On the Linear Dependencies of Interpolation Functions in s-Version Finite Element Method*

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Abstract

In this paper, we first point out that the finite element interpolation functions sometime lose their linear independencies when the elements are superposed each other in s-Version finite element (s-FEM) analyses. This problem has been known among engineers/researchers who have studied and used the s-FEM for solid mechanics analyses. Once the interpolation functions lose their linear independencies, resulting global stiffness matrix becomes singular and, therefore, the solutions lose their uniqueness. Although it is rare for an analyst to come across such a problem, it degrades the robustness of s-FEM. This paper describes a way to judge the occurrence of such a problem and propose a method to circumvent it.

Key words: Finite Element Method (FEM), Stress Intensity Factor, Numerical Analysis, Structural Analysis, Elasticity, s-Version FEM (s-FEM)

1. Introduction

In last two decades, researchers and engineers in the field of computational mechanics have paid much attention to methodologies that can reduce the tasks to generate finite element meshes. The element free Galerkin method (EFGM)⁽¹⁾⁽²⁾ and moving least-square Petrov-Galerkin method (MLPG)⁽³⁾⁽⁴⁾ totally eliminate the finite element meshes for interpolating the displacements. Free Mesh Method (FMM)⁽⁵⁾⁽⁶⁾ is a node base finite element method for which analysts do not need to provide the meshes. Extended finite element method (x-FEM)⁽⁷⁾⁽⁸⁾ that used enrich functions to represent the discontinuous displacements and singular strain fields in crack problems was proposed. x-FEM eliminates needs for placing very fine meshes at the crack tip. Although the s-Version Finite Element Method (s-FEM)⁽⁹⁾⁽¹⁰⁾ does need meshes as its inputs, s-FEM allows us to build an analysis model by generating the so-called global model for representing the structure as whole and the local model for the local features of the structure such as holes, cracks, etc., separately, making model generations very tractable. Then, the local model is superposed on the global model, as shown in Fig. 1. The s-FEM was first proposed by Fish⁽⁹⁾. The

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method was then adopted by many researchers. Nakasumi et al.⁽¹¹⁾ applied the s-FEM to perform the global-local analyses on ship-structures. Okada et al.⁽¹²⁾ performed two-dimensional fracture mechanics analyses and the accuracy of s-FEM was critically discussed. Takano et al.⁽¹³⁾⁽¹⁴⁾ applied the s-FEM to perform multi-scale analyses on composite/porous materials. Okada et al.⁽¹⁵⁾⁽¹⁶⁾ presented analyses on composite materials. In their approach, the second phase material such as reinforcing particle or fiber and its immediate vicinity are represented by a local model and many local models are distributed in the analyses region. To do so, their multiple overlappings are allowed to occur.

Although there have been many successful applications of s-FEM as mentioned above, present authors would like to point out its inherent problem. That is, because two or more finite meshes are superposed, the interpolation functions (shape functions) are not guaranteed to be linearly independent from each other. In other words, for the same strain field, we can have more than one set of nodal displacements. If such an analysis model is used, resulting global stiffness matrix becomes singular⁽¹⁷⁾.

From our experiences in s-FEM analyses⁽¹²⁾⁽¹⁵⁾⁽¹⁶⁾, it is rare to come across such a problem. However, once the problem arises, linear equation solver using iterative method such as conjugate gradient method does not converge or is extremely slow to converge to the solution. From the authors' point of view, this is a problem in the robustness of s-FEM as a numerical method. Ways to circumvent this problem were sought and are presented in this paper. The idea is very similar to what was presented by Tanaka and Okada⁽¹⁸⁾ for wavelet FEM. The linear independencies are checked by means of preprocessing.

In this paper, the formulations of s-FEM are first presented and the problem of the linear independence of interpolation functions is pointed out. Then, methods to judge if the interpolation functions lost their linear independencies are presented. Finally, some numerical demonstrations and concluding remarks are given.

2. The Formulations of s-FEM

In this section, the formulations of s-FEM are briefly presented. It is assumed that the region of the local model is embedded within that of the global model. The displacements are represented by the finite elements of global and local models, independently. They are written to be:

$$u_i^G = u_i^G(\mathbf{x}) \quad \mathbf{x} \in \Omega \quad (1)$$

$$u_i^L = u_i^L(\mathbf{x}) \quad \mathbf{x} \in \Omega^L \quad (2)$$

where Ω and Ω^L are the regions of global and local models, as depicted in Fig. 1, respectively. $\mathbf{x}$ denotes the position vector of a material point. u_i^G and u_i^L are the displacement functions that are expressed by the shape functions of elements of the global and the local models, respectively.

The displacements $u_i = u_i(\mathbf{x})$ are expressed by u_i^G and u_i^L in the following fashion.

$$u_i = u_i(\mathbf{x}) = \begin{cases} u_i^G(\mathbf{x}) & (\mathbf{x} \in \Omega - \Omega^L) \\ u_i^G(\mathbf{x}) + u_i^L(\mathbf{x}) & (\mathbf{x} \in \Omega^L) \end{cases} \quad (3)$$

Here, it is noted that $u_i^L(\mathbf{x})$ are set to be zeros at the outer boundary $\partial\Omega^L$ of the local model region Ω^L when it is inside of domain Ω . When the boundary $\partial\Omega^L$ coincides with that $\partial\Omega$ of domain Ω , $u_i^L(\mathbf{x})$ do not have to be set to be zero. In general, we can write:

$$u_i^L(\mathbf{x}) = 0 \quad (\mathbf{x} \in \partial\Omega^L) \quad (4)$$

The displacement functions are used in the statement of virtual work principle, that is written to be:

$$\int_{\Omega} \frac{\partial \delta u_i}{\partial x_j} E_{ijkl} \frac{\partial u_k}{\partial x_l} d\Omega = \int_{\partial\Omega'} \delta u_i \bar{t}_i d(\partial\Omega') + \int_{\Omega} \delta u_i b_i d\Omega \quad (5)$$

In Eq. (5), $\delta u_i = \delta u_i(\mathbf{x})$ are the variations of the displacements and they are expressed in the same manner as in Eq. (3). E_{ijkl} are the components of the fourth order tensor that represents the Hooke's law. $\bar{t}_i$ and b_i are the prescribed tractions at $\partial\Omega'$ and the body force per unit volume, respectively. Thus, the left hand side of Eq. (5) can be expressed, as:

$$\begin{aligned} \int_{\Omega} \frac{\partial \delta u_i}{\partial x_j} E_{ijkl} \frac{\partial u_k}{\partial x_l} d\Omega &= \int_{\Omega} \frac{\partial \delta u_i^G}{\partial x_j} E_{ijkl} \frac{\partial u_k^G}{\partial x_l} d\Omega + \int_{\Omega^L} \frac{\partial \delta u_i^G}{\partial x_j} E_{ijkl} \frac{\partial u_k^L}{\partial x_l} d\Omega^L \\ &+ \int_{\Omega^L} \frac{\partial \delta u_i^L}{\partial x_j} E_{ijkl} \frac{\partial u_k^G}{\partial x_l} d\Omega^L + \int_{\Omega^L} \frac{\partial \delta u_i^L}{\partial x_j} E_{ijkl} \frac{\partial u_k^L}{\partial x_l} d\Omega^L \end{aligned} \quad (6)$$

Then, Eq. (6) is discretized by using appropriate finite elements for the global and the local models. The global stiffness matrix is constructed, as:

$$\begin{bmatrix} \mathbf{K}^{GG} & \mathbf{K}^{GL} \\ (\mathbf{K}^{GL})^T & \mathbf{K}^{LL} \end{bmatrix} \begin{Bmatrix} \mathbf{u}^G \\ \mathbf{u}^L \end{Bmatrix} = \begin{Bmatrix} \mathbf{t}^G \\ \mathbf{t}^L \end{Bmatrix} + \begin{Bmatrix} \mathbf{b}^G \\ \mathbf{b}^L \end{Bmatrix} \quad (7)$$

where $\mathbf{K}^{GG}$ and $\mathbf{K}^{LL}$ are derived from $\int_{\Omega} \frac{\partial \delta u_i^G}{\partial x_j} E_{ijkl} \frac{\partial u_k^G}{\partial x_l} d\Omega$ and $\int_{\Omega^L} \frac{\partial \delta u_i^L}{\partial x_j} E_{ijkl} \frac{\partial u_k^L}{\partial x_l} d\Omega^L$, respectively, and they are the ordinary global stiffness matrices arising from the global and local finite element models. $\mathbf{K}^{GL}$ and $(\mathbf{K}^{GL})^T$ are from $\int_{\Omega^L} \frac{\partial \delta u_i^G}{\partial x_j} E_{ijkl} \frac{\partial u_k^L}{\partial x_l} d\Omega^L$ and $\int_{\Omega^L} \frac{\partial \delta u_i^L}{\partial x_j} E_{ijkl} \frac{\partial u_k^G}{\partial x_l} d\Omega^L$, respectively. $\{\mathbf{u}^G\}$ and $\{\mathbf{u}^L\}$ are the nodal displacements of the global and the local finite

Fig. 1 Schematic illustration for the model superposition procedure in the s-Version FEM

element models. $\{t^G\}$, $\{t^L\}$, $\{b^G\}$ and $\{b^L\}$ are the equivalent force vectors of external and body forces.

3. Linear Dependencies between Nodal Degrees of Freedoms

3.1 Linear Dependency Test Using Scalar Functions-Idea and Procedures

Eq. (7) is solved for unknown displacements. A unique solution exists only when the coefficient matrix, that is nothing but the global stiffness matrix, is not singular. However, it is known among engineers/researchers who applied the s-FEM to their problems that the coefficient matrix in the left hand side of Eq. (7) sometimes becomes singular (see, for example, Nakasumi⁽¹⁷⁾). When the global stiffness matrix suffers from such a problem, a zero displacement field can be represented by a set of non-zero nodal displacements. We can examine the interpolation functions themselves instead of the coefficient matrix in Eq. (7). To do so, we make use of scalar functions.

We denote the scalar functions based on the global and the local meshes by $\phi^G = \phi^G(x)$ ($x \in \Omega$) and $\phi^L = \phi^L(x)$ ($x \in \Omega^L$), respectively. They are expressed by appropriate interpolation functions and nodal parameters. We can write:

$$\phi^G = \phi^G(x) = \{N^G(x)\}^T \{q^G\} \quad (x \in \Omega) \quad (8)$$

$$\phi^L = \phi^L(x) = \{N^L(x)\}^T \{q^L\} \quad (x \in \Omega^L) \quad (9)$$

where $\{N^G(x)\}$ and $\{N^L(x)\}$ are the vectors whose components are the interpolation functions based on the global and the local meshes, respectively. $\{q^G\}$ and $\{q^L\}$ are the nodal quantities at the nodes of the global and of the local meshes. If the interpolation functions for the global and the local models are not linearly independent from each other, the following equations can be satisfied even when $\{q^G\}$ and $\{q^L\}$ are not zero vectors.

$$\{N^G(x)\}^T \{q^G\} = \{N^L(x)\}^T \{q^L\} \quad (x \in \Omega^L) \quad (10)$$

Eq. (10) states that the scalar functions ϕ^G and ϕ^L are the same throughout the region Ω^L . To express such a condition, one can state:

$$0 = \int_{\Omega^L} \left(\{N^G(x)\}^T \{q^G\} - \{N^L(x)\}^T \{q^L\} \right)^2 d\Omega^L \quad (11)$$

Equating Eq. (11), one can reach:

$$0 = \begin{Bmatrix} \{q^G\} \\ \{q^L\} \end{Bmatrix}^T \begin{bmatrix} A^{GG} & -A^{GL} \\ -(A^{GL})^T & A^{LL} \end{bmatrix} \begin{Bmatrix} \{q^G\} \\ \{q^L\} \end{Bmatrix} \quad (12)$$

$$\begin{aligned} [A^{GG}] &= \int_{\Omega^L} \{N^G(x)\} \{N^G(x)\}^T d\Omega^L, \quad [A^{LL}] = \int_{\Omega^L} \{N^L(x)\} \{N^L(x)\}^T d\Omega^L \\ [A^{GL}] &= \int_{\Omega^L} \{N^G(x)\} \{N^L(x)\}^T d\Omega^L \end{aligned} \quad (13)$$

Equal sign in Eq. (12) can only be satisfied when the coefficient matrix is

singular. Therefore, non-zero vector $\begin{Bmatrix} q^G \\ q^L \end{Bmatrix}$ that satisfies the following equation must exist.

$$\{\theta\} = \begin{bmatrix} A^{GG} & -A^{GL} \\ -(A^{GL})^T & A^{LL} \end{bmatrix} \begin{Bmatrix} q^G \\ q^L \end{Bmatrix} \quad (14)$$

Vector $\begin{Bmatrix} q^G \\ q^L \end{Bmatrix}$ that satisfies Eq. (14) indicates the linear dependencies between the nodal quantities. By applying the fully-pivoted Gauss-Jordan method⁽¹⁹⁾ and appropriately reordering the right hand side vector of Eq. (14), we obtain:

$$\{\theta\} = \begin{bmatrix} I & \tilde{\kappa} \\ O & O \end{bmatrix} \begin{Bmatrix} \tilde{q}^I \\ \tilde{q}^D \end{Bmatrix} \quad (15)$$

Here, I and O indicate that they are the identity and zero matrices. $\tilde{q}^D$ is the vector of nodal parameters that are expressed in terms of linear combinations of the elements of $\tilde{q}^I$ which is the vector of independent nodal quantities. According to Eq. (15), we have the relationship, as:

$$\{\tilde{q}^I\} + [\tilde{\kappa}]\{\tilde{q}^D\} = \{\theta\} \quad (16)$$

It is clear that $\tilde{q}^I$ and $\tilde{q}^D$ are linearly dependent from each other.

In present investigation, we let the components of $\tilde{q}^D$ include the elements of q^G only, so that we can take the freedoms associated with the global mesh away. To do so, we take slightly modified procedures based on the fully-pivoted Gauss-Jordan scheme. First, we rewrite Eq. (14) so that the nodal parameters q^L associated with the local mesh are placed above those q^G with the global mesh, as:

$$\{\theta\} = \begin{bmatrix} A^{LL} & -(A^{GL})^T \\ -A^{GL} & A^{GG} \end{bmatrix} \begin{Bmatrix} q^L \\ q^G \end{Bmatrix} \quad (17)$$

Then, the Gauss-Jordan elimination is performed for the rows that contain the elements of A^{LL} without pivoting. Thus, we have:

$$\{\theta\} = \begin{bmatrix} I & B^{LG} \\ O & B^{GG} \end{bmatrix} \begin{Bmatrix} q^L \\ q^G \end{Bmatrix} \quad (18)$$

where B^{GG} and B^{LG} are the matrices that appear after the elimination. Finally, the fully-pivoted Gauss-Jordan elimination is performed on B^{GG} and B^{LG} and we reach at Eq. (15). By adopting these procedures, $\tilde{q}^D$ in Eq. (15) contains the elements of q^G .

As the result of these processes, we find unnecessary nodal degrees of freedoms associated with the global finite element model. Thus, they should be eliminated.

3.2 Numerical Implementation

While applying the fully-pivoted Gauss-Jordan method to the coefficient matrix, the diagonals with respect to nodal degrees of freedoms that are not linearly independent become zeros, as stated in Eq. (15). However, although the diagonal terms should become zeros, they tend to become very small numbers. Therefore, one should write:

$$\{\theta\} = \begin{bmatrix} I & \tilde{\kappa} \\ O & p \end{bmatrix} \begin{Bmatrix} \tilde{q}^I \\ \tilde{q}^D \end{Bmatrix} \quad (19)$$

where $[p]$ is a matrix whose elements are very small. Thus, we introduce a threshold value ε to judge zero. If the diagonal term of matrix $[p]$ were smaller than the threshold value ε , the associated nodal quantities are judged to be linearly independent. The degrees of freedoms associated with those nodes are eliminated.

Some discussions are given on the appropriate choice for ε in the sections of numerical examples.

3.3 Linear Dependency Test Using Scalar Functions-A Simpler Way

In this section, a simpler way to check the linear dependencies between the nodal parameters is presented. First, we define a function $\pi(\{q^G\}, \{q^L\})$ of the nodal parameters $\{q^G\}$ and $\{q^L\}$, as:

$$\pi(\{q^G\}, \{q^L\}) = \begin{Bmatrix} q^G \\ q^L \end{Bmatrix}^T \begin{bmatrix} A^{GG} & -A^{GL} \\ -(A^{GL})^T & A^{LL} \end{bmatrix} \begin{Bmatrix} q^G \\ q^L \end{Bmatrix} \quad (20)$$

As discussed previously, when $\{q^G\}$ and $\{q^L\}$ that satisfy $\pi(\{q^G\}, \{q^L\}) = 0$ are found, the interpolation functions of the global and the local finite element models are not linearly independent from each other. On the other hand, the function $\pi(\{q^G\}, \{q^L\})$ must equal or be greater than zero by its definition. Thus, the problem can be recast to be a minimization problem. If the minimum value of $\pi(\{q^G\}, \{q^L\})$ were zero, the linear independency is judged to be lost.

Here, we only consider a special case such that one nodal parameter of the global mesh can be expressed in terms of a linear combination of those of the local mesh. First, we let the components of $\{q^G\}$ be known and set them to be:

$$\{q^G\} = (q_1^G, q_2^G, q_3^G, \dots, q_k^G, \dots, q_{n-2}^G, q_{n-1}^G, q_n^G)^T = (0, 0, 0, \dots, 1, \dots, 0, 0, 0)^T \quad (21)$$

Since all the components of $\{q^G\}$ are known, the variation $\{\delta q^G\}$ of $\{q^G\}$ is a zero vector.

$$\{\delta q^G\} = \{\theta\} \quad (22)$$

Thus, the variation $\delta\pi(\{q^G\}, \{q^L\})$ of the function $\pi(\{q^G\}, \{q^L\})$ is written to be:

$$\delta\pi(\{q^G\}, \{q^L\}) = 2\{\delta q^L\}^T \begin{bmatrix} -(A^{GL})^T & A^{LL} \end{bmatrix} \begin{Bmatrix} q^G \\ q^L \end{Bmatrix} \quad (23)$$

When $\pi(\{q^G\}, \{q^L\})$ takes the minimum value, its first variation must be zero. Therefore, we have:

$$\{0\} = \begin{bmatrix} -(A^{GL})^T & A^{LL} \end{bmatrix} \begin{Bmatrix} q^G \\ q^L \end{Bmatrix} \quad (24)$$

Therefore, we arrive at:

$$\{q^L\} = [A^{LL}]^{-1} [A^{GL}]^T \{q^G\} \quad (25)$$

It is noted here that $\{q^G\}$ is given as Eq. (21) and $\{q^L\}$ is determined by Eq. (25). They are substituted in Eq. (20) and the value of the function $\pi(\{q^G\}, \{q^L\})$ is computed. If the value of the function $\pi(\{q^G\}, \{q^L\})$ were zero, the none zero component q_k^G can be expressed by a linear combination of the components of $\{q^L\}$. Therefore, the nodal freedoms associated with the global node can be eliminated. However, since the value of $\pi(\{q^G\}, \{q^L\})$ may depend on the mesh size, we normalize it by $\{q^G\}^T [A^{GG}] \{q^G\}$ and define $\bar{\pi}(\{q^G\}, \{q^L\})$ by the following expression.

$$\bar{\pi}(\{q^G\}, \{q^L\}) = \pi(\{q^G\}, \{q^L\}) / \{q^G\}^T [A^{GG}] \{q^G\} \quad (26)$$

However, there is a shortcoming in this approach. That is, when more than one nodal parameters of global mesh and those of local mesh constitute a set of nodal parameters of linear dependency, this approach fails to work.

4. Numerical Tests

4.1 A Plane Strain Plate Subject to Uniaxial Tension-1

In this section, we present a very simple but important test problem and it is demonstrated that eliminating the nodal degrees of freedoms as proposed in previous chapter serves reasonable results.

Here, we set up a problem that obviously suffers from the problems of linear independencies of the interpolation functions. The boundary value problem, the global mesh and the local mesh that is superposed on the global one are shown in Figs. 2 (a), (b) and (c), respectively. For both, the global and the local models, 4-node bi-linear finite element is adopted. The elements of the local model are placed in such a manner that four elements constitute a square shape that coincides with an element of the global model, as shown in Fig. 2 (c). There are a total of 400 elements and 441 nodes in both the global and the local meshes.

The processes described in section 3.3 are applied to check if the displacements of the global model could be expressed by those of the local model. The none-zero component $q_k^G = 1$ of vector $\{q^G\}$ is set at a node which locates inside and just outside of the overlapping region Ω^L . Then, after the computations as described in Eqs. (21)~(26) are performed, the value of $\bar{\pi}(\{q^G\}, \{q^L\})$ is computed. In Fig. 3, the values of $\bar{\pi}(\{q^G\}, \{q^L\})$ are indicated when the corresponding components q_k^G of the global nodes are set to be unity. As expected, $\bar{\pi}(\{q^G\}, \{q^L\})$ takes very small number which can be regarded to be zero

Fig. 2 A problem of tension block for the demonstration of proposed methodology [(a) boundary value problem, (b) global mesh and (c) local mesh which is superposed on the global mesh]

numerically at the nodes within the overlapping region Ω^L .

Then, we eliminated nodal degrees of freedoms of global finite element model and examined the solutions. First, we eliminated the degrees of freedoms associated with those nodes whose $\bar{\pi}(\{q^G\}, \{q^L\})$ values are less than 10^{-14} . In other word, the displacements of those nodes are constrained. In Fig. 4(a), the distribution of vertical displacement is depicted. The contributions of the global and the local finite element models and their total are shown. Within the region where the displacements of the global mesh are set to be zero, those of the local mesh completely reproduce the linear variation of vertical displacement. Thus, the total of the global and the local meshes gives the correct solution.

Next, we also constrained the global nodes with $\bar{\pi}(\{q^G\}, \{q^L\}) = 0.5$. Then, the variations of the displacements are plotted in Fig. 4(b). It is seen that the total displacement is far from the right solution. Therefore, it is concluded that if the DOFs were inappropriately taken away, the solution loses its accuracy.

4.2 A Plane Strain Plate subject to Uniaxial Tension-2

In this section, the uniaxial tension problem, as depicted in Fig. 2 (a), is dealt with. But, more general approach than previous subsection, as discussed in section 3.1, is adopted. Furthermore, we discuss about the convergence of iterative linear equation solver, namely Conjugate Gradient Solver with diagonal scaling.

In Figs. 5 (a) and (b), the global and the local meshes for the uniaxial tension problem are depicted. The total numbers of elements and nodes are 400 and 441 in the global model, and 240 and 273 in the local model, respectively. Then, they are superposed as shown in Fig. 5 (c). Which nodes should be given nodal constraints is not so obvious in this example problem. The approach of section 3.1 was used to find the nodes. However, an appropriate threshold value for judging the diagonal components to be zero was not known. Thus, we tried several values.

In Figs. 6 (a), (b) and (c), the nodes of the global mesh whose displacements are constrained are depicted by solid circles. The threshold values were set to be 10^{-14} , 10^{-3} and 10^{-2} , respectively. Then, error in tensile stress was evaluated at the center of the plate, as indicated in Fig. 5 (c), and is summarized in Table 1. The tensile stress is the average of that in four elements surrounding the central point in the global or local model. It is seen that except for the threshold value being equal 10^{-14} , the stress solutions contain unacceptably large error. Therefore, the threshold value must be very small. It is noted here that a small amount of error always arises even in a uniaxial tension problem, due to the occurrence of unavoidable numerical error when the coupling stiffness matrices are evaluated in s-FEM analysis.

Fig. 7 shows the convergences of conjugate gradient method with and

without eliminating the nodal degrees of freedoms. The threshold value is 10^{-14} . It is clearly seen in Fig. 7 that the convergence rate is very slow when the nodal displacements are not constrained. For the case that present approach is applied to eliminate the nodal degrees of freedoms, the convergence rate is much faster. It is considered that the global stiffness matrix of s-FEM is ill-conditioned when there are nodal degrees of freedoms that are not linearly independent.

5. Fracture Mechanics Applications

It is considered that fracture mechanics is among the most interesting field of application of s-FEM. In this chapter, analyses on three-point bend test problems are presented.

In Fig. 8, the three-point bend test is illustrated. Here, two cases, mode I and mixed mode problems, are considered. The global and local meshes to perform the s-FEM analyses are presented in Fig. 9. There are a total of 561 nodes and 500 linear quadrilateral finite elements in the global mesh and the local model

Fig. 3 The value of $\pi(q^G, q^L)$ associated with the nodes of global

Fig. 4 The variations of vertical displacements along A'-A when (a) DOFs with $\pi(q^G, q^L) < 10^{-14}$ and (b) $\pi(q^G, q^L) = 0.5$ are eliminated

Fig. 5 A problem of tension block for the demonstration of proposed methodology [(a) global mesh and (b) local mesh and (c) their superposition]

Fig. 6 The nodes of global mesh that are constrained by letting the threshold value ε be (a) 10^{-14} , (b) 10^{-3} and (c) 10^{-2}

Table 1 Error in tensile stress at the center of the plate

Threshold value ε	10^{-14}	10^{-03}	10^{-02}
No. of Iterations*	353	237	186
Error (%) in Global Elements	-1.08	-57.03	-64.96
Error (%) in Local Elements	-1.09	-57.21	-65.15

*Convergence criterion for the residual is 10^{-10} .

Fig. 7 The convergence of residuals from the conjugate gradient solver

has 1261 nodes and 1200 linear quadrilateral elements. The crack is modeled by the local mesh only as free surfaces and the global mesh models the beam only. The displacement constraints at the outer boundaries of the local model are given as shown in Fig. 10. The bottom of the local model is set to be free from constraints. Then, the displacement constraints are given to all the other parts of the outer boundary of the local model. The applied force is given to the global model only, as the displacements of the local mesh are constrained at its top portion.

The method presented in section 3.1 was applied to examine the linear dependencies of DOFs of the global mesh. However, the diagonals occasionally take negative sign whose absolute values are very small. Such components are judged to be zero and, for positive values, the threshold is set to be 10^{-10} . The nodes of the global model whose DOFs are eliminated are indicated by solid circles in Fig. 11.

In Fig. 12, the convergence of conjugate gradient method with diagonal scaling is shown, for the mode I problem. It is seen that the convergence was not achieved when the DOFs of the global model were not eliminated by using present method. When the DOFs are eliminated, good convergence rate was achieved.

The stress intensity factors (SIFs) were evaluated by using the Virtual Crack Closure-Integral Method (VCCM)⁽²⁰⁾. In the computation of energy release rates, only the local model was used as described by Okada, Endoh and Kikuchi⁽¹²⁾. Based on the energy release rates, the SIFs were computed. The evaluated SIFs are summarized in Table 2. They compare favorably with the theoretical and ordinary finite element solutions. It is noted that the finite element mesh discretization for the ordinary finite element analysis has the same arrangement as the local mesh of s-FEM at the vicinity of the crack.

In Table 2, the SIFs that are evaluated by an ordinary finite element analysis with VCCM and with a direct method are also given for the mode I case. The same finite element mesh as that for the mixed mode problem was used. The direct approach is the displacement correlation method by using the crack opening displacement at the nearest node to the crack tip on the crack face. The results indicate that the SIFs that are obtained by the s-FEM and the FEM with VCCM are almost the same. The SIF by the direct method (FEM-Direct) may not be as accurate as the s-FEM and the FEM with VCCM.

6. Conclusion

In this paper, systematic ways to eliminate DOFs of global finite element model of s-FEM, are presented and demonstrated for their effectiveness. Present authors believe that many researchers/engineers have known the problems associated with the linear independencies of DOFs of the global and the local

Fig. 8 Three-point bend test

Fig. 9 Local and global meshes for the analysis of three-point bend problem [(a) Global mesh and (b) Local mesh that is superposed on the global one].

Fig. 10 Displacement constraints that are applied to the outer boundaries of local mesh

Fig. 11 Nodes of global mesh at which the displacement constraints are enforced

Fig. 12 The convergence of conjugate gradient method with diagonal scaling, for the mode I problem

Table 2 Summary of the results for the stress intensity factor evaluations

Mode I problem		Mixed mode problem		
case	K_I / K_I^*	case	K_I / K_I^*	K_{II} / K_I^*
Theoretical	1.000	FEM-VCCM	0.889	0.057
s-FEM-VCCM	0.977	s-FEM-VCCM	0.889	0.057
Error (%)	-2.267	Difference (%)	0.009	0.075
FEM-VCCM	0.979	K_I^* : Theoretical value of the mode I stress intensity factor ⁽²¹⁾		
FEM-Direct	0.906			

finite element models. In many cases, an ad-hoc approach that is to slightly modify the models has been adopted. Present approach systematically finds the linear dependencies of DOFs by means of pre-processing using scalar functions.

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