \inf_{X,Y} \gamma \text{ subject to } L(X,Y) > 0, u^TXu - \gamma < 0, v^TVv - \gamma < 0.
\]

By this fact in mind, in the following, we prove that \( \gamma^* \) given in (A.5) can be represented analytically as \( \gamma^* = \max \{|u^TV|, |v^TV|\} \).

Now we are ready to prove Lemma 2.

**Proof of Lemma 2:** We first note that, if \( u = 0 \) or \( v = 0 \), it is obvious that \( \gamma^* = 0 = \max \{|u^TV|, |v^TV|\} \) and hence the assertion holds. Therefore, in the following, we assume that \( u \neq 0 \) and \( v \neq 0 \).

Let us denote \( u = u_R + j \alpha \) and \( v = v_R + j \beta \) where \( u_R, u_I, v_R, v_I \in \mathbb{R}^n \). We further define \( U := [u_R, u_I] \in \mathbb{R}^{n \times 2} \) and \( V := [v_R, v_I] \in \mathbb{R}^{n \times 2} \). Then, we can rewrite (A.5) as in
\[
\gamma^* := \inf_{X,Y} \gamma \text{ subject to } L(X,Y) > 0, \quad \text{trace} \left( UU^TX \right) - \gamma < 0, \quad \text{trace} \left( VV^TY \right) - \gamma < 0.
\]

By Lagrange dual [20], the optimum \( \gamma^* \) can also be characterized as
\[
\gamma^* := \sup_{f,g} \text{trace}(H_{12}) \text{ subject to } f + g \geq 2, \quad \left[ \begin{array}{cc} f & \Omega \\ H_{12}^T & gVV^T \end{array} \right] \succeq 0.
\]

The above nonnegativity constraint implies that \( H_{12} = U\Omega V^T \) for some \( \Omega \in \mathbb{R}^{2 \times 2} \), and hence we have
\[
\gamma^* := \sup_{f,g} \text{trace}(U\Omega V^T) \text{ subject to } f + g \geq 2, \quad \left[ \begin{array}{cc} U & 0 \\ 0 & V \end{array} \right] \left[ \begin{array}{cc} f & \Omega \\ H_{12}^T & gVV^T \end{array} \right] \succeq 0.
\]

To complete the proof, we let us denote by \( U = U_L U_R^T \) and \( V = V_L V_R^T \) the full-rank factorizations of \( U \) and \( V \) where \( U_L U_R = I \) and \( V_L V_R = I \). Note that the size of \( U_L \) and \( U_R \) are \( U_L \in \mathbb{R}^{n \times 2} \) and \( U_R \in \mathbb{R}^{2 \times 2} \) or \( U_L \in \mathbb{R}^{n \times 2} \) and \( U_R \in \mathbb{R}^{2 \times 1} \). Similarly for \( V_L \) and \( V_R \). Under these notations, (A.8) can be rewritten as
\[
\gamma^* := \sup_{f,g} \text{trace}(U_L U_R^T \Omega \Omega^T V_L^T V_R) \text{ subject to } f + g \geq 2, \quad \left[ \begin{array}{c} fI \\ \Omega \Omega^T \\ H_{12}^T \end{array} \right] \succeq 0.
\]

If we define \( \Omega^T = U_R^T \Omega \), we have
\[
\gamma^* := \sup_{f,g} \text{trace}(V_L^T U_L \Omega^T) \text{ subject to } f + g \geq 2, \quad \|\Omega\| \leq \sqrt{f+g}.
\]

Since \( \sqrt{f+g} \) takes the maximal value 1 when \( f = g = 1 \) under the constraint \( f + g = 2 \), we have
\[
\gamma^* := \sup_{f,g} \text{trace}(V_L^T U_L \Omega^T) \text{ subject to } \|\Omega\| \leq 1.
\]

From Lemma 4, we see that \( \gamma^* = \sum \sigma_i(V_L^T U_L) \). Moreover, in view of the fact that
\[
(V^T U)^T(V^T U) = U^T V V^T U = U_R^T V_L^T U_L^T U_R U_L^T
\]
and \( U_L^T U_R = I \), it is clear that the (non-zero) eigenvalues of \( (V_L^T U_L)^2 \) and \( (V^T U)^2 \) are the same. It follows that \( \gamma^* = \sum \sigma_i(V_L^T U_L) = \sigma_1(V^T U) + \sigma_2(V^T U) \). Here, it is obvious to see that
\[
V^T U = \left[ \begin{array}{c} v_L^T u_R \\ v_I^T u_R \end{array} \right].
\]

Hence, from Lemma 5, we conclude that
\[
\gamma^* = \sigma_1(V^T U) + \sigma_2(V^T U)
\]
\[
= \max \left\{ \sqrt{(v_L^T u_R + v_I^T u_R)^2 + (v_L^T u_I - v_I^T u_I)^2}, \sqrt{(v_L^T u_I - v_I^T u_I)^2 + (v_L^T u_I + v_I^T u_I)^2} \right\}
\]
\[
= \max \{|v^Tu_I|, |v^TV|\} = \max \{|u^TV|, |v^TV|\}.
\]

This completes the proof.

**Appendix B** **Proof of Lemma 3**

In view of the fact that (21) has been proved, it suffices to show that (24b) holds if (16) holds. Before getting into the details of this proof, we need several preliminary results. The first preliminary result is concerned with the property of the linear combination \( \sum_{i=1}^d a_i u_i \) where \( u_i (i = 1, \ldots, d) \) are given by (12) and \( a_i \in \mathbb{C} \) (\( i = 1, \ldots, d \)). We see for this linear combination that, if (16) (or more precisely (16b)) holds, then
\[
\left( \sum_{i=1}^d a_i u_i \right)^T (X^T X + X^T A^T) \left( \sum_{i=1}^d a_i u_i \right) \leq 0
\]
holds for any \( a_i \in \mathbb{C} \) (\( i = 1, \ldots, d \)) since \( u_i^* B = 0 \) (\( i = 1, \ldots, d \)) as noted in (14). The second preliminary result is concerned with the specific property of \( a_{d+1} = (-1)^{d-i+1} c_{-i} \) (\( i = 1, \ldots, d \)) given by (25). To see this, let us consider the polynomial \( f_d(t) = (t - 1)^d \). Then, it is very clear that \( a_{d+1} \) corresponds to the coefficient of \( t^{d-1} \) in \( f_d(t) \). With this fact and \( f_d(t) = f_{d-1}(t)(t-1) \) \( (d \geq 2) \), we see
\[
\begin{align*}
\alpha_{d+1} &= 1 \\
\alpha_{d+1-i} &= \alpha_{d+1-i-1} \quad (i = 1, \ldots, d-1) \quad (B.2)
\end{align*}
\]
We have proved that (24b) holds for $d = k + 1$. We complete the proof by induction with respect to $d(\geq 1)$. Suppose (24b) holds for $d = k$ (≥ 1) or equivalently,

$$F_k := (\ast)^* X \left( \sum_{i=1}^{k+1} a_{k,i} x_{i}^k u_t \right) - s_c y < 0. \quad (B.3)$$

Then, what we want to prove is

$$F_{k+1} = (\ast)^* X \left( \sum_{i=1}^{k+2} a_{k+1,i} x_{i}^{k+1} u_t \right) - s_c y < 0. \quad (B.4)$$

Note that a zero of degree $d = k + 1$ can be regarded as a zero of degree $d = k$ as well and hence we can use the result (B.3) when dealing with a zero of degree $d = k + 1$.

Our proof is based on the decomposition of $F_{k+1}$ given in (B.4). First, from (B.2), we see

$$\begin{align*}
\sum_{i=1}^{k+1} a_{k+1,i} x_{i}^{k+1} u_t &= x_{k+1}^1 u_t + \sum_{i=1}^{k} a_{k+1,i} x_{i}^{k+1} u_t \\
&= x_{k+1}^1 u_t + \sum_{i=1}^{k} \left( a_{k+1,i-1} x_{i}^{k+1} u_t - a_{k+1,i} x_{i}^{k+1} u_t \right) \\
&= \sum_{i=1}^{k} a_{k+1,i-1} x_{i}^{k+1} u_t - \sum_{i=1}^{k} a_{k+1,i} x_{i}^{k+1} u_t.
\end{align*}$$

It follows that

$$F_{k+1} = (\ast)^* X \left( \sum_{i=1}^{k+1} a_{k+1,i} x_{i}^{k+1} u_t - \sum_{i=1}^{k} a_{k+1,i} x_{i}^{k+1} u_t \right) - s_c y \quad \text{He} \left( \sum_{i=1}^{k+1} a_{k+1,i} x_{i}^{k+1} u_t \right)^* X \left( \sum_{i=1}^{k} a_{k+1,i} x_{i}^{k+1} u_t \right) + (\ast)^* X \left( \sum_{i=1}^{k} a_{k+1,i} x_{i}^{k+1} u_t \right) - s_c y \quad (B.4)$$

$$G_{k+1}^{[1]} - G_{k+1}^{[2]} = (\ast)^*(AX + XA^T) \left( \sum_{i=1}^{k+1} a_{k+1,i} x_{i}^{k+1} u_t \right) + G_{k+1}^{[2]} - G_{k+1}^{[2]}$$

$$= \left( A^T X + AX \right) \left( \sum_{i=1}^{k+1} a_{k+1,i} x_{i}^{k+1} u_t \right)$$

where we used (13) to prove the fourth equality. Moreover, it is not hard to see that the last term above can be rewritten as

$$G_{k+1}^{[1]} = \left( A^T X + AX \right) \left( \sum_{i=1}^{k+1} a_{k+1,i} x_{i}^{k+1} u_t \right)$$

$$< 0$$

where the last inequality comes from (B.1). This completes the proof.

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