New Method of Finite Element Sensitivity Analysis for Coupled Problems and Its Application to Structural Optimization Technology*

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Abstract

This paper describes a newly developed optimization system based on a new Finite Element Sensitivity Analysis (FESA) of mechanical structures with coupled problems concerning heat conduction and stress analysis. In this system, variation in a material property, for example heat conductivity, is considered as a structural change because changing a material property locally or partially is difficult. Like FESA, this system requires just one calculation and then offers efficient designing products. The effectiveness of the system was verified by results showing that optimizations of structural design were given efficiently when the system was applied to the design of actual industry products: a turbine blade and a cooling jacket of computer modules.

Key words: Finite Element Sensitivity Analysis, Stress, Coupled Problem, Perturbation Method

1. Introduction

In general, industrial commodity are required to offer high performance and to be operated at maximum efficiency. Therefore, developing a method of determining design parameters that satisfy design requirements exactly is an important subject in the design process. The Finite Element Method (FEM)(1) is one of the essential analytical tools used to identify these demanding requirements. Typically, the FEM helps in building an object model that has the desired design parameters such as shape, dimensions, material factor, external force, and support conditions. However, in the development process, several design parameters are changed to yield optimum values. Therefore, evaluating the impact on the overall solution is important when these parameters are changed.

The parameter survey based on the FEM or Finite Element Sensitivity Analysis (FESA)(2-8) is the most common method for evaluating the impact resulting from changed parameters. However, the parameter survey could increase the number of man-hours in the design process because the parameter survey is an iterative calculation with various design parameters that are determined by the designer. Such a survey is not efficient because there is difficulty in identifying which and to what extent design parameters need to be changed to satisfy design requirements. On the other hand, FESA is more efficient than the
A parameter survey based on the FEM because the calculation is simpler. After creating a
stiffness matrix that incorporates the variation in design parameters, FESA requires only a
one-time calculation based on the perturbation technique. In addition, FESA simultaneously
generates defined solutions for the displacement, strain, stress, and their sensitivities, which
can be utilized to identify the impact on the solution resulting from the variation in each
design parameter.

Hence, the impact of changed parameters is efficiently evaluated by FESA during the
design process. However, FESA is only formulated for individual physical phenomena. In
actual structures, various physical phenomena are coupled with each other. Therefore,
developing a new method of FESA that incorporates coupling phenomena is necessary. In
this research, based on the existing FESA\cite{2,3}, which has been studied by Nakakiri et al. in
terms of individual physical impacts, a new method of FESA of the coupling problem is
formulated. Subsequently, using the newly formulated FESA, a system for evaluating the
impact of design parameters was developed. The effectiveness of the system was verified
using configuration designs of a gas turbine blade and a multi-chip module of a large-scale
computer.

2. Formulation of FESA for Coupling Problem

In this section, a description of a new FESA is given. The new FESA is formulated for a
coupling problem that is characterized by various physical phenomena such as the problem
of heat conduction and thermal stress.

2.1 Formula for Heat Conduction and Thermal Stress

When solving the equation of heat conduction using the FEM, the heat conduction
equation of the element can be expressed as:

\[
[k_q] \{t\} = \{q\}
\]  

(1)

where \([k_q]\) is the heat conduction matrix that includes the heat conductivity coefficient \(\lambda\), and \([t]\) and \([q]\) are temperature and heat flux vectors, respectively.

On the other hand, the element stiffness matrix of the FEM for the thermal stress
analysis that includes the heat-load can be expressed as:

\[
[k_E] \{u\} = \{f\} + \{f'\}
\]  

(2)

where \([k_E]\) is the element stiffness matrix that includes Young's modulus \(E\), \([u]\) and
\([f]\) are the displacement and load vector, respectively, and \([f']\) is the heat-load vector
yielded by the heat strain.

2.2 Description of Uncertain Variable

If material constants \(M_{xi}\) and \(M_{yi}\) (\(i = 1\sim n\): \(n\) is the number of uncertain variables of
material), which represent anisotropic materials in the x- and y- directions, are
given as design parameters that influence both heat conduction and thermal stress, the
variation in these variables is assumed to be expressed by the following
equations in FESA:

\[
M_{si} = \bar{M}_{si}(1 + \xi_i)
\]  

(3)

\[
M_{ji} = \bar{M}_{ji}(1 + \zeta_j)
\]  

(4)
where $\bar{M}_{xi}$ and $\bar{M}_{yi}$ are definite values and $\xi_i$ and $\zeta_i$ are infinitesimal uncertain variables.

2.3 Variation in Heat Conduction and Stiffness Matrix

If values of uncertain variables that are expressed in Eqs. (3) and (4) fluctuate, their heat conductivity and the stiffness matrix that contains these variables would also fluctuate. Here, these matrices are called material matrices $[k(M)]$. Then, given that the external force terms $\{g\}$ and $\{f\}$ do not fluctuate, the variation in $[k(M)]$ is given by Taylor expansion with $\xi_i$ and $\zeta_i$, and the following equation is derived after eliminating the first term. (10) to (11)

$$[k(M)] = [\bar{k}(M)] + \sum_{i=1}^{n} ([k(M)_{xi}^i] \xi_i + [k(M)_{yi}^i] \zeta_i)$$

(5)

In Eq. (5), $[k(M)_{xi}^i]$ and $[k(M)_{yi}^i]$ are the first-order sensitivities of the material matrix in the $x$- and $y$- directions, respectively, and they are expressed as:

$$[k(M)_{xi}^i] = \frac{\partial[k(M)_{xi}]}{\partial \xi_i} \bigg|_{\xi_1=\xi_2=\ldots=\xi_n=0}$$

(6)

$$[k(M)_{yi}^i] = \frac{\partial[k(M)_{yi}]}{\partial \zeta_i} \bigg|_{\xi_1=\xi_2=\ldots=\xi_n=0}$$

(7)

Hereafter, the subscript $I$ refers to first-order sensitivity.

2.4 Variation of Temperature Vector

If the material constant that is associated with thermal conduction fluctuates, the heat conduction matrix will also fluctuate, as expressed in the material matrix Eq. (5) as well as the temperature vector $\{t\}$. Therefore, $\{t\}$ can be expressed in a similar manner:

$$\{t\} = \{\bar{t}\} + \sum_{i=1}^{n} \left( [t_{xi}^i] \xi_i + [t_{yi}^i] \zeta_i \right)$$

(8)

As stated above, the heat flux vector $\{g\}$ does not fluctuate in this assumption. Therefore, by substituting Eqs. (5) and (8) into Eq. (1) and then applying the perturbation method, the definite values of temperature $\{t\}$ and sensitivities $\{t(M)_{xi}^i\}$ and $\{t(M)_{yi}^i\}$ are given as follows:

$$\{\bar{t}\} = \left[\bar{k}_{xx}\right]^{-1} \{g\}$$

(9)

$$\{t_{xi}^i\} = -\left[\bar{k}_{xx}\right]^{-1} \left[ k_{xx} \right] \{\bar{t}\}$$

(10)

$$\{t_{yi}^i\} = -\left[\bar{k}_{xx}\right]^{-1} \left[ k_{yy} \right] \{\bar{t}\}$$

(11)

2.5 Variation of Displacement Vector

Likewise, by assuming that the heat load vector $\{f\}$ fluctuates due to the fluctuations of $M_{xi}^i$ and $M_{yi}^i$, the following equation is given.
\[
\{f'\} = \{f\} + \sum_{i=1}^{n} (\{f_i'\} + \{f_{i}''\}) \zeta_i 
\]  
(12)

The displacement vector \(\{u\}\) can also be expressed as follows:

\[
\{u\} = \{u\} + \sum_{i=1}^{n} (\{u_i'\} + \{u_{i}''\}) \zeta_i 
\]  
(13)

As above, the load vector \(\{f\}\) does not fluctuate in this assumption. Therefore, by substituting Eqs. (12) and (13) into Eq. (2) and then applying the perturbation method, the definite values of the displacement and the sensitivities are given by the following series of equations:

\[
[k_e]\{u\} = \{f\} + \{f'\} 
\]  
(14)

\[
[k_e]\{u_i'\} = \{f_i''\} 
\]  
(15)

\[
[k_e]\{u_{i}''\} = \{f_{i}'''\} 
\]  
(16)

### 2.6 Formulation of Heat Load Vector for Coupling Problem

With respect to heat conduction, temperature vector \(\{\xi\}\) that is given in Section 2.4 influences the heat-load vector \(\{f'\}\) of the thermal stress problem. Therefore, \(\{f'\}\) fluctuates with respect to variation in both \(M_{ii}\) and \(M_{ij}\). This section refers to the formulation of the variation in \(\{f'\}\).

In the elastic problem, the heat-load vector is given by the following equation:

\[
\{f'\} = \int_{V} [B]^T[D]\{\varepsilon'\}dV 
\]  
(17)

where \([D]\) is the stress-strain matrix, \([B]\) is the strain-nodal displacement matrix, \(\{\varepsilon'\}\) is the heat-strain vector, \(T\) is transposed, and \(V\) is the volume of the element.

In the case of the plane stress, the stress-strain matrix \([D]\) of the two-dimensional surface can be expressed as follows:

\[
[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} 
\]  
(18)

On the other hand, in the case of the plane strain, \([D]\) can be expressed as follows:

\[
[D] = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} 
\]  
(19)

where \(\nu\) is Poisson's ratio. In the case of the plane stress, the heat-strain vector \(\{\varepsilon'\}\) with respect to anisotropic material can be expressed as follows:
\[
\varepsilon' = \begin{bmatrix}
\varepsilon'_x \\
\varepsilon'_y \\
\varepsilon'_z 
\end{bmatrix} = \begin{bmatrix}
\alpha t \\
0 \\
0 
\end{bmatrix}
\]

(20)

On the other hand, in the case of the plane strain, \( \varepsilon' \) can be expressed as follows:

\[
\varepsilon' = \begin{bmatrix}
\frac{(1 + \nu)\alpha t}{1 + \nu} \\
\frac{(1 + \nu)\alpha t}{1 + \nu} \\
0 
\end{bmatrix}
\]

(21)

where \( \alpha \) is the linear expansion coefficient.

The strain-nodal displacement matrix \([B]\) can be expressed by using the shape function matrix with the local coordinate system. Therefore, in the same way as that of Eq. (13), the variation in the heat-strain matrix \( \varepsilon' \) with respect to variation in \( M_{xi} \) and \( M_{yi} \) can be obtained as follows:

\[
\varepsilon' = \varepsilon' + \sum_{i=1}^{n} (\varepsilon'_{xi}) \xi_i + (\varepsilon'_{yi}) \eta_i
\]

(22)

In addition, by substituting Eq. (22) into Eq. (17), the heat-load vector can be expressed as follows:

\[
\{ f' \} = \int_V [B]^T [\overline{D}] \varepsilon' dV + \sum_{i=1}^{n} \int_V ([B]^T [D_{xi}'] \varepsilon' + [B]^T [\overline{D}] \varepsilon'_{xi}) \xi_i + [B]^T [D_{yi}'] \varepsilon'_{yi} \xi_i + [B]^T [\overline{D}] \varepsilon'_{yi} \eta_i dV
\]

(23)

The first term of the right-hand side of Eq. (23) is a definite value, and the remaining terms are sensitivities. Thus, by comparing Eqs. (23) and (12), the sensitivities \( \{ f'_{xi} \} \), \( \{ f'_{yi} \} \), and \( \{ f'_{xi} \} \) of the heat-load vector are given.

Likewise, the following equation is obtained by Taylor-expanding the stress variation from the variation in \( M_{xi} \) and \( M_{yi} \) by \( \xi_i \) and \( \eta_i \) and then by eliminating the first term.

\[
\{ \sigma \} = \overline{\sigma} + \sum_{i=1}^{n} (\sigma_{xi}') \xi_i + (\sigma_{yi}') \eta_i
\]

(24)

The definite values \( \{ \overline{\sigma} \} \) of stress, and sensitivities \( \{ \sigma_{xi}' \} \) and \( \{ \sigma_{yi}' \} \) are given by using Eqs. (24) and (14) and based on the relational equation of stress-strain as follows:

\[
\{ \overline{\sigma} \} = [\overline{D}] [B] \{ \overline{u} \}
\]

(25)

\[
\{ \sigma_{xi}' \} = [\overline{D}] [B] \{ u_{xi}' \} + [D_{xi}'] [B] \{ \overline{u} \}
\]

(26)

\[
\{ \sigma_{yi}' \} = [\overline{D}] [B] \{ u_{yi}' \} + [D_{yi}'] [B] \{ \overline{u} \}
\]

(27)

2.7 Benefit of FESA for Coupling Problems

The equations described above were formulated with respect to an element, but later, by considering the boundary condition as used in the conventional FEM and by superposing the boundary condition, an equation for the overall system can be obtained.
Then, by applying the inverse matrix calculation only once definite values of temperature and displacement and sensitivity can be obtained. More specifically, by using the perturbation method, calculating the definite value and the variation in the neighborhood of the definite value of a structural object that contains uncertain variables in a short period of time became possible. This is particularly beneficial for the coupling analysis that requires the calculation of more than two sets of equations.

In addition, by including anisotropy in the design parameter of the material constant, identifying the impact on the solution caused by stress or other factors due to the variation in a material constant whose shape was also included in the design parameters becomes possible. For example, if sensitivities that have an impact on x- and y-directions of the heat conduction have negative values, the thermal stress value can be decreased by increasing the rate of heat conduction. On the other hand, if sensitivities that have an impact on x- and y-directions of the heat conduction have positive values, the thermal stress value can be decreased by decreasing the rate of heat conduction.

Fig. 1 Design system

3. Proposal of Impact Evaluation System with Respect to Design Parameter Variation Using FESA

A design system was developed to determine guidelines for the structure optimization by evaluating the impact of various design parameters, as illustrated in Fig. 1.

First, design parameters for a standard model were defined, and Eqs. (15) and (16) of FESA were used to obtain the defined solution with respect to variation in the initial design parameters and the sensitivity coefficient. Then, with respect to all design parameters in which the variation was taken into consideration, the defined solutions and sensitivities of all nodes were obtained. As illustrated in Block A of Fig. 1, a representative function was added to the system to immediately display Node a, where the design tolerance was not satisfied according to the defined solution with respect to all nodes, and to display Parameter b, which had the largest impact on the defined solution at Node a. Using this system, the designer determines which design parameters to modify, instead of being overwhelmed by the enormous quantity of data.

In addition, although only the design parameter that had the most impact was extracted in Block A, by using the sensitivity coefficient obtained in FESA, the displacement vector \( \{u\} \) of the neighborhood solution of Node a was obtained by using Eq. (14).

To facilitate the process, as shown in Block A of Fig. 1, likewise, another function of
the graph display was added to this system. Variation in the solution at Node \( a \) caused by variation in design parameter \( b \) is displayed. In addition, a graph is displayed that illustrates whether the solution at Node \( a \) had satisfied the design requirement and if the parameters could be designed.

Moreover, to determine the optimum design parameter value, sensitivities were summed (in descending order of sensitivity) until the sum became equal to or greater than the base value that was a result of subtracting the expected design tolerance \( \sigma \) from the definite value \( \sigma_0 \) obtained from Eq. (25). Equation (28) describes this process, and the design parameter that was selected to satisfy Eq. (28) will change.

\[
\bar{\sigma}_{a} - \sigma_{\text{design},a} \leq \sum_{i=1}^{n} \left( \sigma_{x} \xi_{i} + \sigma_{y} \xi_{i} \right) \geq \sigma_{0} - \sigma
\]  

(28)

For example, clearly, if the variation variable satisfies Eq. (28) when \( \xi_{i} = X \) and \( \xi_{i} = X \), the design parameter that was selected in Eqs. (3) and (4) needs to be multiplied by \((1 + X)\). In other words, the value was decreased to the targeted thermal stress value by multiplying the material constant by \((1 + X)\) in the x- and y- directions of the element of the design parameter selected in the same manner as that of Eqs. (3) and (4). However, changing the material constant in an actual design was difficult. Therefore, identifying the relationship between the material constant and the shape is necessary. For example, the heat-conductivity coefficient \( \lambda \) is expressed as follows:

\[
\lambda = \frac{Q}{St(T_{1} - T_{2})} \frac{L}{S}
\]  

(29)

where \( S \) is the cross-sectional area, \( L \) is thickness, and \( T_{1} \) and \( T_{2} \) are temperatures on the plane.

Equation (29) indicates that the heat conductivity coefficient \( \lambda \) of each element is inversely proportional to the area of the element in the two-dimensional FEM. Therefore, multiplying the heat conductivity coefficient by \((1 + X)\) is equivalent to multiplying the area of the element by \( S/(1 + X)\). Given this finding, the element of heat conduction in the x- and y- directions of the selected design parameter was converted to the area of the element in the x- and y- directions, and the shape also changed.

The above serves to summarize this system for evaluating the impact of design parameter variation. The optimum design parameter was determined in only one analysis by obtaining the sensitivity coefficient using FESA. Guideline principles for structure optimization \((13)-(15)\) can be easily obtained by using this system, which is equipped with the result-display function.

4. Calculation Example

As described in the previous section, based on the FESA formula stated with respect to the problem of coupling heat conduction and thermal stress, the system for evaluating the impact of variation in design parameters was developed. This section describes the result of verifying the effectiveness of this system for the structure optimization by using actual examples of a gas turbine blade and a multi-chip module for a large-scale computer.

4.1 Structure Optimization of Gas Turbine Blade

Developing the life design of a gas turbine blade from the aspects of both cooling design and strength design is important because the peripheral part of a gas turbine blade is exposed to high-temperature combustion gas. A two-dimensional cross-section diagram of a typical gas turbine blade is illustrated in Fig. 2. As shown in Fig. 3, high-thermal stress is yielded locally because of the difference in temperature between the inside and outside
surface area due to the high-temperature gas and internal cooling.

In this verification, the design of the blade was modified according to the temperature and sensitivities of stress with respect to the variation in the heat conduction in the $x$- and $y$-directions. This was performed to optimize the blade structure by using per-element heat conduction in the material constants of the $x$- and $y$-directions as a design parameter.

To be more specific, the impact of the temperature on the variation in the per-element heat conduction in the $x$- and $y$-directions was obtained from Eqs. (10) and (11). Then, the obtained temperature sensitivities were used in Eqs. (15) and (16) to obtain displacement sensitivities. Subsequently, stress sensitivities were obtained from Eqs. (26) and (27). Based on these results, stress sensitivities that were related to variation in per-element heat conduction at Node $a$ (Fig. 2) where the highest thermal stress was yielded (Fig. 4) were displayed in Block $A$ (Fig. 1). The element of the design parameter with the largest stress sensitivity is the peripheral element on the cooling hole of the blade. Thus, determining that the stress at Node $a$ is sensitive to the heat conduction of a certain element was easy.

The optimum design parameter was obtained by solving Eq. (28) to decrease the stress design tolerance on Node $a$ to 0.7 (dimensionless stress value). The variation in the identified design parameter was converted to the per-element area from the heat conduction, and the peripheral shape of the blade hole was determined by changing the area of the element in the $x$- and $y$-direction. A two-dimensional cross-section diagram of the blade after the shape was modified is illustrated in Fig. 5. The result of re-analyzing the modified blade shape is shown in Fig. 6. In this figure, the relationship between stress and temperature is shown. The modified blade satisfies the design tolerance, and the maximum stress was decreased by 20%.

![Fig. 2 2D model of turbine blade (Initial configuration)](image)

![Fig. 3 Distribution of stress vs. temperature (Initial configuration)](image)
Thus, this method requires only a one-time analysis to identify the proper direction for the shape of a cooling hole. The cross-sectional shape of the blade can be determined in a short period of time without the analysis being overwhelmed by a time-consuming parameter survey.

4.2 Structure Optimization of Multi-Chip Module

In another application, this system was used to optimize the structure of a multi-chip module for a large-scale computer in which the density of connections on the chip is increasing because of increased integration. The reliability of solder connections inside a multi-chip module was highly dependent on variation in design parameters, such as the heat conductivity coefficient, shape, material constant, and boundary conditions of the cooling
component. The impact of the variation in the design parameter that influenced the solution was evaluated. This was performed by obtaining the temperature sensitivity and stress sensitivity with respect to variation in the design parameter of the module. The module shape was modified according to the result.

An analytical model of the subject module is shown in Fig. 7. This model had the coupling problem where both temperature and stress fluctuated due to the variation in thickness $t$ of a cooling jacket base. Determine boundary conditions of this model an large-scale integrated chip (LSI) was heated, and water pressure was exerted on the jacket base by means of a water-cooling system. As shown in Fig. 7, this water-cooling system was very effective in terms of cooling performance because the jacket and chip (the heat source) were directly connected. However, a high solder strain on the controlled collapse bonding (CCB) connection (Point $P$) was caused due to thermal deformation as a result of the internal temperature gradient and deformation due to the water pressure.

Identifying which deformation (internal temperature gradient or water pressure) had a greater impact on the deformation of the CCB connection was necessary to reduce the strain on the connection while increasing the cooling performance.

First, by referring to Block $A$ (Fig. 1), the sensitivity of the component with respect to the variation in heat conduction, Young's modulus, and thickness of the component were obtained. Based on the level of impact, a design variable was determined. However, in this scenario, the strain on the CCB connection was minimized under the condition that the design variable was the thickness $t$ of the jacket because modifying the design was easier, and the manufacturing limit of the sheet thickness was 1.2 mm or greater.

Block $B$ (Fig. 1) was used to calculate the variation in the valuation strain that is yielded by the variation in the design variable. A graph of the first-order approximation solution and the re-analyzed solution of the variation in strain vs. the variation in jacket base thickness is shown in Fig. 8. This data is based on the jacket base (Fig. 7) with the definite value of 2 mm. As shown in Fig. 8, the strain decreases when the uncertain variable becomes negative (smaller thickness) because the variation in strain vs. variation in the jacket thickness $t$ is a positive value. This means that the impact of the heat variation was

![Fig. 7 2D model of large computer module](image)

![Fig. 8 Variation in strain at node p vs. variation in jacket thickness](image)
more influential than the impact of pressure variation. Hence, the optimum thickness \( t \) of the jacket was 1.2 mm, and the solder strain on the CCB connection (Point \( P \)) was reduced by 16%.

The application of the system to designing the shape of a gas turbine blade and a multi-chip module of a large-scale computer demonstrated that the system can efficiently optimize the structure.

5. Conclusion

In this research, in order to realize structure optimization by considering the impact of design parameter variation, a new FESA equation was formalized with respect to the coupling problem between the heat conduction and the thermal stress. Subsequently, the variation impact system of the design parameter that is based on the FESA was introduced, and through evaluation using shape designs of a gas turbine blade and a multi-chip module for a large-scale computer, the structure was efficiently optimized in a short period of time. Thus, the effectiveness of this system was verified.

In the future, by incorporating the variation of the design parameters into the optimum design process for the coupling problem, this system can be further developed for the optimization-based design with respect to reliability.

References