Evaluation of Intensity of Singularity for Three-Materials Joints with Power-Logarithmic Singularities using an Enriched Finite Element Method*

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Abstract
In this study, the enriched finite element method (enriched FEM) is applied to evaluate the intensity of singularity for three-material joints with power-logarithmic singularities. Using this method, the intensity of singularities can be directly evaluated and very refined meshes around the singular point are unnecessary. Analyses for eigenvalue and eigenvector are conducted to calculate the order of stress singularities and the asymptotic displacement fields on the enriched elements. Accuracy of the results for different mesh types (4-node and 8-node elements) and different sizes of the enriched region are compared with the value obtained from extrapolation for stress distribution based on FEM with very fine mesh. Finally, the models with various lengths and thicknesses are investigated to study an influence of geometry on the singular stress field.

Key words: Stress Singularity, Enriched Finite Element Method, Intensity of Singularity, Power-Logarithmic Singularity

1. Introduction

Dissimilar material joints are frequently used in electronic components since development of new technology leads to compact and light packages. Dissimilar material joints have singularities created by discontinuities in material properties across an interface. A mismatch in material properties of the joints may cause high stresses and lead to fracture and failure, so the investigation of stresses in dissimilar material joints is important. It is difficult to calculate accurately stresses near a singular point using conventional FEM (Finite Element Method). There are some special elements or methods developed for solving singular stress problems. For examples, the methods based on developing a special element are; a hybrid element (Tong et al.\(^{13}\)), an enriched FEM (Benzley\(^{22}\)), the extended/generalized finite element method (XFEM/GFEM) (Belytschko and Black\(^{3}\), Fries and Belytschko\(^{4}\)), or the methods based on a post process for calculating the intensity of singularity are; a stress extrapolation (Munz and Yang\(^{5}\)), a conservative integral (Banks-Sills and Sherer\(^{6}\)).

In this study, the enriched FEM is selected for analysis. Advantages of this method are; directly evaluated the intensity of singularities and unnecessary to create a very refined mesh around the singular point. Furthermore, this method is based on conventional FEM that may be easy to apply on a commercial FEM in the future.

The enriched FEM was firstly developed by Benzley\(^{22}\) for cracks in 2D elastic solids.
After that, Pageau and Biggers\(^{(7,8)}\) presented the formulation in two major points. Firstly, an eigenvalue analysis by FEM (Pageau and Biggers\(^{(9)}\)) was used for calculating the order of singularity and the asymptotic displacement fields on an enriched element. Secondly, multiple layers of enriched elements were applied around the singular point.

In the present study, the singular stress field in three-material joints with power-logarithmic singularities is analyzed by an enriched finite element method. To the author’s knowledge, studies on power-logarithmic singularities are limited; e.g., Dempsey and Sinclair\(^{(10,11)}\), Pageau et al.\(^{(12)}\), Gadi et al.\(^{(13)}\). Most of these investigations had studied on the order of singularity and the angular functions only although the intensity of singularity is also an important parameter for predicting failure in a structure.

The enriched element formulation for power-logarithmic singularities was firstly developed by Luangarpa and Koguch\(^{(14)}\). In order to improve the accuracy of the results, the effect of changing element shape function, \(i.e.,\) changing from 4-node to 8-node elements, on the intensity of singularity is investigated in this study. Furthermore, enriched element size and area are varied to study the convergence of results and to find a suitable element for each case.

In order to study the relationship between the intensity of singularity and model’s size, the joint models with various lengths and thicknesses are investigated. Comparison between the intensities of singularity in power-logarithmic singularities form and real singularities form are considered.

This paper is composed of 4 main-sections as follows. The first section is introduction. Analytical formula; the enriched FE formulation for power-logarithmic singularities, is described in section 2. Section 3 shows numerical analyses in 6 sub-sections. Firstly, details of model and boundary condition for analysis are explained. Next, the results of the order of singularity with various Young’s modulus of material 3 are shown in sub-section 3.2. After that, asymptotic displacements are analyzed by eigenvector analysis and transformed by stress-strain relation to be angular functions for enriched FEM in sub-section 3.3. Then, meshes for enriched FEM analysis are explained in the next sub-section. In sub-section 3.5, the enriched FEM results with different mesh type, size and enriched area are shown and the effect of those parameters on the accuracy of results is discussed. Finally, the results for various model lengths and thicknesses are discussed. The conclusion is presented in section 4.

**Nomenclature**

- \(\sigma_{ij}\): Stress, MPa
- \(\tau_0\): External shear load, MPa
- \(\lambda\): The order of singularity
- \(K\): Intensity of singularity for power-logarithmic singularities
- \(K_0\): Intensity of singularity for 2-real singularities
- \(k\): Dimensionless intensity of singularity for power-logarithmic singularities
- \(k_0\): Dimensionless intensity of singularity for 2-real singularities
- \(h_{ij}(\theta)\): Angular function
- \(E\): Young’s modulus, GPa
- \(\nu\): Poisson’s ratio
- \(r\): distance from a singular point, mm
- \(u\): displacement, mm
- \(a\): size of enriched region, mm
- \(b\): size of enriched element, mm
2. Analytical formula

2.1 Element type

In this method, three types of element; enriched, transition and standard elements, are employed (see Fig. 1). The enriched element (type A element) is the element close to the singular point which is inside a singularity field, the transition element (type B element) is a connector between an enriched element and a standard element. The standard element (type C element) is the element located far from the singular point that is assumed that it receives no effect from singularity, so this element is used as a conventional element.

Fig. 1 Element model for enriched FEM

2.2 Enriched FEM

Following Pageau et al.\(^{(12)}\) and Gadi et al.\(^{(13)}\), 2D singular stress field around the singular point with power-logarithmic stress singularities can be described by

$$\sigma_{ij} = \bar{K}_1 r^{-\lambda} h_{ij1}(\theta) + \bar{K}_2 r^{-\lambda} \left[ -\ln(r) h_{ij1}(\theta) + h_{ij3}(\theta) \right]$$

where \( r \) is the radial distance from the singular point, \( \lambda \) is the order of stress singularity. \( \bar{K}_1, \bar{K}_2 \) and \( h_{ij1}(\theta), h_{ij3}(\theta); i, j = r \) or \( \theta \) are the intensities of singularity and angular functions, respectively.

The displacement assumption in an enriched element is of form

$$u_i = \sum_{e=1}^{n} g_{xe} \bar{u}_{ne} + \bar{K}_1 \left[ Q_{a1} - \sum_{e=1}^{m} g_{xe} \bar{Q}_{a1} \right] +$$

$$\bar{K}_2 \left[ -\ln(r)Q_{a1} + Q_{a3} - \sum_{e=1}^{m} g_{xe} (-\ln(r)\bar{Q}_{a1} + \bar{Q}_{a3}) \right]$$

(2)

In Eq. (2), \( u_1 \) and \( u_2 \) represent the displacements of a point within the element in the \( x \) and \( y \) directions, respectively. \( Q_{a1} \) and \( Q_{a3} \) are the asymptotic displacement fields related to the angular functions \( h_{ij1} \) and \( h_{ij3} \) in Eq. (1), respectively. \( \bar{u}_{ne}, \bar{Q}_{a1}, \bar{Q}_{a3} \) are the values of \( u_k, Q_{a1}(r, \theta) \) and \( Q_{a3}(r, \theta) \) evaluated at node \( n \). \( g_{xe} \) is the shape function of standard element with the number of node of \( m \).

The displacement field in a transition element is given by the relationship

$$u_i = \sum_{e=1}^{n} g_{xe} \bar{u}_{ne} + R(\xi, \eta) \left[ \bar{K}_1 \left[ Q_{a1} - \sum_{e=1}^{m} g_{xe} \bar{Q}_{a1} \right] + \right.$$

$$\bar{K}_2 \left[ -\ln(r)Q_{a1} + Q_{a3} - \sum_{e=1}^{m} g_{xe} (-\ln(r)\bar{Q}_{a1} + \bar{Q}_{a3}) \right] \right]$$

(3)

where \( R(\xi, \eta) \) is set a ‘zeroing’ function, equals 1 along ‘enrich’ boundaries and equals 0
3. Numerical analysis

3.1 Model for analysis

The model for analysis is the three-material joint model fixed on the bottom side and applied shear load ($\tau_0 = 1$ MPa) on the top surface, as shown in Fig. 2. Plane strain condition is considered. Material properties are shown in Table 1 (Material properties of materials 1 and 2 are fixed, while Young’s modulus of material 3 is varied. All materials have the same Poisson’s ratio of 0.3).

![Fig. 2 Model for analysis](image)

<table>
<thead>
<tr>
<th>Material</th>
<th>Young’s Modulus (GPa)</th>
<th>Poisson’s ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material 1</td>
<td>160.0</td>
<td>0.3</td>
</tr>
<tr>
<td>Material 2</td>
<td>4.0</td>
<td>0.3</td>
</tr>
<tr>
<td>Material 3</td>
<td>varied</td>
<td>0.3</td>
</tr>
</tbody>
</table>

3.2 Eigenvalue analysis

In order to analyze the order of stress singularities and asymptotic displacement fields for enriched elements, the eigen analysis developed by Pageau and Bigger\(^9\) is carried out. The eigen equation is expressed as:

$$ (p^2[A] + p[B] + [C])\{\mathbf{u}\} = 0 $$

where $[A]$, $[B]$ and $[C]$ are matrices composed of Young’s modulus and Poisson’s ratio, $p = 1-\lambda$ and $\{\mathbf{u}\}$ is the eigenvector of displacement.

An element model for eigen analysis is exhibited in Fig. 3. The model has 72 equidimensional quadratic sectorial elements.

Figure 4 shows the order of stress singularity against $E_3$. As can be seen in this figure, when $E_3$ increases from 1 to nearly 7, there are 2-real singularities. In case that $E_3$ is between 7 and 16.1, the singularity changes to be complex. After that, it changes to be 2-real singularities again when $E_3$ is more than 16.1. This study is focused on the power-logarithmic stress singularity that occurs on transition between real and complex singularities. In this analysis, the model with $E_3 = 16.123$ is selected (Material properties are shown in Table 2). The result obtained from eigenanalysis shows that this model has 2-real singularities, $\lambda_1 = 0.3474$ and $\lambda_2 = 0.3466$, which are very close together.
3.3 Displacement and angular functions from eigenvector analysis

After calculating the order of singularities from eigenvalue analysis, the angular variations of displacement are obtained by eigenvector analysis. Figures 5 and 6 show the angular variations of displacement for \( \lambda_1 = 0.3474 \) and \( \lambda_2 = 0.3466 \), respectively. After that, the angular variations of displacement are converted to the angular functions of stresses by a stress-strain relation. Referring to Pageau et al.\(^{(12)}\), the angular functions in forms for 2-real singularities, \( h_{\theta \theta 1}(\theta) \) and \( h_{\theta \theta 2}(\theta) \), are defined that \( h_{\theta \theta 1}(0) = h_{\theta \theta 2}(0) = 1 \) (See Figs. 7 and 8). Since two values for the order of singularity are very close together, the angular functions, \( h_{ij1}(\theta) \) and \( h_{ij2}(\theta) \), are similar to each other. In power-logarithmic singularities cases, only one of the angular functions \( h_{ij1}(\theta) \) is directly represented in Eq. (1) as \( h_{ij1}(\theta) \), the other one, \( h_{ij2}(\theta) \), is the derivative of the functions with respect to \( \lambda \) as follows\(^{(12)}\):

\[
h_{ij3} = \frac{d h_{ij}}{d \lambda} \approx \frac{h_{ij1} - h_{ij2}}{\lambda_1 - \lambda_2}
\]

(5)

Profiles for the angular function, \( h_{ij1}(\theta) \), are shown in Fig. 9.
Fig. 5 Displacement field for $\lambda_1$

Fig. 6 Displacement field for $\lambda_2$

Fig. 7 Angular functions, $h_{ij1}$

Fig. 8 Angular functions, $h_{ij2}$

Fig. 9 Angular functions, $h_{ij3}$

3.4 Meshes for enriched FEM analysis

For enriched finite element analysis, 2 types of element are employed: 4-node and 8-node elements. In order to study the influence of mesh refinement, 4 models; model (a), (b), (c) and (d), with different size of enriched elements ($b$) from 0.05, 0.1, 0.5 and 1.0 mm, respectively, are examined. The meshes around the singular point are shown in Fig. 10. In addition, the enriched area ($a$) is varied from 0.5 to 4.0 mm to study an effect of the enriched region size on the result convergence (see Table 3).
Table 3 Details of models for the enriched FEM analysis

<table>
<thead>
<tr>
<th>Model</th>
<th>Size of enriched elements ($b \times b$)</th>
<th>The enriched area ($a$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0.05x0.05</td>
<td>0.5, 1.0</td>
</tr>
<tr>
<td>(b)</td>
<td>0.1x0.1</td>
<td>0.5, 1.0</td>
</tr>
<tr>
<td>(c)</td>
<td>0.5x0.5</td>
<td>0.5, 1.0, 2.0, 3.0</td>
</tr>
<tr>
<td>(d)</td>
<td>1.0x1.0</td>
<td>1.0, 2.0, 3.0, 4.0</td>
</tr>
</tbody>
</table>

3.5 Results for changing element types, enriched area and element sizes

Following the models shown in Sec. 3.4, the results of the intensity of singularities, $K_1$ and $K_2$, with various enriched element sizes and various enriched areas are shown in Figs. 11-14. The obtained $K_1$ and $K_2$ are compared with the results determined from the stress distribution following a fitting technique proposed by Munz and Yang (5), whose method requires very refine mesh near the singular point and a post-process for calculating the intensities of singularities (See Appendix A; $K_1 = 1.609$ and $K_2 = 0.244$). Figures 11 and 13 show the results of $K_1$ and $K_2$ using 4-node element models. As can be seen in these Figures, the intensities of singularity converge to the result from the fitting of stress distribution when the element size and enriched area reduce. In case of 4-node element, the error originated from the stress jump between different types of element, e.g., the transition element and the standard element, may be large. This is a reason why the results for small element models are better than those for large element models. The error may be reduce by using a higher order polynomial function. After changing element type into 8-node elements, as shown in Figs. 12 and 14, the convergence rate of these results are better than that of 4-node elements. Furthermore, an influence of enriched area’s size on the intensity of singularities is small using 8-node elements.
3.6 Results for various model lengths and thicknesses

In order to investigate a relationship between the intensity of singularity and model’s size, the models with various lengths and thicknesses are analyzed. The first model is the one fixed its thickness ($h = 10$ mm) and varied its length ($L$) from 60 to 300 mm, and the second model is the one fixed its length ($L = 60$ mm) and varied its thickness ($h$) from 2 to 10 mm.

For enriched finite element analysis, the models with enriched element size, $b$, of 0.1 mm and the enriched region size, $a$, of 0.5x0.5 mm$^2$ are employed. Figures 15 and 16 show the intensities of singularity, $K_1$ and $K_2$, respectively. These results show that, for the model with fixed $h$, the intensities of singularity are constant for various values of $L$. However, for the model with fixed $L$, the intensities of singularity gradually decrease with decreasing $h$.

From the results for various lengths and thicknesses, it is difficult to explain the effect of the model size on the intensities of singularity since the units of $K_1$ and $K_2$ depend on the order of singularity and the dimensions of the joint. These results are rearranged by
using a dimensionless intensity of singularity defined as follows:

\[ \bar{K}_i = \frac{K_i}{\tau_0 h^\lambda} \]  

(6)

By using the dimensionless intensity of singularity, the results with various lengths and the results with various thicknesses can be arranged as shown in Figs. 17 and 18, respectively. The results show that \( \bar{K}_2 \) from both the models with fixed \( h \) and fixed \( L \) can be arranged to be the same line; and the results are nearly constant, but \( \bar{K}_1 \) for fixed \( h \) and fixed \( L \) are different. Comparing \( \bar{K}_1 \) for the same \( L/h \), the magnitude of \( \bar{K}_1 \) for fixed \( L \) are lower than the magnitude of \( \bar{K}_1 \) for fixed \( h \). Furthermore, the results for fixed \( L \) model decrease with increasing \( L/h \).

Following Pageau et al.\(^{(12)}\), the intensity of singularity in power-logarithmic singularities form can be modified to be 2-real singularities form as follows:

\[ \sigma_y = K_i r^{-i}h_{y_1}(\theta) + K_2 r^{-i}h_{y_2}(\theta) \]  

(7)

\[ K_1 = \bar{K}_1 + \frac{K_2}{\lambda_1 - \lambda_2} \]  

(8)

\[ K_2 = -\frac{K_2}{\lambda_1 - \lambda_2} \]  

(9)

The same as power-logarithmic singularities, a dimensionless intensity of singularity in 2-real singularities form is given by

\[ k_i = \frac{K_i}{\tau_0 h^\lambda} \]  

(10)

Figures 19 and 20 show the dimensionless intensities of singularity in 2-real singularities form as Eq. (10). It is clearly shown that the dimensionless intensities of singularity can be arranged to be the same line for both \( 1^{st} \) and \( 2^{nd} \) intensity of singularity, if the results are modified to be in 2-real singularities form. Considering the magnitudes of the intensities of singularity in 2-real singularities form, \( k_i \), the magnitudes are quite large. In addition, \( k_1 \) and \( k_2 \) are nearly the same values but different in their signs, positive and negative.

Finally, in case of long model (\( L/h \) more than 20), the variation rate of \( k_i \) against \( L/h \) decreases and the results approach to the values corresponding to material combinations when the model is sufficiently long. This characteristic is similar to 2-real singularities cases in Luangarpa and Koguchi\(^{(14)}\).

\[ \]
Fig. 17 The 1\textsuperscript{st}-dimensionless intensity of singularity, $\kappa_1$, against $L/h$

Fig. 18 The 2\textsuperscript{nd}-dimensionless intensity of singularity, $\kappa_2$, against $L/h$

Fig. 19 The 1\textsuperscript{st}-dimensionless intensity of singularity, $\kappa_1$, against $L/h$

Fig. 20 The 2\textsuperscript{nd}-dimensionless intensity of singularity, $\kappa_2$, against $L/h$

4. Conclusion

The enriched FEM was developed for calculating the intensity of singularity for power-logarithmic singularities. The results from the enriched FEM were agreed with those using the extrapolation for stress distribution based on FEM.

The accuracy of the results could be improved by changing from 4-node to 8-node elements. In addition, using 8-node element, the intensity of singularity was not a function of the enriched area. However, the accuracy of the results increased when smaller elements were used.

The results for various lengths and thicknesses showed that the intensities of singularity were constant with increasing lengths but the intensities of singularity decrease with decreasing thicknesses. Finally, the intensities of singularity for various lengths and thicknesses could be well arranged by changing to be the form of dimensionless 2-real singularities.

References


**Appendix A**

The extrapolation following Munz and Yang (1993)’s fitting technique for stress distribution is considered for calculating the intensities of singularity. The method for calculating $K_1$ and $K_2$ is given by

$$
\Pi_y = \frac{1}{M} \sum_{i=1}^{M} \left[ \sigma_y^{ex}(r_i, \theta) - \sigma_y(r_i, \theta) \right]^2
$$

where $M$ is the number of points used for determining $K$, $\sigma_y^{ex}(r_i, \theta)$ is a stress calculated from the conventional FEM.
The magnitude of $K$ is obtained by the least squares method as

$$\frac{\partial \Pi}{\partial K} = 0$$  \hspace{1cm} (12)

Under isotropic elastic materials and plane strain condition, the stresses, $\sigma_{ij}^{FE}(r, \theta)$, are calculated by a conventional FEM (Marc) with extremely refined meshes near the singular point, and then the intensity of singularity is determined. 8-node quadrilateral elements (total 7,500 elements), are employed. Figures 21 and 22 show the results of the intensities of singularity, $K_1$ and $K_2$, against $r$ (distance from singular point) by fitting the stresses, $\sigma_{ij}^{FE}(r, \theta)$, with angular functions derived from eigenanalysis (the results from section 3.2 and 3.3). In this case, the singular stresses can be expressed as follows:

$$\sigma_{ij} = 1.609 r^{-0.3474} h_{801}(\theta) + 0.244 r^{-0.3474} [-\ln(r) h_{801}(\theta) + h_{801}(\theta)]$$  \hspace{1cm} (13)

where the unit of $r$ is mm and the unit of $\theta$ is radian.

Fig. 21 The 1st-intensity of singularity, $K_1$, by extrapolation

Fig. 22 The 2nd-intensity of singularity, $K_2$, by extrapolation