Adaptive Polyhedral Mesh Generation Method for Compressible Flows*

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Abstract

A method for generating a solution-adaptive mesh is proposed. It uses the adaptive mesh refinement (AMR) method and polyhedral mesh, which, as an unstructured mesh, is better suited for complicated bodies than triangular mesh. Furthermore, polyhedral mesh with hundreds of edges is more effective for generating adaptive mesh. Application of the AMR method to a test problem demonstrated that the adaptive mesh had a resolution about 30 times that of the initial mesh. Application of the proposed mesh generation method to compressible flows with a shock wave showed that using an artificial value for making the division judgment instead of the derivative value is feasible and highly effective.

Key words: Numerical Method, Compressible Flow, Adaptive Mesh Refinement, Unstructured Mesh, Polyhedral Mesh

1. Introduction

A solution-adaptive mesh can be used, for example, to accurately compute the collapse to very small length scales and the rapid loss of regularity of a time-evolving solution. Many methods for generating a solution-adaptive mesh have been proposed. The one to use depends on the flow field to which it will be applied. An all-purpose method would eliminate any confusion about which method to use, but such a method is not realistic. Therefore, methods should be developed targeting specific fields.

We have developed a more effective method for generating a solution-adaptive mesh that is targeted at compressible flows. Our goal was to develop a method that achieves high resolution even with only a few grid points. We achieved this goal by doing two things in particular. First, we greatly reduced the number of grid points in the domain where the change in the flow is small. For example, the number of grid points in front of the first shock wave in a supersonic flow was reduced to almost zero. Second, we expressed a sudden change in a flow by using the minimum number of grid points. As in a shock capturing method, the shock wave is computed. (Of course, shock capturing method is not actually used.)

Two functions are needed to develop a solution-adaptive mesh generation method: "mesh having high flexibility" and "ability to adapt to a flow". We first listed up, from the viewpoint of high flexibility, the unstructured mesh generation methods used to form triangular meshes. Then, with the goal of high flexibility, we identified a suitable polyhedral mesh [1–2], i.e., a super-polyhedral mesh having element cells with several hundred edges.
In addition, from the viewpoint of the ability to adapt to a change in the flow field, we identified a solution-adaptive mesh generation method.

However, a solution-adaptive mesh generation method is one case, and it is not a thing limited to the necessary mesh generation technology. There are various kinds of solution-adaptive mesh generation methods, including moving grid line methods [3–4] and overset grid methods [5]. Considering compatibility with a polyhedral mesh, we used an adaptive mesh refinement [6–7] method that not only performs mesh refinement but also performs mesh merging. We used this method because our objective includes excluding the limit of the mesh. Although the definition of the individual computational elements is complicated with this method, the number of elements needed for computation is less than with other methods.

In this paper, we demonstrate the validity of the mesh generation method with adaptive mesh refinement using a super-polyhedral mesh. We also show that using a pseudo-function for the flow field information instead of actual information enables a more effective mesh to be generated.

2. Polyhedral mesh

2.1. Mesh generation

We developed an effective solution-adaptive mesh generation method that uses a polyhedral mesh as a multiplex element mesh. The key feature of a polyhedral mesh is able to use any cells in which each side is connected as sum of normal vectors of each side is zero. In particular, a super-polyhedral mesh with several hundred sides has great flexibility. Here, in this paper, it is dealt with for two-dimensional inviscid compressible flows. Thus, it is not actually a polyhedral mesh; it is a polygonal mesh. However, we plan to extend it to three-dimensional flows in the near future. Thus, to prevent confusion, these meshes are referred to as "polyhedron" here.

The initial polyhedral mesh can be made using conventional methods. First, a triangular mesh, as shown in Fig. 1(a), is created, for example, by using the Delaunay method, the advancing front method, or another geometric method. Next, as shown in Fig. 1(b), vertices are plotted in the center of the elements, and vertices are added to the middle and edge points along the boundary. Then, each vertex is connected as shown in Fig. 1(c). Finally, a polyhedral mesh is obtained by removing the triangular sides except for the boundary sides, as shown in Fig. 1(d).

![Fig. 1 Procedure for making initial polyhedral mesh](image-url)
2.2. Numerical method

We used the cell-centered finite volume method for the Euler equation in the two-dimensional coordinate system. The flux vectors for a cell-centered polyhedral mesh were evaluated in the same way as for the vertex-centered triangular method. They were estimated using the Roe flux difference splitting scheme [8]. Primitive variables $Q$ for a higher-order approach were reconstructed using the cell average values given by Eq. (1). The $Q^-_{ij}$ and $Q^+_{ij}$ are on the boundary, as shown in Fig. 2.

\[
Q^-_{ij} = Q_i + \Phi_i \nabla Q_i \cdot (r_{ij} - r_i)
\]
\[
Q^+_{ij} = Q_j + \Phi_j \nabla Q_j \cdot (r_{ij} - r_j),
\]

where $r$ is the position vector. Element $j$ is adjacent to element $i$, so $r_{ij}$ is the position vector for the boundary between elements $i$ and $j$. The primitive variable gradient, $\nabla Q_i$, is given by Eq. (2), and $\Phi$ is defined as Venkatakrishnan’s limiter [9].

\[
\nabla Q_i = \frac{1}{S_i} \sum_{j \in \partial S_i} \frac{1}{2} (Q_i + Q_j) n_{ij} \Delta l_{ij},
\]

where $S_i$, $n_{ij}$, and $\Delta l_{ij}$ are the area of element $i$, the outward normal vector from element $i$ and the length of the edge between elements $i$ and $j$.

3. Adaptive mesh refinement using polyhedral mesh

3.1. Procedure

The adaptive mesh refinement (AMR) process is outlined in Fig. 3. As shown on the left-hand side, at the $n^{th}$ step, the flow variables $q^n$ (e.g., pressure and density) and coordinate data $r^n$ (in this case, the $x$, $y$ positions) are given. The $q^n$ are the results after computation on mesh $r^n$. The $q^n$ and $r^n$ are replaced with $q'^n$ and $r'$ and sent to the AMR process. Then the AMR process creates new flow variables $q'^{n+1}$ and new coordinate data $r'^{n+1}$ as $n+1$ step solutions. The $n+1$ step solutions for a steady flow are the final result.

As shown on the right-hand side, the derivative of flow value $\Delta q$ is calculated. Although the flow value is not limited, the density or pressure value is often chosen. Here, if maximum $\Delta q$ is greater than a parameter of the user appointment $\Delta q_{User}$, the AMR-process is finished. The cells and sides that should be divided and combined are identified on the basis of the calculated derivative value. These cells and sides are divided and combined, resulting in a new mesh, data $r'^{n+1}$. Flow value $q'^{n+1}$ is calculated on new...
cells. This process ends when inner iteration number $\nu$ exceeds the user set value, $\nu_{\text{User}}$.

![Fig. 3 AMR process](image)

### 3.2. Division and combination

The AMR method was selected because it is compatible with a polyhedral mesh and it has large generalization for mesh generation. Both refinement and recombination are used in this method.

A cell should be divided on the basis of its characteristics. We created a division method that combines two conventional division methods. One divides a cell into two cells by adding a line ("two-division method"), as shown in Fig. 4(a). The line can be placed at any angle. It is therefore placed perpendicularly to obtain the largest derivative of the flow value. The second method divides a cell into several cells by adding a vertex at the center of the cell ("multipartite method"), as shown in Fig. 4(b). In this case, a cell is divided into several cells at once. Our newly created method has two main steps.

1. Calculate gradients of flow variables with respect to adjacent elements. The maximum for each gradient is defined as $\Delta q_{\text{max}}$, and the mean value of each gradient is defined as $\Delta q_{\text{mean}}$.

2. Divide the cell if $\Delta q_{\text{max}} > \Delta q_{\text{User}}$, where $\Delta q_{\text{User}}$ is a user set threshold value. Otherwise, do not divide the cell.
   - Use the two-division method if $\Delta q_{\text{max}}$ is greater than $c_{\text{User}} \Delta q_{\text{mean}}$, where $c_{\text{User}}$ is a user set threshold value. Divide the cell by placing a line parallel to a side having $\Delta q_{\text{max}}$.
   - Use the multipartite method if $\Delta q_{\text{max}}$ is less than $c_{\text{User}} \Delta q_{\text{mean}}$. Divide the cell by placing a vertex at the center of gravity of the cell.

If a vertex cannot be placed at the center of gravity of the cell due to the skewed shape of the polyhedral mesh (for example, in the case of remeshing for merged polyhedron), the two-division method is used instead of the multipartite method.

The procedure for combination is simple. If the calculated gradient of a flow variable with respect to an adjacent element is less than a user set threshold value, the side is removed.
3.3. Test problem

We applied the AMR method to a test problem in which there was a static condition: low pressure inside a circle and high pressure outside the circle, as shown in Fig. 5. The resolution of the derivative is estimated by refining the mesh around the circle using the AMR method. The total number of cells can be reduced by combining the mesh except for the portion on the boundary of the circle. The evaluation metric was how sharply the mesh caught the circle with the fewest number of the cells possible. Again, In this case, there in no flow, for static condition, resolution is estimated in this method.

Figure 6(a) shows the initial mesh. The meshes constructed using the AMR method are shown in Figs. 6(b) to (f). There is a refined mesh around the circle. Then huge cell except boundary of circle is seen. This is a super-polyhedral mesh, which achieves high efficiency.

Figure 7(a) shows the pressure distribution estimated for the initial mesh. The pressure isoline is heavy and fuzzy because the cell is course. Figure 7(b) shows the pressure distribution estimated for the final mesh (Fig. 6(f)). The pressure isoline is thin and sharp. The number of vertices, cells, and sides on the adaptive mesh are the same as for the initial mesh due to having repositioned the grid points effectively. Nevertheless, the adaptive mesh had a resolution about 30 times that of the initial mesh.
4. Application to supersonic flow around a cylinder

We applied our super-polyhedral mesh generation method to a practical flow, a supersonic flow around a circle. The computational domain was a rectangle with a length about 20 times the circle diameter. (The vertical and horizontal lengths were almost equal.) The inlet Mach number was 2.0, so a bow shock wave was generated in front of the circle. Figure 8(a) shows the initial mesh and corresponding solution, and Fig. 8(b) shows the adaptive mesh and corresponding solution.

The adaptive mesh was refined in the domain of the shock wave. Therefore, the shock wave on the adaptive mesh was sharply captured in comparison with the solution for the initial mesh. The very large computational element evident in the forward domain of the shock wave (upper half of Fig. 8(b)) is the super-polyhedral mesh.
However, the mesh on the shock wave was not sufficiently refined, particularly the mesh on the tail of the shock wave. The mesh in front of the circle was fully combined, meaning that a super-polyhedral mesh was generated. The mesh behind the circle was not. The control for refinement and combination thus needs to be improved.

This control deficiency was due to too few cells being divided. In other words, judgment using the derivative value reduced the number of divided cells, as shown in Fig. 9(a). Furthermore, the number of cells combined was reduced due to reducing the threshold value to supplement that. There was thus a vicious circle. Therefore, instead of using the derivative value itself, we used an artificial value that exists in a wide range, as shown in Fig. 9(b). The artificial value $\tilde{q}$ is given by Eq. (3), where $\varphi(0 \leq \varphi \leq 1)$ is a user set parameter. Use of this artificial value should not only improve the division but also improve the combination.

$$\tilde{q}_i = \varphi q_i + (1 - \varphi) \frac{\sum_{j=1}^{N_c} q_j \Omega_j}{\sum_{j=1}^{N_c} \Omega_j}$$  \hspace{1cm} (3)

Figure 10 shows the resulting improved adaptive mesh. The mesh over the whole area of the shock wave was refined. Furthermore, the mesh was combined behind the circle. In comparison with the result with the conventional approach (Fig. 9(b)), the improved method caught the shock wave sharply and efficiently. Table 1 shows the stagnation density in front of the circle and the error from the exact solution. The stagnation density obtained with the improved adaptive mesh agrees quantitatively with the exact solution. This result demonstrates the validity of using the improved adaptive mesh.
5. Conclusion

A solution-adaptive mesh generation method using a super-polyhedral mesh has been developed. A test problem demonstrated the validity of using the AMR method. Application to an actual problem showed that using an artificial value for making the division judgment instead of the derivative value is feasible and highly effective.

References