Distributed Optimization for Energy Flow Problem by Price Coordination Using Augmented Lagrangian Method

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Abstract: Recently, an orthodox decomposition method for optimization problems by pricing mechanism based on Lagrange multipliers method has been interested as a distributed optimization approach for electric power supply problems with an electricity trading market. However, the approach can be applied to a convex case only, where the objective function is convex and the constraint set is convex, that is, Lagrange multipliers method cannot be used to nonconvex cases theoretically. In this paper, we propose to utilize the augmented Lagrangian method available to nonconvex cases. However, it is impossible to separate the augmented Lagrangian into mutually independent sub-objective functions, because squared penalty terms are added to the Lagrangian. Therefore, we regard a group of the mutually dependent sub-problems with interfered objective functions as a game problem, and present a new decomposition method in which Nash equilibrium as a rational solution is required. Availability of the presented decomposition method is verified by applying to simple energy flow problems with non-convexity, for example, electric power flow interchanging through nonlinear converter from gas energy and also market is allocated on the power flow.

Key Words: augmented Lagrangian method, price coordination, Nash equilibrium, steepest descent method, particle swarm optimization (PSO).

1. Introduction

Many efforts are under way to solve energy problems by energy system planning and control in cities or regions incorporating flexible interchange of energy forms and comprehensive response to fluctuations in energy demand [1]–[3]. Such efforts generally center on electric energy, particularly demand response [4], together with maintenance of the supply-demand balance through price coordination [5]. One approach that has been proposed for electricity price coordination that includes electricity storage systems is distributed optimization with decomposition solution of the optimization problem by the use of the method of Lagrange multipliers [6],[7]. This is a classical technique that was applied to decomposing optimization problems at a time when computer performance levels were far lower than at present, but in recent years this technique is being proposed not simply as a means of distributed optimization calculations but rather as a means of achieving distributional processing systems with price coordination in electricity supply systems. The proposals, however, have largely been directed toward energy supply-demand coordination involving electricity alone as the energy supply form, and little research has been conducted on inclusion of interchange between different forms of energy via energy converters. The method of Lagrange multipliers is particularly difficult to apply to decomposing an optimization problem into the sub-problems for the individual energy forms in cases involving nonlinear energy conversion, since the method requires convexity [8].

In the present study, we construct a network model representing an energy flow involving different energy forms inter-

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sarily contribute to computational efficiency, and the presented method using an augmented Lagrangian method is the same. In this paper, when plural decision-makers exist in the energy flow optimization problems and the demand-supply adjustment is performed through market mechanisms, and even if convexity assumption cannot be required, it is shown that the theoretical market prices and the corresponding trading energy amounts can be obtained by using the procedure based on heuristic algorithms such as PSO with the augmented Lagrangian.

The objective here is not to directly handle energy flow in “Smart Communities” or other large-scale concrete systems but to model the basic framework of the energy flow problem for cases involving interchange between different forms of energy in networks that include a market function, as well as to propose the basic principles of a new distributional processing system based on the model. For that purpose, we begin by considering in detail the basic principles of the method of distributional solution by price coordination using an augmented Lagrangian method and how it should be applied to energy flow. We then proceed to test the effectiveness of the proposed model and methodology by simulation with a simple static energy flow problem and with a dynamic flow problem involving a time series of fluctuations.

2. Decomposition Method by Augmented Lagrangian

2.1 Game Decomposition Problem by Augmented Lagrangian

To simplify the discussion, let us consider application of the optimization problem

\[
\begin{align*}
\min_{x_1, x_2} & \quad f_1(x_1) + f_2(x_2) \\
\text{subj. to} & \quad h_1(x_1) + h_2(x_2) = 0
\end{align*}
\]

(1a)

(1b)

to the energy flow problem, where the variables \(x_1 \in \mathbb{R}^{N_1}\) and \(x_2 \in \mathbb{R}^{N_2}\) are mutually interfering variables under the equality constraint. We denote the equality value by \(L\). Then, for \(h_1\) and \(h_2\), let \(h_1 : \mathbb{R}^{N_1} \rightarrow \mathbb{R}^L\) and \(h_2 : \mathbb{R}^{N_2} \rightarrow \mathbb{R}^L\). The number of differing variable types can generally be extended to \(P\), in which case the objective functions and the equality constraint functions are \(f_p(x_p), p = 1, \ldots, P\), and \(h_p(x_p), p = 1, \ldots, P\), respectively. Here let us discuss the case of \(P = 2\).

The augmented Lagrangian for the primal problem (1) is then

\[
M(x_1, x_2, \varphi, \rho) = f_1(x_1) + f_2(x_2) + \rho \sum_{i=1}^{L} \left[ h_i(x_1) + h_2(x_2) \right] + \varphi^T \left[ h_1(x_1) + h_2(x_2) \right] \]

(2)

which is a Lagrangian with an augmented objective function penalty containing a squared penalty term as the last term on the right-hand side, and even if the objective functions \(f_1(x_1)\) and \(f_2(x_2)\) in the primal problem (1) are not convex and the equality constraint functions \(h_1(x_1)\) and \(h_2(x_2)\) are not linear, given an appropriate penalty with a sufficiently large value \(\rho > 0\), there exists an optimum Lagrange multiplier \(\varphi\) and it is known that in the augmented Lagrangian optimization problem (hereinafter abbreviated as the “augmented Lagrangian problem”), the local optimum solution of problem

\[
\begin{align*}
\min_{x_1, x_2, \varphi, \rho} & \quad M(x_1, x_2, \varphi, \rho) \\
\end{align*}
\]

(3)

coincides with the local optimum solution of the primal problem (1) [8]. Taking this penalty coefficient \(\rho\) as fixed in this appropriate value and \((\bar{x}_1(\varphi), \bar{x}_2(\varphi))\) as the solution in the augmented Lagrange problem (3) for the Lagrange multiplier \(\varphi\), then \(\varphi\) is the optimum Lagrange multiplier if it satisfies the equality constraint

\[
h_1(\bar{x}_1(\varphi)) + h_2(\bar{x}_2(\varphi)) = 0.
\]

(4)

When \(\rho = 0\), the Lagrangian becomes

\[
L(x_1, x_2, \varphi) = f_1(x_1) + f_2(x_2) + \varphi^T \left[ h_1(x_1) + h_2(x_2) \right]
\]

(5)

and the above relation also holds for the Lagrange problem

\[
\min_{x_1, x_2, \varphi} L(x_1, x_2, \varphi)
\]

(6)

provided that the objective functions \(f_1(x_1)\) and \(f_2(x_2)\) are convex and the equality constraint functions \(h_1(x_1)\) and \(h_2(x_2)\) are linear, thus fulfilling the “convexity prerequisite”. Furthermore, the Lagrangian (5) can be decomposed for variables and the minimization problem (6) can then be rewritten in the additive form

\[
\begin{align*}
\min_{x_1, x_2, \varphi} & \quad L(x_1, x_2, \varphi) = \min_{x_1} \left[ f_1(x_1) + \varphi^T h_1(x_1) \right] \\
& \quad + \min_{x_2} \left[ f_2(x_2) + \varphi^T h_2(x_2) \right].
\end{align*}
\]

(7)

Application of the Lagrange multiplier \(\varphi\) is then separated into the two suboptimization problems

\[
\begin{align*}
\min_{x_1} & \quad L_1(x_1, \varphi) = f_1(x_1) + \varphi^T h_1(x_1) \\
\min_{x_2} & \quad L_2(x_2, \varphi) = f_2(x_2) + \varphi^T h_2(x_2).
\end{align*}
\]

(8a)

(8b)

These are the basic principles of decomposition solution based on the Lagrangian method.

However, the penalty term on the right-hand side of the augmented Lagrangian in Eq. (2) is

\[
\|h_1(x_1) + h_2(x_2)\|^2 = \sum_{i=1}^{L} \left( h_i(x_1) + h_2(x_2) \right)^2
\]

\[
= \sum_{i=1}^{L} \left( h_1(x_1) + h_2(x_2) \right)^2 + 2h_1(x_1)h_2(x_2) + h_2(x_2)^2.
\]

(9)

This cannot be separated to an additive form for the variables and thus tends to preclude direct application of a decomposition solution. To resolve this impasse, we add a shared penalty term and decompose the augmented Lagrangian problem (2) into subaugmented Lagrangians containing interdependent variables, as

\[
\begin{align*}
M_1(x_1, x_2, \varphi, r) &= f_1(x_1) + \varphi^T h_1(x_1) + r \|h_1(x_1) + h_2(x_2)\|^2 \\
M_2(x_1, x_2, \varphi, r) &= f_2(x_2) + \varphi^T h_2(x_2) + r \|h_1(x_1) + h_2(x_2)\|^2
\end{align*}
\]

(10a)

(10b)

and for the augmented Lagrangian problem (3) substitute the interdependent suboptimization problem set

\[
\begin{align*}
\min_{x_1} & \quad M_1(x_1, x_2, \varphi, r) \\
\min_{x_2} & \quad M_2(x_1, x_2, \varphi, r).
\end{align*}
\]

(11a)

(11b)

The problems (11a) and (11b) involve interference with \(x_1\) by \(x_2\) and with \(x_2\) by \(x_1\), respectively. We therefore approach
this optimization problem set containing mutually interfering variables as a game problem, and given the Lagrange multiplier \( \varphi \) and the squared penalty \( r \) attempt to find a Nash equilibrium solution \((x^N_1, x^N_2)\) that is a game-problem rational solution

\[
\begin{align*}
M_1\left( x^N_1, x^N_2, \varphi, r \right) &= \min_{x_1} M_1 \left( x_1, x^N_1, \varphi, r \right) \quad (12a) \\
M_2\left( x^N_1, x^N_2, \varphi, r \right) &= \min_{x_2} M_2 \left( x^N_1, x_2, \varphi, r \right). \quad (12b)
\end{align*}
\]

In the following, we assume that an appropriate and sufficiently large value is always given for the squared penalty \( r \) and specify dependence only on the Lagrange multiplier \( \varphi \) and reformulate the Nash equilibrium solution as \((x^N_1(\varphi), x^N_2(\varphi))\). We refer to the game problem (11) as the “game decomposition problem” for the primal problem (1) that uses the augmented Lagrangian.

The Nash equilibrium \((x^N_1(\varphi), x^N_2(\varphi))\) that satisfies Eq. (12) and the augmented Lagrangian problem (3) optimum solution \((\bar{x}_1(\varphi), \bar{x}_2(\varphi))\) coincide in the sense that their Karush-Kuhn-Tucker necessary condition equations are equivalent. Thus, based on the differentiability of functions \( M_1 \) and \( M_2 \), if \((x^N_1, x^N_2)\) (the argument \( \varphi \) is taken as implied here and below, for brevity) is the Nash equilibrium solution satisfying Eq. (12), then

\[
\nabla_{x_1} M_1 \left( x^N_1, x^N_2, \varphi, r \right) = \nabla f_1 (x^N_1) + \varphi^T \nabla h_1 (x^N_1) + 2r \left( h_1 (x^N_1) + h_2 (x^N_2) \right)^T \nabla h_1 (x^N_1) = 0 \quad (13a)
\]

\[
\nabla_{x_2} M_2 \left( x^N_1, x^N_2, \varphi, r \right) = \nabla f_2 (x^N_2) + \varphi^T \nabla h_2 (x^N_2) + 2r \left( h_1 (x^N_1) + h_2 (x^N_2) \right)^T \nabla h_2 (x^N_2) = 0 \quad (13b)
\]

hold, and if \((\bar{x}_1, \bar{x}_2)\) (the argument \( \varphi \) is taken as implied here and below, for brevity) is the optimum local solution for augmented Lagrange problem (3), then

\[
\nabla_{x_1} M(\bar{x}_1, \bar{x}_2, \varphi, r) = \nabla f_1 (\bar{x}_1) + \varphi^T \nabla h_1 (\bar{x}_1) + 2r \left( h_1 (\bar{x}_1) + h_2 (\bar{x}_2) \right)^T \nabla h_1 (\bar{x}_1) = 0 \quad (14a)
\]

\[
\nabla_{x_2} M(\bar{x}_1, \bar{x}_2, \varphi, r) = \nabla f_2 (\bar{x}_2) + \varphi^T \nabla h_2 (\bar{x}_2) + 2r \left( h_1 (\bar{x}_1) + h_2 (\bar{x}_2) \right)^T \nabla h_2 (\bar{x}_2) = 0 \quad (14b)
\]

hold and the two necessary-condition equations coincide.

### 2.2 Decomposition of Augmented Lagrangian Using Nash Equilibrium Solution

The optimum solution \((\bar{x}_1(\varphi), \bar{x}_2(\varphi))\) of the Lagrange problem (6) with the Lagrange multiplier \( \varphi \) satisfying the constraint (4) is the optimum solution of the primal problem (1), and it is known that, under the above assumption of convexity, the solution of the max-min dual problem

\[
\max_{\varphi} \min_{(x_1, x_2)} L(x_1, x_2, \varphi) \quad (15)
\]

is known [8]. Denoting by \( v(\varphi) \) the minimum value of the inner minimization problem \( \min_{x_1, x_2} L(x_1, x_2, \varphi) \) which is given by the sum of \( v_1(\varphi) \) and \( v_2(\varphi) \), the minimum values of the suboptimization problems (8a) and (8b), respectively, the max-min dual problem (15) may be rewritten

\[
\max_{\varphi} \{ v_1(\varphi) + v_2(\varphi) \} \quad (16)
\]

However, the dual problem of the game problem (12) for the subaugmented Lagrangian (10) remains undefined, and therefore given the equivalence of optimum solutions \( \bar{x}_1(\varphi) \) and \( \bar{x}_2(\varphi) \) for \( \bar{x}_1(\varphi) \) and \( \bar{x}_2(\varphi) \) and the Nash equilibrium solution \((x^N_1(\varphi), x^N_2(\varphi))\) for the problem (12), we replace Eq. (4) with a minimization problem for the Lagrange multiplier \( \varphi \) that satisfies

\[
\min_{\varphi} \left\| h_1 \left( x^N_1(\varphi) \right) + h_2 \left( x^N_2(\varphi) \right) \right\|^2 \quad (17)
\]

directly solve this minimization problem.

\[
\min_{\varphi} \left\| h_1 \left( x^N_1(\varphi) \right) + h_2 \left( x^N_2(\varphi) \right) \right\|^2 \quad (18)
\]

Also taking Eq. (12) containing the minimization operation that defines the Nash equilibrium solution \((x^N_1(\varphi), x^N_2(\varphi))\) as the equality constraint condition, we formulate the substitute problem as

\[
\begin{align*}
\min_{\varphi} \left\| h_1 \left( x^N_1(\varphi) \right) + h_2 \left( x^N_2(\varphi) \right) \right\|^2 \\
\text{sub. to}
\begin{cases}
M_1 \left( x^N_1(\varphi), x^N_2(\varphi), \varphi, r \right) = \min_{x_1} M_1 \left( x_1, x^N_1(\varphi), \varphi, r \right) \quad (19b) \\
M_2 \left( x^N_1(\varphi), x^N_2(\varphi), \varphi, r \right) = \min_{x_2} M_2 \left( x^N_1(\varphi), x_2, \varphi, r \right). \quad (19c)
\end{cases}
\end{align*}
\]

We refer to this as the “game decomposition price coordination problem”, with the following assumption.

**Assumption 1** There is one and only one Nash equilibrium solution \((x^N_1(\varphi), x^N_2(\varphi))\) that satisfies Eqs. (19b) and (19c) for the Lagrange multiplier \( \varphi \).

To find the Nash equilibrium solution \((x^N_1(\varphi), x^N_2(\varphi))\) given the Lagrange multiplier \( \varphi \) satisfying Eqs. (19b) and (19c), we may, for example, apply the steepest descent method. The rule for updating the trial points \((x_1(l), x_2(l))\) is then

\[
\begin{align*}
x_1(l + 1) &= x_1(l) + c \nabla_{x_1} M_1 \left( x_1(l), x_2(l), \varphi, r \right) \\
&= x_1(l) + c \nabla f_1 (x_1(l)) + c \varphi^T \nabla h_1 (x_1(l)) + 2r \left( h_1 (x_1(l)) + h_2 (x_2(l)) \right)^T \nabla h_1 (x_1(l)) \quad (20a)
\end{align*}
\]

\[
\begin{align*}
x_2(l + 1) &= x_2(l) + c \nabla_{x_2} M_2 \left( x_1(l), x_2(l), \varphi, r \right) \\
&= x_2(l) + c \nabla f_2 (x_2(l)) + c \varphi^T \nabla h_2 (x_2(l)) + 2r \left( h_1 (x_1(l)) + h_2 (x_2(l)) \right)^T \nabla h_2 (x_2(l)) \quad (20b)
\end{align*}
\]

where \( l \) is the iteration number and \( c \) is a positive number. The stability of this update rule under sufficient tolerance in precision establishes the conditions (13), and in the sense of satisfying Eq. (13) may be taken to represent the Nash equilibrium solution. Given the trial points \((x_1(l), x_2(l))\) in the \( l \)-th iteration, moreover, for the different optimization problems for variables \( x_1 \) and \( x_2 \)

\[
\begin{align*}
\min_{x_1} M_1 \left( x_1, x_2(l), \varphi, r \right) \quad (21a)
\end{align*}
\]
\[
\min_{x_1} M_2(x_1(l), x_2, \varphi, r) \quad (21b)
\]

the steepest descent method is applied to each as the updating rule. With \( r = 0 \) in this updating rule by steepest descent, it then becomes an update rule for mutually independent steepest descents for the suboptimization problems (8a) and (8b) in decomposition by the Lagrangian method. For the problem (21), a conjugate-gradient or quasi-Newton method can be applied as the updating rule for finding the Nash equilibrium solution. Alternatively, it has been proposed that a heuristic method such as PSO can be utilized to obtain Nash equilibrium solutions \([11]\), and if the minimized function is non-differentiable or if it is a multimodal function, then PSO can be applied.

Explicit mathematical expression of the Nash equilibrium solution \((x_1^\varphi(\varphi), x_2^\varphi(\varphi))\) satisfying Eqs. (19b) and (19c) as a function of the Lagrange multiplier \(\varphi\) is generally difficult, and mathematical expression of the objective function in Eq. (19) of the problem as a function of \(\varphi\) is not feasible, therefore requiring a heuristic method for its solution. If, for example, PSO is applied, then the procedure for finding the Nash equilibrium solution is a nested function, essentially as follows.

**Game decomposition price coordination problem by PSO**

1. Randomly select the initial value of the Lagrange multiplier \(\varphi\) in \(P\) types, and let \(\varphi^0(1), p = 1, \ldots, P\), with the iteration number \(k = 1\) and \(K\) as the maximum number of iterations.
2. Let \(\varphi = \varphi^0(k)\), solve game decomposition problem (12) using the steepest descent rule (20), and find the Nash equilibrium solution \((x_1^\varphi(\varphi^0(k)), x_2^\varphi(\varphi^0(k)))\). Do for \(p = 1, \ldots, P\).
3. Compute objective function value

\[
D(\varphi^0(k)) = \left[ h_1\left( x_1^\varphi(\varphi^0(k)) \right) + h_2\left( x_2^\varphi(\varphi^0(k)) \right) \right]^2, \quad p = 1, \ldots, P \quad (22)
\]

and find

\[
\varphi^p_{\text{best}}(k) = \arg\min_{\varphi(i)} \{ D(\varphi^0(i)) \} \quad (23a)
\]

\[
\varphi^{\text{best}}(k) = \arg\min_{\varphi^{\text{best}}(k)} \{ D(\varphi(i)) \} \quad (23b)
\]

4. If \(k = K\), then end the computation and take \(\varphi^{\text{best}}(k)\) as the optimum solution for the price coordination problem (19), and otherwise proceed to Step 5.
5. Update the \(P\) types of Lagrange multiplier \(\varphi^p(k), p = 1, \ldots, P\) to

\[
\varphi^p(k+1) = \varphi^p(k) + \lambda \left( \varphi^p(k) - \varphi^p(k-1) \right) + c_1(\varphi^p_{\text{best}}(k) - \varphi^p(k)) + c_2(\varphi^{\text{best}}(k) - \varphi^p(k)) \quad (24)
\]

and return to Step 2 as \(k = k + 1\).

**3. Distributed Optimization of Energy Flow Problem**

**3.1 Formulation of a Simple Static Energy Flow Problem**

Let us consider an energy flow such as that of Fig. 1 that, having satisfied a supply-demand balance for energy in forms A and B, can supply energy from side B to side A by converting energy B to energy A via converter T and can be formulated as

\[
\min_{u_1} M_2(x_1(l), x_2, \varphi, r) \quad (21b)
\]

where \(u_1\) is the cost function for the amount of energy A supplied \(u_1\) and \(f_2(u_2)\) is the cost function for the amount of energy B supplied \(u_2\). The conversion function \(h_3(x_3)\) represents the conversion property of the converter. For the problem (25), the decomposition by converter T may be located on either side A or side B, but in the following, let us consider application of the augmented Lagrangian method for distributed optimization in the case of decomposition on side A. We assume that energy A is electricity, energy B is gas, and converter T is a gas turbine. In the operation for energy interchange from the gas side to the electricity side, following conversion of gas to electricity by the gas turbine, decomposition is performed by the Lagrange multiplier \(\varphi\) with price correspondence incorporated by the price coordination function of the assumed market. In Fig. 2, the decomposition is represented by a capacitor symbol. Following the conversion of gas to electricity, denoting the desired sale amount to market by \(y_{31}\) and the desired purchase amount from the market by \(x_{31}\), in place of the problem (25c) in the problem (25) we substitute the equality constraint to be fulfilled as

\[
x_{31} = y_{31}, \quad (26a)
\]

\[
y_{31} = h_3(x_{31}), \quad (26b)
\]

The Lagrange multiplier \(\varphi\) is incorporated into the constraint (26a), which represents the market trade, and the augmented Lagrangian with the objective function (25a) incorporating this thus becomes

\[
M(u_1, u_2, x_{31}, y_{31}, \varphi, r) = f_1(u_1) + f_2(u_2) + \varphi(x_{31} - y_{31}) + r(x_{31} - y_{31})^2. \quad (27)
\]
In this way, we obtain the subaugmented Lagrangians by decomposition in the market segment as
\begin{align}
M_1(u_1, x_{11}, y_{11}, \varphi, r) &= f_1(u_1) + \varphi x_{11} + r(x_{11} - y_{11})^2 \tag{28a} \\
M_2(u_2, x_{11}, y_{11}, \varphi, r) &= f_2(u_2) - \varphi y_{11} + r(x_{11} - y_{11})^2 \tag{28b}
\end{align}

The game decomposition price coordination problem corresponding to the problem (19) is then
\begin{align}
&\min_{\varphi} D(\varphi) = (x_{11}^N(\varphi) - y_{11}^N(\varphi))^2 \tag{29a} \\
&\text{subj. to } M_1(u_1^N(\varphi), x_{11}^N(\varphi), y_{11}^N(\varphi), \varphi, r) \\
&= \min_{(u_1, x_{11})} \left\{ f_1(u_1) + \varphi x_{11} + r(x_{11} - y_{11})^2 \right\} \tag{29b} \\
&\text{subj. to } u_1 + x_{11} = v_1 \tag{29c} \\
&u_1, x_{11} \geq 0, \ v_1; \ \text{given} \tag{29d} \\
M_2(u_2^N(\varphi), x_{11}^N(\varphi), y_{11}^N(\varphi), \varphi, r) \\
&= \min_{(u_2, x_{22})} \left\{ f_2(u_2) - \varphi y_{11} + r(x_{11} - y_{11})^2 \right\} \tag{29e} \\
&\text{subj. to } u_2 = x_{22} + v_2 \tag{29f} \\
&y_{11} = h_3(x_{22}) \tag{29g} \\
&u_2, x_{22} \geq 0, \ v_2; \ \text{given.} \tag{29h}
\end{align}

With electricity and gas as independent variables $u_1$ and $u_2$, respectively, we eliminate $x_{11}, y_{11}$, and $x_{22}$ using Eqs. (29c), (29f), and (29g) and solve the problem (29) under their non-negativity.

In applying the PSO method of solution described in the previous section to the game decomposition price coordination problem (29), the distributional solution is obtained as follows. Since PSO is a multipoint search method, in Step, multiple (P-type) price candidates $\varphi^p$, $p = 1, \ldots, P$, are selected by the market manager and all presented to the electricity-side manager and the gas-side manager. At these prices, the electricity- and gas-side managers, with information on the counterpart’s flow, separately perform minimization of their own energy flow cost using the subaugmented Lagrangian and find the stabilization point for their respective flows by the steepest descent method.

Next, in Step 2, for each price candidate $\varphi^p$, the electricity and gas sides show the market manager the amount of electricity $x_{11}^N(\varphi^p)$ it wishes to buy from the market and the amount of electricity $y_{11}^N(\varphi^p)$ it wishes to sell to the market, respectively. This is followed by Step 3, in which the market manager calculates for each price candidate the difference $D(\varphi^p)$ between the electricity amounts for sale $x_{11}^N(\varphi^p)$ and for purchase $y_{11}^N(\varphi^p)$ in order to obtain p-best and g-best.

In Step 5, the market manager updates the multiple price candidates $\varphi^p$, $p = 1, \ldots, P$, by applying the PSO updating rule based on this information. During Step 3, if a price with $D(\varphi^p) = 0$ ($x_{11}^N(\varphi^p) = y_{11}^N(\varphi^p)$) is found, then that price becomes the trade price, and otherwise new price candidates $\varphi^p$, $p = 1, \ldots, P$, are shown in Step 2 to the electricity-side manager and the gas-side manager. Iteration of this procedure constitutes the distributed optimization by PSO.

3.2 Example of Numerical Values in a Simple Static Energy Flow Problem

3.2.1 Case of nonconvex objective function

If the objective functions in the problem (29) are
\begin{align}
f_1(u_1) &= \sqrt{u_1}, \tag{30a} \\
f_2(u_2) &= 0.5 \sqrt{u_2} \tag{30b}
\end{align}

and the conversion function is
\begin{align}
h_3(x_{22}) &= 0.8x_{22} \tag{31}
\end{align}

with the squared penalty $r = 5.0$ and electricity and gas demand amounts $v_1 = 6.0$ and $v_2 = 4.0$, then set an upper constraint $u_2 \leq 15.0$ for concave function minimization. The concave cost functions can be often seen as rational examples in which marginal cost per unit energy supply decreases in accordance with the increase of the supplies. Applying the Lagrange multiplier $\varphi$ corresponding to the market price, we solve for the electricity supply amount $u_1$ and the gas supply amount $u_2$ by steepest descent while considering their maximum and minimum constraints, find the Nash equilibrium ($u_{1,2}^N(\varphi), u_{2,1}^N(\varphi), \varphi$), and compute the corresponding $(x_{11}^N(\varphi), y_{11}^N(\varphi))$ from the equality constraints (29c), (29f), and (29g). Figure 3 shows the values obtained in assessing $D(\varphi^p)$ of the objective function (29a) for the price $\varphi$ in the game decomposition price coordination problem. The Lagrange multiplier $\varphi$ is a single variable, and we find the trading price $\bar{\varphi} = 0.096$ as the optimum solution to the problem (29) by the linear search method, in which the Nash equilibrium solution in the problem (29b)–(29h) is $u_{1,2}^N(\bar{\varphi}) = 0.0, u_{2,1}^N(\bar{\varphi}) = 11.5$, the supply and demand in the market coincide, and the trade amount is thus $x_{11}^N(\bar{\varphi}) = y_{11}^N(\bar{\varphi}) = 6.0$. In this numerical example, the electricity demand happens to be met entirely from the gas side.

Let us compare this with application of the corresponding non-augmented Lagrangian method to the same case, by assuming a squared penalty value of $r = 0.0$. We similarly assess the value of $D(\varphi)$, the objective function in (29a), in the decomposition price coordination problem and plot it against the market price $\varphi$, as shown in Fig. 4. Based on this plot, we find...
Nash equilibrium solution

When the conversion function is nonlinear, with the numerical values use consider a case in which the objective function is convex but and the conversion function was linear. In the next example, let sidered a case in which the objective function was nonconvex

3.2.2 Case of nonlinear conversion function

In the above example of convexity non-fulfillment, we con-
considered a case in which the objective function was nonconvex
and the conversion function was linear. In the next example, let us consider a case in which the objective function is convex but the conversion function is nonlinear, with the numerical values

\[f_1(u_1) = u_1^2, \quad f_2(u_2) = 0.5u_2^2, \quad h_3(x_3) = 0.2(x_3)^2.\] (32a)

The constraint (25c) representing conversion from gas to elec-
tricity is nonlinear. In order to make optimization problems eas-
ily solvable, cost functions are often approximated by quadratic functions as Eq. (32). Nevertheless, approximation of a nonlin-
eral equality constraint by a quadratic function as Eq. (32) would change to non-convex type problems to which the Lagrangian method cannot be applied. The squared penalty \(r\) and the electricity and gas demands \(v_1, v_2\) are the same as in the previous example. Under application of the Lagrange multiplier \(\varphi\) corresponding to the market price in the same manner as in the previous section, Fig. 5 shows a plot of the \(D(\varphi)\) value for the objective function in Eq. (29a) in the decomposition price coordi-
nation problem against the market price.

3.3 Numerical Example of Dynamic Energy Flow with Storage Facility

In this section, let us consider a dynamic energy flow in which the supply-demand relation changes over time. For this purpose, we assume that over a discrete time interval \([1, \ldots, T]\) the electricity and gas demands are given as the time series \(v_1(t), t = 1, \ldots, T, v_2(t), t = 1, \ldots, T\). Without a storage fa-
cility in this flow, the optimization problem can be decomposed to a problem of static flow at each time in the discrete time interval; however, with a storage facility, decomposition to in-
dividual times is not possible, because past energy flow can be folded into the present flow by the storage facility. As an ex-
ample of this kind of energy flow, let us consider the flow in the network shown in Fig. 9 with a storage facility (a gas tank) located in the network at point 4 between point 2 and the gas
demand. We denote the amount in storage at point 4 at time \(t\) as \(x_3(t)\), which as indicated in Fig. 2 is obtained following the most recent gas conversion to electricity by the market function. The energy flow optimization problem with the assumed

\[x_{N}^T(\varphi) = y_{N}^T(\varphi) = 3.57.\] Figure 6 shows the market electricity demand and supply amounts \(x_{N}^T(\varphi), y_{N}^T(\varphi)\) versus the market price \(\varphi\). The higher demand against the market price in this case is apparently an effect of the penalty term in the augmented Lagrangian.

Figures 7 and 8 show plots obtained when we applied the corresponding non-augmented Lagrangian method by setting the squared penalty at \(r = 0.0\), in terms of the \(D(\varphi)\) value of the objective function in (29a) in the decomposition price coordi-
nation problem and the market electricity supply and demand amounts. As shown in Fig. 7, the minimum value of \(D(\varphi)\) oc-
curs at \(\varphi = 5.07\) and is thus non-zero, but the market electricity supply and demand amounts \(x_{N}^T(\varphi) = 3.47, y_{N}^T(\varphi) = 0.0\) do not coincide and thus no trade occurs.

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market is then formulated as

\[
\begin{align*}
\min & \sum_{t=1}^{T} \left[ f_1(u_1(t)) + f_2(u_2(t)) + f_3(x_3(t)) \right] \\
\text{subject to} & \quad x_3(t) = y_3(t), t = 1, \ldots, T \\
& \quad y_3(t) = b_3(x_3(t)), t = 1, \ldots, T \\
& \quad u_1(t) + x_3(t) = v_1(t), t = 1, \ldots, T \\
& \quad u_2(t) = x_3(t) + x_3(t), t = 1, \ldots, T \\
& \quad s_4(t) = x_4(t) - v_2(t), t = 1, \ldots, T \\
& \quad s_4(0) = s_4(T); \text{given} \\
& \quad s_4(t) \in S_4, t = 1, \ldots, T \\
& \quad (\text{all variables}) \geq 0 \\
& \quad v_1(t), v_2(t), t = 1, \ldots, T; \text{given}
\end{align*}
\]  

(34a)

where \( f_3(x_3) \) is the cost function of \( x_3 \) and each time series is a vector variable, for example, \( u_1 = (u_1(1), \ldots, u_1(T))^T \), \( x_3 = (x_3(1), \ldots, x_3(T))^T \), and \( s_4 = (s_4(1), \ldots, s_4(T))^T \). We composed the augmented Lagrangian by introducing Lagrange multiplier \( \varphi(t) \), \( t = 1, \ldots, T \), into the constraints (34b) in the problem (34) representing the market trading and incorporate this into the objective function in (34a) to compose the game decomposition price coordination problem using this, as

\[
\begin{align*}
\min & \quad D(\varphi) = \| x_3^N(\varphi) - y_3^N(\varphi) \|^2 \\
\text{subject to} & \quad M_1 \left( u_1^N(\varphi), x_3^N(\varphi), y_3^N(\varphi), \varphi, r \right) \\
& \quad \min_{(u_1,x_3)} \left\{ \sum_{t=1}^{T} f_1(u_1(t)) \right. \\
& \quad + \varphi^T x_3(t) + r \left\| x_3(t) - y_3(t) \right\|^2 \left. \right\} \\
& \quad \text{subject to Eq. (34d)} \\
& \quad (\text{all variables}) \geq 0 \\
& \quad v_1(t), t = 1, \ldots, T; \text{given} \\
& \quad M_2 \left( u_2^N(\varphi), y_3^N(\varphi), x_3^N(\varphi), \varphi, r \right) \\
& \quad \min_{(u_2,x_3)} \left\{ \sum_{t=1}^{T} f_2(u_2(t)) + f_3(x_3(t)) \right. \\
& \quad - \varphi^T y_3(t) + r \left\| x_3(t) - y_3(t) \right\|^2 \left. \right\} \\
& \quad \text{subject to Eqs. (34c) and (34e)-(34h)} \\
& \quad (\text{all variables}) \geq 0 \\
& \quad v_2(t), t = 1, \ldots, T; \text{given}
\end{align*}
\]  

(35a)

In this problem, we also organize the Lagrange multiplier term as the vector \( \varphi = (\varphi(1), \ldots, \varphi(T))^T \). In Eqs. (35b)–(35d), which represent the electricity-side problem, no storage facility is involved. In the objective function in Eq. (35b), \( \varphi^T x_3(t) = \sum_{t=1}^{T} \varphi(t)x_3(t) \) and \( \| x_3(t) - y_3(t) \|^2 = \sum_{t=1}^{T} (x_3(t) - y_3(t))^2 \), and we thus have a problem amenable to decomposition at each time. Therefore, we added this as a number sequence to the functions of Eqs. (32) and (33) as

\[
f_3(x_3) = 0.01 x_3^2
\]

(36)

satisfying \( s_4 = \{ s_4 \mid s_4 \geq 1 \} \) and \( s_4(0) = s_4(T) = 5 \). In the time series, as shown in Fig. 10, in time interval \( T = 24 \), the electricity and gas demands \( v_1 \) and \( v_2 \) were assumed to peak in midday for electricity and in midmorning and evening for gas. For the number series, to find the Nash equilibrium solution of problem (35b)–(35i) for \( \varphi \), after eliminating dependent variables \( x_3 \) and \( y_3 \) using the equality constraint, we used the steepest descent method in the total dependent-variaule \( (u_1, u_2, s_4) \) 72-dimensional interval, and since \( \varphi \) is also a 24-dimensional variable in the time series, for updating market-price \( \varphi \) that minimizes the objective function in (35a), we therefore applied the computing procedure by PSO given in Section 2.2. With \( P = 25 \) as the number of search points in PSO, we generated the initial value using uniform random numbers in the range \( 0 \leq \varphi \leq 5 \) and, in order to terminate the computation, took \( K = 1000 \) as the maximum number of iterations. The algorithm was a basic simple-PSO algorithm with the updating formula (24). Here, a coefficient \( \lambda \) is set as \( \lambda = 0.73 \) (constant), and coefficients \( c_1, c_2 \) are set equal \( (c_1 = c_2) \) and randomized uniformly within \([0,1.4995]\), which are recommended values in the simple PSO. Considering the PSO is one of random search methods, we perform 100 trials with resetting the initial searching points. Then it is confirmed that convergences of the g-best points to the optimal market prices \( \varphi = \frac{e^{-b(e)}}{(K)} \) where the objective function values of Eq. (35a) are zeros in all trials. Figure 11 shows the time series of the market price \( \varphi = \frac{e^{-b(e)}}{(K)} \) at the optimum search points at the end of the PSO computation, and Fig. 12 shows the time series of the Nash equilibrium solution \( (u_1^N(\varphi), u_2^N(\varphi), s_4^N(\varphi)) \) for the corresponding game decomposition problem (35b)–(35i). In the time series, as also shown in Fig. 11, the market electricity demand and supply amounts are determined by these coincide as \( x_3^N(\varphi) = y_3^N(\varphi) \). As indicated by these results, preparation is performed for the peaks in electricity and gas demand by gradually increasing the amount stored in the storage facility from late night to early morning and then using it in correspondence with the change in the electricity demand together with the highly weighted electricity cost function, which holds down the electricity supply amount and its fluctuation. In concert with this, the gas in storage is appropriately drawn down, converted to electricity, and increased in price, and the gas supply is gradually increased to meet high evening demand and then to build the storage amount to the required terminal level in preparation for the next day. For this reason, as shown in the figures, the market price remained high in the trades through late afternoon even as the trade volumes via the market gradually decrease in correspondence with the decrease in electricity demand.

4. Conclusion

In this study, we have constructed a network model representing an energy flow comprising different energy forms and their interchange in a converter and incorporating a market function that performs price coordination in the conversion segment, and proposed a distributional processing system in which an agent performs operations management for distribution that optimizes the flow of each energy form based on that price coordination. Since Lagrangian methods generally cannot be applied to price coordination for optimization problems involving nonconvex objective functions or nonlinear equality constraint functions, we propose a distributional solution method incorporating a new price coordination function in an augmented Lagrangian method that is effective for nonconvex optimization problems and thus eliminates the convexity prerequisite. The
augmented Lagrangian includes a squared penalty term that precludes decomposition into sub-problems free from variable interdependence, so we also propose a new decomposition solution method in which sub-problems containing variables inducing interdependence are treated as a game problem with the goal of finding a Nash equilibrium solution. The effectiveness of this decomposition method was assessed and verified in energy flow optimization problems for flows involving nonconvex objective functions and nonlinear functions for conversion between different energy forms.

Figure 13 as a reference shows changes of the objective function value $D(\varphi)$ (Eq. (35a)) in the problem (35) in response to the market price $\varphi$ in the PSO’s iteration process. The average objective value of the g-best point at each iteration in 100 trials is plotted, and the value reaches $9.12 \times 10^{-7}$ at an iteration number $k = 300$. From the results, it is certified that the PSO is effective to decisions of the optimal market prices and the corresponding optimal trading energy flows in the distributed optimization for energy flow problems.

References


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