Computational Modeling of Superelastic Behaviors of Shape Memory Alloy Devices Under Combined Stresses*

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Abstract

The three-dimensional incremental finite element formulation for the multiaxial behavior of shape memory alloy devices is proposed in the present study by considering the coupling effect of the axial and torsional behaviors of shape memory alloys. The previously proposed one-dimensional constitutive model for shape memory alloy devices is extended to take account of the multiaxial stress state introducing some new material constants. The calculated results are compared with the uniaxial, purely torsional and multiaxial test results for NiTi tubes to illustrate the validity of the proposed computational modeling.

Key words: Combined Stress, Computational Mechanics, Constitutive Equation, Finite Element Method, Shape Memory Materials

1. Introduction

The shape memory alloy (abbreviated as SMA) devices have been practically applied in various fields such as mechanical and medical engineering. The superelasticity (shape recovery by the unloading) as well as the shape memory effect (shape recovery by the heating) is utilized in the applications of the SMAs. The SMAs have a high energy absorbing capacity as they store the deformation energy by the phase transformation. The SMAs have also been developed as damping materials. The development of a computational tool to support the design process is necessary for the efficient development of the SMA devices with a complicated shape and mechanical characteristics.

As a research for the constitutive equations of the SMAs and their applications to the finite element analysis, Brinson formulated a one-dimensional constitutive equation and applied it to the finite element analysis\(^{(1)}\)\(^{(2)}\). Kawai et al.\(^{(3)}\), Trochu and Qian\(^{(4)}\), Auricchio and Taylor\(^{(5)}\), Keefe et al.\(^{(6)}\), Tokuda and Sittner\(^{(7)}\), Qidwai and Lagoudas\(^{(8)}\) formulated the constitutive equations of the SMAs, some of which have been applied to the finite element analysis. However, the standard computational procedure has not yet been established.

The basic constitutive equation model in the present study is the one-dimensional constitutive equation\(^{(9)}\) which is an extension of Brinson’s modeling to take account of the asymmetric tensile and compressive deformation. The gradients of the critical stresses for starting and finishing martensite and austenite transformations with respect to the temperature are independently assumed to have a better correspondence of the superelastic behavior of the SMAs to the material test result. The evolution equation for the martensite volume fraction is expressed in terms of Drucker-Prager equivalent stress. The one-dimensional constitutive equation is extended to the multi-axial modeling by coupling the normal and the shear deformation. The incremental finite element procedure is
formulated, based on the tangential stiffness method. The validity of the proposed constitutive
modeling is studied by comparing the calculated results with the experimental results(10)-(12).

The multiaxial constitutive equation of the SMAs and the incremental finite element procedure
are formulated in the section 2 and the section 3, respectively. The calculated results are compared
with the experimental results in the section 4. The last section 5 is the concluding remarks.

2. Multiaxial Constitutive Equation of the SMAs

The mechanical property of the SMAs discussed in the present study is schematically shown in
Fig. 1. Figure 1(a) shows the superelastic behavior (bold line arrow) and the shape memory effect
dotted line arrow), while Fig. 1(b) is the relations between the critical transformation stresses and the
temperature. The following notations are used in Fig. 1: \( \sigma \); the stress, \( \varepsilon \); the strain,
\( T \); the temperature, \( \sigma_{f}^{cr} \) and \( \sigma_{s}^{cr} \); the critical stresses for finishing and starting martensite transformation,
\( M_{f} \) and \( M_{s} \); the temperatures for finishing and starting martensite transformation, \( A_{s} \) and \( A_{f} \);
the temperatures for starting and finishing austenite transformation. The superelastic behavior as
shown in Fig. 1(a) occurs, when the stress loading and unloading takes place at the constant
temperature higher than \( A_{f} \) in Fig. 1(b).

\( C_{M_{s}} \) and \( C_{M_{f}} \) are the gradients of the critical stresses for starting and finishing martensite
transformation with respect to the temperature, while \( C_{A_{s}} \) and \( C_{A_{f}} \) are the gradients of the critical
stresses for starting and finishing austenite transformation with respect to the temperature. The
following assumptions are assumed in Brinson’s modeling(1):

\[ C_{M_{s}} = C_{M_{f}} \quad \text{and} \quad C_{A_{s}} = C_{A_{f}} \]  \hspace{1cm} (1)

In the present formulation, all the gradients are assumed to be independent material constants
as follows:

\[ C_{M_{s}} \neq C_{M_{f}} \]  \hspace{1cm} (2)

\[ C_{A_{s}} \neq C_{A_{f}} \]  \hspace{1cm} (3)

The formal extension of the one-dimensional stress-strain relation for the SMAs leads to the
three-dimensional modeling as given by the following equation:

\[ \{ \sigma \} = [D] \{ \varepsilon \} + \{ \varepsilon_{s} \} \{ \Omega \} + T \{ \theta \} \]  \hspace{1cm} (4)

\[ \{ \sigma \}^{T} = \begin{bmatrix} \sigma_{x} & \sigma_{y} & \sigma_{z} & \tau_{xy} & \tau_{yz} & \tau_{zx} \end{bmatrix} \]  \hspace{1cm} (5)

\[ \{ \varepsilon \}^{T} = \begin{bmatrix} \varepsilon_{x} & \varepsilon_{y} & \varepsilon_{z} & \gamma_{xy} & \gamma_{yz} & \gamma_{zx} \end{bmatrix} \]  \hspace{1cm} (6)

(a) Superelastic behavior and shape memory effect

(b) Critical stresses for transformation vs. temperature

Fig. 1  Mechanical properties of shape memory alloys
where the following notations are used: \( \{\sigma\} \); the stress vector, \( \{\varepsilon\} \); the strain vector, \([D]\); the stress-strain matrix, \(\{\Omega\}\); the transformation tensor, \(\xi_s\); the stress-induced martensite volume fraction, \(\{\theta\}\); the thermal elastic coefficient and \(T\); the temperature.

The stress-strain matrix \([D]\) is given by the following equation:

\[
[D] = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix}
1 - \nu & \nu & 0 & 0 & 0 \\
\nu & 1 - \nu & 0 & 0 & 0 \\
\nu & \nu & 1 - \nu & 0 & 0 \\
0 & 0 & 0 & \frac{1 - 2\nu}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{1 - 2\nu}{2}
\end{bmatrix}
\] (7)

where \(E\) and \(\nu\) are Young’s modulus and Poisson’s ratio, respectively.

Young’s modulus \(E\) is expressed by the following equation as a function of the total martensite volume fraction \(\xi\):

\[
E = E_a + \xi (E_m - E_a)
\] (8)

where \(E_m\) and \(E_a\) are Young’s modulus of the martensite phase and the austenite phase, respectively.

Similarly, the shear modulus \(G\) is also expressed by the following equation as a function of the total martensite volume fraction \(\xi\):

\[
G = G_a + \xi (G_m - G_a)
\] (9)

where \(G_m\) and \(G_a\) are the shear modulus of the martensite phase and the austenite phase, respectively. The following relations hold between Young’s modulus \(E\), \(E_m\), \(E_a\) and the shear modulus \(G\), \(G_m\), \(G_a\).

\[
G = \frac{E}{2(1 + \nu)}, \quad G_m = \frac{E_m}{2(1 + \nu)}, \quad G_a = \frac{E_a}{2(1 + \nu)}
\] (10)

The total martensite volume fraction \(\xi\) is expressed as follows:

\[
\xi = \xi_s + \xi_f
\] (11)

where \(\xi_f\) is the temperature-induced martensite (twinned martensite) volume fraction. \(\xi_s\) and \(\xi_f\) are all functions of the temperature and the stresses.

The transformation tensor \(\{\Omega\}\) is expressed as follows, using the maximum residual strain vector \(\{\varepsilon_L\}\) and the residual strain direction tensor \(\{R_s\}\):

\[
\{\Omega\} = -[D]\{R_s\}\{\varepsilon_L\}
\] (12)

The maximum residual strain vector \(\{\varepsilon_L\}\) is expressed as follows:

\[
\{\varepsilon_L\} = \{\xi_L, \varepsilon_L, \gamma_L, \gamma_L\}
\] (13)

where \(\varepsilon_L\) and \(\gamma_L\) are the maximum residual normal strain and the maximum residual shear strain, respectively.

The residual strain direction tensor \(\{R_s\}\) is given by the following equation expressed in terms of the stress components:
\[
\frac{\partial \sigma_{eq}}{\partial \sigma_y} 0 0 0 0 0 \\
0 \frac{\partial \sigma_{eq}}{\partial \sigma_x} 0 0 0 0 \\
0 0 \frac{\partial \sigma_{eq}}{\partial \tau_{xy}} 0 0 0 \\
0 0 0 \frac{\partial \sigma_{eq}}{\partial \tau_{yz}} 0 \\
0 0 0 0 0 \frac{\partial \sigma_{eq}}{\partial \tau_{zs}} \\
\]  
(14)

where \( \sigma_{eq} \) is von Mises equivalent stress.

The thermo-elastic coefficient vector \( \{ \theta \} \) is expressed as follows:

\[
\{ \theta \} = [D] \{ \alpha \} 
\]
(15)

where the thermal expansion coefficient vector \( \{ \alpha \} \) is given as follows:

\[
\{ \alpha \}^T = [\alpha \ \alpha \ \alpha \ 0 \ 0 \ 0] 
\]
(16)

The following Drucker-Prager equivalent stress is used instead of von Mises equivalent stress in the evolution equations for \( \xi, \xi_S \) and \( \xi_T \) in order to consider the asymmetric tensile and compressive behaviors:

\[
\sigma_{DP} = \sigma_{eq} + \beta (\sigma_x + \sigma_y + \sigma_z) 
\]
(17)

where \( \beta \) is a material constant. In the present calculations, \( \beta = 0.15 \) is assumed according to Auricchio and Taylor(5).

The following evolution equations for the martensite transformation process and the inverse austenite transformation process can be obtained by substituting Eqs. (2), (3) and (17) into the existing evolution equations(1) for \( \xi, \xi_S \) and \( \xi_T \):

(i) Transformation process to the martensite phase

- In the case when \( T > M_s \) and \( \sigma_{\sigma'} (1 + \beta) + C_{M_s} (1 + \beta) (T - M_s) < \sigma_{DP} < \sigma_{\sigma'} (1 + \beta) + C_{M_s} (1 + \beta) (T - M_s) \):

\[
\xi_S = \frac{1 - \xi_{S0}}{2} \cos \left( \frac{\pi}{C_{M_s} (\sigma_{\sigma'} - \sigma_{\sigma'})} \left[ \frac{\sigma_{DP}}{(1 + \beta) - \sigma_{\sigma'} - C_{M_s} (T - M_s)} \right] \right) + \frac{1 + \xi_{S0}}{2} 
\]
(18)

\[
\xi_T = \xi_{T0} - \frac{\xi_{T0}}{1 - \xi_{S0}} (\xi_S - \xi_{S0}) 
\]
(19)

where

\[
C_{M_s} = \frac{(C_{M_s} - C_{M_s}) (T - M_s)}{\sigma_{\sigma'} - \sigma_{\sigma'}} + 1
\]
(20)

- In the case when \( T < M_s \) and \( \sigma_{\sigma'} (1 + \beta) < \sigma_{DP} < \sigma_{\sigma'} (1 + \beta) \):

\[
\xi_S = \frac{1 - \xi_{S0}}{2} \cos \left( \frac{\pi}{C_{M_s} (\sigma_{\sigma'} - \sigma_{\sigma'})} \left[ \frac{\sigma_{DP}}{(1 + \beta) - \sigma_{\sigma'} - C_{M_s} (T - M_s)} \right] \right) + \frac{1 + \xi_{S0}}{2} 
\]
(21)

\[
\xi_T = \xi_{T0} - \frac{\xi_{T0}}{1 - \xi_{S0}} (\xi_S - \xi_{S0}) + \Delta_{T_T} 
\]
(22)
where, if $M_f < T < M_s$ and $T < T_0$

\[ Z_{T_0} = \frac{1 - \frac{T}{T_0}}{2} \{1 + \cos[a_M(T - M_f)]\} \quad (23) \]

else

\[ Z_{T_0} = 0 \quad (24) \]

(ii) Transformation process to the austenite phase

In the case when $T > A_f$ and $C_{A_f}(1 + \beta(T - A_f)) < \sigma^{dp} < C_{A_f}(1 + \beta(T - A_f))$

\[ \xi = \frac{\xi_0}{2} \left\{1 + \cos\left[\frac{a_A}{C_{A_f}}(C_{A_f}(T - A_s) - \frac{\sigma}{1 + \beta})\right]\right\} \quad (25) \]

\[ \xi_s = \xi_{s0} - \frac{\xi}{\xi_s} (\xi_0 - \xi) \quad (26) \]

\[ \xi_f = \xi_{f0} - \frac{\xi}{\xi_f} (\xi_0 - \xi) \quad (27) \]

where

\[ C_{A_f} = \frac{C_{A_f}(T - A_f) - C_{A_f}(T - A_f)}{A_f - A_s} \quad (28) \]

\[ a_M \quad \text{and} \quad a_A \quad \text{are defined by the following equations:} \]

\[ a_M = \frac{\pi}{M_s - M_f}, \quad a_A = \frac{\pi}{A_f - A_s} \quad (29) \]

3. Finite Element Formulations

The stress-strain relation (4) is rewritten to the following incremental form of equation to formulate the incremental finite element procedure by the tangential stiffness method:

\[ \{\Delta \sigma\} = [D]\{\Delta \varepsilon\} + [\Delta D]\{\Delta \varepsilon\} + \Delta \varepsilon_s [\Delta \Omega] + \Delta \varepsilon_s [\Delta \Omega] + \Delta \Omega [\Delta \theta] + T[\Delta \theta] \]

\[ = [D]\{\Delta \varepsilon\} + \Delta \varepsilon_s [\Delta D]\{\Delta \varepsilon_s\} - \Delta \varepsilon_s [D]\{\Delta R_s\}\{\Delta \varepsilon_s\} - \Delta \varepsilon_s [\Delta D]\{\Delta R_s\}\{\Delta \varepsilon_s\} - \Delta \varepsilon_s [D]\{\Delta R_s\}\{\Delta \varepsilon_s\} - \Delta \varepsilon_s [\Delta D]\{\Delta R_s\}\{\Delta \varepsilon_s\} - \Delta \varepsilon_s [D]\{\Delta R_s\}\{\Delta \varepsilon_s\} - \Delta \varepsilon_s [\Delta D]\{\Delta R_s\}\{\Delta \varepsilon_s\} \]

\[ - \Delta T[D]\{\Delta \varepsilon\} - T[\Delta \varepsilon_s [\Delta D]\{\Delta \varepsilon_s\} \]

\[ \{\Delta \sigma^{dp}\} \quad \text{is expressed by the following equation:} \]

\[ \Delta \sigma^{dp} = \frac{2\sigma_s - \sigma_f - \sigma_s}{2\sigma_eq} + \beta \frac{2\sigma_s - \sigma_f - \sigma_s}{2\sigma_eq} + \beta \frac{2\sigma_s - \sigma_f - \sigma_s}{2\sigma_eq} + \beta \frac{3\tau_{xy}}{\sigma_eq} \frac{3\tau_{xy}}{\sigma_eq} \frac{3\tau_{xy}}{\sigma_eq} \]

\[ = \{\sigma^{dp}\}\{\Delta \varepsilon\} \quad (33) \]
Using $\Delta \sigma^{dp}$, $\Delta \xi$ and $\Delta \xi_s$ are expressed as follows:

$$\Delta \xi = A_1 \left[ \sigma^{dp} \right] \left( \Delta \sigma \right) + A_2 \Delta T$$

(34)

$$\Delta \xi_s = B_1 \left[ \sigma^{dp} \right] \left( \Delta \sigma \right) + B_2 \Delta T$$

(35)

Substituting Eqs. (34) and (35) into Eq. (30), the incremental stress-strain relation is expressed by the following equation:

$$\{ \Delta \sigma \} = [X]^{-1} \left[ D \right] \{ \Delta \varepsilon \} + [X]^{-1} \left[ Y \right] \Delta T = [D_f] \{ \Delta \varepsilon \} + \left[ \Theta_f \right] \Delta T$$

(36)

where $[X]$ and $[Y]$ are defined by the following equations:

$$[X] = [Y] - (A_1 \Delta D_f) \left[ \varepsilon \left[ \sigma^{dp} \right] - B_1 [D] \left[ \sigma^{dp} \right] + A_1 \xi_s [\Delta D_f] \left[ \varepsilon_L \right] \left[ \sigma^{dp} \right] - A_1 T \left[ \Delta D_f \right] \left[ \sigma^{dp} \right] - B_1 \xi_s [D] \left[ \Delta R_s \varepsilon_L \right])$$

$$[Y] = A_1 \Delta D_f [\varepsilon] - B_2 [D] \left[ \varepsilon_L \right] - [D] \left[ \varepsilon \right] - A_1 \xi_s [\Delta D_f] \left[ \varepsilon_L \right] - A_2 T \left[ \Delta D_f \right] \left[ \varepsilon \right]$$

(37)

$$A_1, A_2, B_1 \text{ and } B_2 \text{ in Eqs. (37) and (38) are expressed as follows:}$$

(i) Transformation process to the martensite phase

- In the case when $T > M_s$, $\sigma^\sigma (1 + \beta) + C_m (1 + \beta) (T - M_s) < \sigma^{dp} < \sigma_f^\sigma (1 + \beta) + C_m (1 + \beta) (T - M_s)$ and $\Delta \sigma^{dp} - C_m (1 + \beta) \Delta T$:

$$B_1 = \frac{1 - \xi_{s0}}{2} \sin \frac{\pi}{C_M \left( \sigma_{s0}^\sigma - \sigma_f^\sigma \right)} \left( \frac{\sigma^{dp} - \sigma_f^\sigma - C_m (1 + \beta) (T - M_s)}{\sigma_f^\sigma - C_m (1 + \beta) (T - M_s)} \right)$$

(39)

$$B_2 = \frac{1 - \xi_{s0}}{2} \sin \frac{\pi}{C_M \left( \sigma_{s0}^\sigma - \sigma_f^\sigma \right)} \left( \frac{\sigma^{dp} - \sigma_f^\sigma - C_m (1 + \beta) (T - M_s)}{\sigma_f^\sigma - C_m (1 + \beta) (T - M_s)} \right)$$

(40)

$$A_1 = \frac{1 - \xi_{s0}}{1 - \xi_{s0}^{-1}} B_1, \quad A_2 = \frac{1 - \xi_{s0}}{1 - \xi_{s0}^{-1}} B_2$$

(41)

- In the case when $T < M_s$, $\sigma^\sigma (1 + \beta) < \sigma^{dp} < \sigma_f^\sigma (1 + \beta)$ and $\Delta \sigma^{dp} > 0$, or $T > M_s$, $\Delta T < 0$ and $\sigma^\sigma (1 + \beta) < \sigma^{dp} < \sigma_f^\sigma (1 + \beta)$:

$$B_1 = \frac{1 - \xi_{s0}}{2} \sin \frac{\pi}{\sigma_f^\sigma - \sigma_f^\sigma \left( 1 + \beta \right)} \left( \frac{\sigma^{dp} - \sigma_f^\sigma - \sigma_f^\sigma \left( 1 + \beta \right)}{\sigma_f^\sigma - \sigma_f^\sigma \left( 1 + \beta \right)} \right)$$

(42)

$$B_2 = 0$$

(43)

$$A_1 = \frac{1 - \xi_{s0}}{1 - \xi_{s0}^{-1}} B_1$$

(44)

where, if $M_f < T < M_s$ and $T < T_0$

$$A_1 = \frac{1 - \xi_{s0}}{2} \sin \left[ a_m \left( T - M_f \right) \right]$$

(45)

else

$$A_1 = 0$$

(46)

(ii) Inverse transformation process to the austenite phase

- In the case when $T > A_1$, $C_m \left( 1 + \beta \right) (T - A_1) < \sigma^{dp} < C_m (1 + \beta) (T - M_a)$ and
The finite element formulation based on the virtual work principle using the incremental constitutive equation (36) leads to the incremental element stiffness equation expressed by the following equation:

\[
\Delta \sigma^{DP} - C_A \left(1 + \beta \right) \Delta T < 0 : \\
A_1 = \frac{\xi}{2} \sin \frac{\alpha}{C_A \left( T - A_s \right) - \frac{\sigma^{DP}}{1 + \beta}} \frac{\alpha}{C_A \left( T - A_s \right) - \frac{\sigma^{DP}}{1 + \beta}} \\
A_2 = \frac{\xi}{2} \sin \frac{\alpha}{C_A \left( T - A_s \right) - \frac{\sigma^{DP}}{1 + \beta}} \frac{\alpha}{C_A \left( T - A_s \right) - \frac{\sigma^{DP}}{1 + \beta}} \\
B_1 = \frac{\xi + \xi_0}{\xi_0} A_1, \quad B_2 = \frac{\xi + \xi_0}{\xi_0} A_2
\]

The finite element formulation based on the virtual work principle using the incremental constitutive equation (36) leads to the incremental element stiffness equation expressed by the following equation:

\[
[K_T] \{\Delta u\} = \{A\} + \{f\} - \int_{V} [B^T] [\Theta_T] \Delta T dV
\]

where

\[
[K_T] = \int_{V} [B^T] [D] [B] dV
\]

In Eqs. (50) and (51), the following notations are used: \([K_T]\); the tangential stiffness matrix, \([D]\); the stress-strain matrix, \([\Theta_T]\); the nodal displacement increment vector, \([B]\); the strain-nodal displacement matrix, \([f]\); the external force increment vector and \([f]\); the unbalanced force vector. The three-dimensional, eight node isoparametric element is used in the analysis.

4. Results of Finite Element Analysis

4.1 Multiaxial Behavior of the SMAs at Constant Temperature

In the present subsection, the calculated results for the SMA microtubes (56.0Ni-44.0Ti (at.%)) under an axial force and torsion are compared with the experimental results given by Sun and Li(10). The material constants used in the analysis are shown in Table 1. The elastic constants of the austenite phase and the temperatures for the phase transformation in Table 1 are from the literature(10), while the other material constants have been determined so as to fit the constitutive equation proposed in the section 2 with the experimental results(10). The temperature in the analysis is 23°C which is higher than the temperature for finishing austenite transformation (\(f_A\)). Figure 2 shows the relation between the critical transformation stress and the temperature. The material constants in Table 1 are valid when the temperature is lower than 24.64°C.

Figure 3(a) shows the dimensions and the boundary conditions of the SMA bar for the uniaxial tensile analysis. The numbers of elements and nodes are 16 and 45, respectively. The calculated and experimental stress-strain curves are shown in Fig. 4. Although the gradients in the martensite transformation are slightly different in the calculation and the experiment, both are totally in good agreement.

Figure 3(b) shows the dimensions (\(L = 5.0\)mm, \(D_{out} = 1.5\)mm, \(D_{in} = 1.2\)mm) and the boundary conditions of the SMA microtube for the torsional deformation analysis. The numbers of elements and nodes are 240 and 520, respectively. The calculated and experimental stress-strain curves are shown in Fig. 5. The perfect superelastic behavior is observed as in the uniaxial tensile analysis as the temperature is 23°C which is higher than the temperature for finishing austenite transformation. The agreement between the calculation and the experiment is not so good as in the uniaxial tensile analysis.

In the torsional analysis, the shear modulus \(G\) for the austenite phase is assumed to be 18GPa referring to Sun et al.\(^{(10)}\), which does not correspond to the isotropic relation of the material \((G = E/2(1 + \nu))\). This is probably due to the effect of the anisotropy at the crystal scale as the specimen is very small.
The boundary condition for the tensile-torsional analysis is shown in Fig. 3(c). Figure 6 shows the interaction curves of the critical stresses for martensite transformation given by the experiment of Sun and Li\(^{10}\) and the model proposed in the present study. For comparison, von Mises equivalent stress curve is also shown with a bold line. Although the experimental results are only for the tensile side, it is observed that the agreement between the model and the experiment is good when the normal stress is dominant. When the shear stress is dominant, there is about 30% difference at the maximum.

In the next subsection, the model is compared with the experimental results for larger specimens. It will be confirmed from the comparison that the difference between the model and the experiment in the present subsection is mainly due to the effect of anisotropy in small specimens.

<table>
<thead>
<tr>
<th>Moduli</th>
<th>(E_a (\text{MPa}))</th>
<th>(E_m (\text{MPa}))</th>
<th>(C_{\text{ie}} (\text{MPa}^2/\text{C}))</th>
<th>(C_{\text{im}} (\text{MPa}^2/\text{C}))</th>
<th>(C_{\text{le}} (\text{MPa}/\text{C}))</th>
<th>(C_{\text{lm}} (\text{MPa}/\text{C}))</th>
<th>(\alpha (\text{C}^{-1}))</th>
<th>(\beta)</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>(3.0\times10^4) *</td>
<td>(2.0\times10^4)</td>
<td>(5.7)</td>
<td>(6.3)</td>
<td>(9.6)</td>
<td>(37.0)</td>
<td>(1.0\times10^{-6})</td>
<td>(0.15)</td>
</tr>
<tr>
<td>Transformation</td>
<td>(M_s (\text{C}))</td>
<td>(-67.6) *</td>
<td>(M_f (\text{C}))</td>
<td>(-54.2) *</td>
<td>(A_s (\text{C}))</td>
<td>(-3.26) *</td>
<td>(A_f (\text{C}))</td>
<td>(17.4)</td>
</tr>
<tr>
<td>Temperatures</td>
<td>(\sigma_{\text{cr}} (\text{MPa}))</td>
<td>(0.0)</td>
<td>(\sigma_{\text{fr}} (\text{MPa}))</td>
<td>(20.0)</td>
<td></td>
<td></td>
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<tr>
<td>Critical stresses</td>
<td>(\varepsilon_{\text{l}})</td>
<td>(4.4\times10^{-2})</td>
<td>(\gamma_{\text{l}})</td>
<td>(4.4\times10^{-2})</td>
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</table>

| Maximum        | \(\varepsilon_{\text{l}}\) | \(4.4\times10^{-2}\) | \(\gamma_{\text{l}}\)               | \(4.4\times10^{-2}\)               | | | |

Fig. 2  Critical stress for transformation vs. temperature
4.2 Multiaxial Behavior of the SMAs Considering Temperature Change

The critical stress for martensite transformation of the SMAs increases a little by the temperature rising due to the heat generation with the martensite transformation and the latent heat \(^{11,12}\). In the present subsection, the multiaxial behavior analysis of the SMAs (50.8Ni-49.2Ti (at.\%)) is conducted...
considering the temperature change. The calculated results are compared with the experimental results of Lim and Mcdowell\textsuperscript{(12)}.

The material constants used in the analysis are shown in Table 2. The elastic constants of the austenite phase and the temperatures for the phase transformation in Table 2 are from the literatures\textsuperscript{(11)(12)}, while the other material constants have been determined so as to fit the constitutive equation proposed in the section 2 with the experimental results\textsuperscript{(11)(12)}. The temperature change of the SMA tube measured during the transformation process is shown in Table 3.

The boundary conditions for the uniaxial tensile analysis, the torsional analysis and the tensile torsional analysis are the same as in Fig. 3(a), Fig. 3(b) and Fig. 3(c), respectively. The dimension of the specimen is \( L = 35.6 \text{ mm} \), \( D_{out} = 20.4 \text{ mm} \) and \( D_{in} = 16.5 \text{ mm} \).

The calculated and the experimental equivalent stress-strain curves are shown in Figs. 7, 8 and 9. Although the calculated results have all corresponded well with the experimental results, there is a large difference for the behavior at the initial deformation stage. This difference is due to the fact that the transformation to the rhombohedral phase observed in some SMAs is neglected in the present modeling. Table 4 shows the critical stresses for martensite transformation determined in each calculation, depending on the temperature change.

### Table 2  Material constants of SMA (50.8Ni-49.2Ti (at.%))

\( (*) : \text{References (11), (12)}\)

<table>
<thead>
<tr>
<th>Moduli</th>
<th>( E_a (\text{MPa}) )</th>
<th>12.0×10(^4) *</th>
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<tr>
<td></td>
<td>( E_m (\text{MPa}) )</td>
<td>3.0×10(^4)</td>
</tr>
<tr>
<td></td>
<td>( C_{sa} (\text{MPa}/\text{°C}) )</td>
<td>5.1</td>
</tr>
<tr>
<td></td>
<td>( C_{sm} (\text{MPa}/\text{°C}) )</td>
<td>4.5</td>
</tr>
<tr>
<td></td>
<td>( C_{sa} (\text{MPa}/\text{°C}) )</td>
<td>7.0</td>
</tr>
<tr>
<td></td>
<td>( C_{sm} (\text{MPa}/\text{°C}) )</td>
<td>1.5</td>
</tr>
<tr>
<td></td>
<td>( \alpha (\text{°C}^{-1}) )</td>
<td>1.0×10(^{-6})</td>
</tr>
<tr>
<td></td>
<td>( \beta )</td>
<td>0.15</td>
</tr>
</tbody>
</table>

### Transformation Temperatures

| \( M_s (\text{°C}) \) | -55.0 * |
| \( M_f (\text{°C}) \) | -33.0 * |
| \( A_s (\text{°C}) \) | -20.0 * |
| \( A_f (\text{°C}) \) | 0.0 * |

### Critical stresses

| \( \sigma_s^c (\text{MPa}) \) | 0.0 |
| \( \sigma_f^c (\text{MPa}) \) | 240.0 |

<table>
<thead>
<tr>
<th>Maximum residual strains</th>
<th>( \varepsilon_s )</th>
<th>8.0×10(^{-3})</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \gamma_s )</td>
<td>8.0×10(^{-3})</td>
</tr>
</tbody>
</table>

### Table 3  Temperature changes during deformation

<table>
<thead>
<tr>
<th>Forced strain(( \varepsilon_{eq} ))</th>
<th>Temperature(( \text{°C} ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniaxial loading</td>
<td>0 ( \leftrightarrow ) 0.03</td>
</tr>
<tr>
<td>Torsional loading</td>
<td>0 ( \leftrightarrow ) 0.03</td>
</tr>
<tr>
<td>Combined loading</td>
<td>0 ( \leftrightarrow ) 0.03</td>
</tr>
</tbody>
</table>

### Table 4  Critical transformation stresses (MPa)

<table>
<thead>
<tr>
<th></th>
<th>Uniaxial loading</th>
<th>250</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Torsional loading</td>
<td>295</td>
</tr>
<tr>
<td></td>
<td>Combined loading</td>
<td>263</td>
</tr>
</tbody>
</table>
5. Conclusion

The one-dimensional constitutive equation of the SMAs previously proposed by the authors is extended in the present study. The gradients of the critical transformation stresses with respect to the temperature are all assumed to be independent material constants in the improved Brinson’s model considering the asymmetric tensile and compressive behavior with Drucker-Prager equivalent stress. The evolution equation for the martensite volume fraction and the one-dimensional constitutive equation are extended to the multiaxial case by coupling the normal deformation and the shear deformation. The incremental finite element procedure is formulated, based on the tangential stiffness method.

The present formulation has been applied to the multiaxial behavior analyses for the SMA microtube at a constant temperature and for the SMA tube considering the temperature change. The calculated results have been compared with the experimental results. Although there is a room for
quantitative improvement such as the consideration of the R-phase, the present method is valid in practice as a computational procedure for the multiaxial behaviors of the SMA devices.

References


