Over-Range Collocation Analyses of the Linear Elastic Cantilever Beam Problem*

Yong-Ming GUO**, Wataru USHIJIMA** and Shunpei KAMITANI**
**Graduate School of Science and Engineering, Kagoshima University,
1-21-40 Korimoto, Kagoshima City, 890-0065 Japan
E-mail: guoy@mech.kagoshima-u.ac.jp

Abstract
The linear elastic cantilever beam problem is analyzed by using the over-range collocation method (ORCM). Because the over-range points are used only in interpolating calculation, no over-constrained condition is imposed into the solved problems. While the over-range points can be used in interpolating calculation of boundary points, so that the unsatisfactory issue of the positivity conditions of boundary points in collocation methods can be avoided. Convergence studies show that the ORCM possesses good convergence for both the displacement and deformation energy, and quite accurate numerical results have been obtained.

Key words: Cantilever Beam, Meshless Method, Collocation Method, Positivity Conditions, Over-Range Points

1. Introduction
The early representatives of meshless methods are the diffuse element method (1), the element free Galerkin method (2), the reproducing kernel particle method (3), the finite point method (4), the hp-clouds method (5), the partition of unity method (6), the meshless local Petrov-Galerkin (MLPG) approach (7), and the local boundary integral equation method (8). In most meshless techniques, however, complicated non-polynomial interpolation functions are used which render the integration of the weak form rather difficult. Failure to perform the integration accurately results in loss of accuracy and possibly stability of solution scheme. The integration of complicated non-polynomial interpolation function also costs much CPU time. The collocation method has no issues of the integration scheme, the integration accuracy and the integration CPU time. Several collocation methods have been proposed in the literature. Onate et al. (4) have proposed a finite point method. Aluru (9) has presented a point collocation method. Jin, Li and Aluru (10) have shown the robustness of collocation meshless methods can be improved by ensuring that the positivity conditions are satisfied when constructing approximation functions and their derivatives. Atluri, Liu and Han (11) have presented a MLPG mixed collocation method, and shown that the MLPG mixed collocation method is more efficient than the other MLPG implementations, including the MLPG finite volume method. Atluri, Liu and Han (12) have proposed a finite difference method, within the framework of the MLPG approach, for solving solid mechanics problems. Wu, Shen and Tao (13) have used the MLPG collocation method to compute two-dimensional heat conduction problems in irregular domain. Chantasiriwan (14) has provided results of using the multiquadric collocation method to solve the lid-driven cavity flow problem. Wen and Hon (15) have performed a geometrically nonlinear analysis of Reissner-Mindlin plate by using a meshless collocation method based on the smooth radial

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basis functions. Yang et al. \cite{16} have introduced a computational procedure based on meshless generalized finite difference method.

But, the robustness (the level of influence by the number of nodes and the node distribution) of the collocation methods is an issue especially when scattered and random points are used. To improve the robustness of the collocation methods, Nayroles, Touzot and Villon\cite{1} suggested that the positivity conditions (the conditions on the shape function and its second partial derivatives, see \S 2.3 of this paper) could be important when using the collocation methods. It has been shown that the satisfaction of the positivity conditions ensures the convergence of the finite difference method with arbitrary irregular meshes for some class of elliptic problems\cite{17}. Jin, Li and Aluru\cite{10} have proposed techniques, based on modification of weighting functions, to ensure satisfaction of positivity conditions when using a scattered set of points. For boundary points, however, the positivity conditions cannot be satisfied. In this paper, the linear elastic cantilever beam problem with a parabolic-shear end load is analyzed by using the over-range collocation method (ORCM)\cite{18}, in which by introducing some collocation points that are located at outside of domain of the analyzed body, unsatisfactory issue of the positivity conditions of boundary points in collocation methods can be avoided.

2. Principle

2.1 The MLS Approximation with Kronecker-Delta Property

In the classical moving least-square (MLS) approximation, the shape functions have no Kronecker-delta property, so that the essential node condition cannot be imposed on boundaries. In this paper, a modified MLS approximation is used, its shape functions have Kronecker-delta property. Therefore, the unsatisfactory issue of the essential node condition can be avoided in the modified MLS approximation.

Consider a small domain \( \Omega_x \), the neighborhood of a point \( x \), which is located in \( \Omega \) or on \( \Gamma \). Over a number of randomly located nodes \( \{x_i\}, i = 1, 2, \ldots, n \), the MLS approximation \( u^h \) of \( u \) can be defined by

\[
u^h = p^T(x) \alpha, \quad \forall x \in \Omega_x
\]

where \( p^T(x) = [p_1(x) \quad p_2(x) \quad \cdots \quad p_m(x)] \) is a complete monomial basis of order \( m \) which is a function of the space coordinates \( x = [x \quad y \quad z]^T \). \( \alpha \) is a vector of unknown polynomial coefficients.

\[
\alpha = [\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_m]^T
\]

For example, for a 2-D problem,

\[
p^T(x) = [1 \quad x \quad y \quad x^2 \quad xy \quad y^2]
\]

this is a quadratic basis, and \( m=6 \).

A weighted least-square solution is obtained for \( \alpha \) from the following system of \( n \) equations in \( m \) unknown \( (n \) is larger than \( m)\):

\[
u^h = H \alpha
\]

where

\[
u^h = [u^h_1 \quad u^h_2 \quad \cdots \quad u^h_n]^T
\]
is a vector of the nodal MLS approximation of function \( u \), and

\[
H = \begin{bmatrix}
p^T(x_1) \\
p^T(x_2) \\
\vdots \\
p^T(x_n)
\end{bmatrix}_{n \times m}
\]  

(6)

The classical least-square solution of the above over-constrained system does not guarantee exact satisfaction of any of the equations of Eq. (4). Non-satisfaction of the first equation would then mean \( \alpha^h \neq p^T(x_1)\alpha \). Hence, a different approach \(^{(19)}\) to weighted least-squares solution can be adopted: Out of the \( n \) equations of Eq. (4), let the first equation (corresponding to node 1) be satisfied exactly and the rest in the least-square sense. This is done by using the first equation to eliminate \( \alpha \) from the rest of equations:

\[
\alpha_1 = u_1^h - (\alpha_2 x_1 + \alpha_3 y_1 + \alpha_3 x_1^2 + \alpha_5 x_1 y_1 + \alpha_6 y_1^2)
\]  

(7)

Substituting for \( \alpha_1 \) in Eq. (4), the reduced system of equations can be obtained:

\[
\bar{u}^h = \bar{H}\bar{\alpha}
\]  

(8)

where

\[
\bar{u}^h = \begin{bmatrix}
u_2^h - u_1^h \\
u_3^h - u_1^h \\
\vdots \\
u_n^h - u_1^h
\end{bmatrix}^T
\]  

(9)

\[
\bar{H} = \begin{bmatrix}
x_2 - x_1 & y_2 - y_1 & x_2^2 - x_1^2 & x_2 y_2 - x_1 y_1 & y_2^2 - y_1^2 \\
x_3 - x_1 & y_3 - y_1 & x_3^2 - x_1^2 & x_3 y_3 - x_1 y_1 & y_3^2 - y_1^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
x_n - x_1 & y_n - y_1 & x_n^2 - x_1^2 & x_n y_n - x_1 y_1 & y_n^2 - y_1^2
\end{bmatrix} = \begin{bmatrix}
p^T(x_2) \\
p^T(x_3) \\
\vdots \\
p^T(x_n)
\end{bmatrix}
\]  

(10)

\[
\bar{\alpha} = [\alpha_2 \alpha_3 \cdots \alpha_m]^T
\]  

(11)

The coefficient vector \( \bar{\alpha} \) is determined by minimizing a weighted discrete \( L_2 \) norm, defined as:

\[
J = \sum_{i=2}^{n} w(x_i) \left[ p^T(x_i)\bar{\alpha} - \bar{u}_i \right]^2 = \left[ \bar{H}\bar{\alpha} - \bar{u} \right]^T \bar{W} \left[ \bar{H}\bar{\alpha} - \bar{u} \right]
\]  

(12)

\[
\bar{u}_i = \bar{u}_1 - \bar{u}_i, \quad i = 2, 3, \ldots, n
\]

\[
\bar{u} = [\bar{u}_2 - \bar{u}_1 \quad \bar{u}_3 - \bar{u}_1 \quad \cdots \quad \bar{u}_n - \bar{u}_1]^T
\]  

(13)

\[
\bar{w} = \begin{bmatrix}
w(x_2) \\
w(x_3) \\
\vdots \\
w(x_n)
\end{bmatrix}_{\text{(n-1)\times(n-1)}}
\]  

(15)

where \( w(x) \) is the weight function, with \( w(x) > 0 \) for all nodes in the support of \( w(x) \) (the support is considered to be equal to \( \Omega_x \) in this paper), \( x_i \) denotes the value of \( x \) at node \( i \), and the matrices \( \bar{W} \) is defined as

Minimizing \( J \) in Eq. (12) with respect to \( \bar{\alpha} \) yields

\[
\bar{\alpha} = \bar{A}^{-1}\bar{B}\bar{u}
\]  

(16)

\[
\bar{B} = \bar{H}^T\bar{W}
\]  

(17)

\[
\bar{A} = \bar{B}\bar{H}
\]  

(18)
Substituting Eq. (16) into Eq. (8) gives a relation which may be written as the form of an interpolation function, as

\[ \tilde{u}_h = \tilde{H}A^{-1}B\tilde{u} \]  

(19)

Equation (7) can be rewritten as:

\[ \alpha_1 = u_h^1 - \mathbf{s}(\mathbf{x}_1) \bar{\alpha} \]  

(20)

\[ \mathbf{s}(\mathbf{x}_1) = [x_1 \ y_1 \ x_1^2 \ x_1y_1 \ y_1^2] \]  

(21)

Equation (1) can be written as:

\[ u_h = \alpha_1 + \mathbf{s}(\mathbf{x}) \bar{\alpha} \]  

(22)

\[ \mathbf{s}(\mathbf{x}) = [x \ y \ x^2 \ xy \ y^2] \]  

(23)

Substituting Eq. (16) and Eq. (20) into Eq. (22), the following equation can be obtained:

\[ u_h = u_h^1 + \mathbf{q}(\mathbf{x})A^{-1}B\tilde{u} \]  

(24)

\[ \mathbf{q}(\mathbf{x}) = \mathbf{s}(\mathbf{x}) - \mathbf{s}(\mathbf{x}_1) \]  

(25)

Because

\[ \mathbf{q}(\mathbf{x}_1) = 0 \]  

(26)

\[ u_h(\mathbf{x}_1) = u_h^1 \]  

(27)

\[ \tilde{u} \] may be defined as

\[ \tilde{u} = [\tilde{u}_1 \ \tilde{u}_2 \ \cdots \ \tilde{u}_n]^T \]  

(28)

then, from Eq. (24), the following equation may be obtained:

\[ u_h = \mathbf{N}(\mathbf{x})\tilde{u} \]  

(29)

\[ \mathbf{N}(\mathbf{x}) = \begin{bmatrix} 1 - \mathbf{q}(\mathbf{x}) & A^{-1} & \mathbf{B} & 1 \\ 1 & \mathbf{q}(\mathbf{x}) & A^{-1} & \mathbf{B} \\ & & & \mathbf{1} \end{bmatrix} \]  

(30)

In Eq. (30), \( \mathbf{1} \) is vector of dimension (n-1) with all entries being equal to unity.

Recall from Eq. (26), using this result in Eq. (30), the Kronecker-delta property of \( \mathbf{N}(\mathbf{x}) \) may be established:

\[ \mathbf{N}(\mathbf{x}_1) = [1 \ 0 \ 0 \ \cdots \ 0] \]  

(31)

It means that at node 1, the shape function for node 1 takes a value of unity and all other shape function take zero values. Therefore, Eq. (30) is the shape functions of the MLS approximation with Kronecker-delta property.

From Eq. (29) and Eq. (27), the following result can be obtained:

\[ \tilde{u}_1 = u_h(\mathbf{x}_1) = u_h^1 \]  

(32)

In this paper, the weight functions \( w(\mathbf{x}) \) may use a spline function as follows:
where $d = |\mathbf{x} - \mathbf{x}_1|$ is the distance from point $\mathbf{x}$ to the center node $\mathbf{x}_1$, and $r$ is the radius of $\Omega_x$, which is taken as a circle for a 2-D problem and its center is the point $\mathbf{x}_1$.

2.2 The Local Coordinate System

As anisotropy of the point distribution in $\Omega_x$, matrix $\mathbf{A}$ in Eq. (18) becomes ill-conditioned and the quality of the approximation deteriorates. In order to prevent such undesirable effect, a local coordinate system $\xi, \eta$ is chosen with origin at the node $\mathbf{x}_1$ for a 2-D problem,

$$\xi = \frac{x - x_1}{R_x}, \quad \eta = \frac{y - y_1}{R_y}$$

where $R_x$ and $R_y$ denote maximum distances along $x$ and $y$ measured from the point $\mathbf{x}_1$ to exterior nodes in $\Omega_x$. In Eq. (33a), the spline function has now the following form in terms of the local coordinates:

$$w(\xi) = 1 - 6 \left( \frac{\xi^2 + \eta^2}{\rho^2} \right) + 8 \left( \frac{\xi^2 + \eta^2}{\rho^2} \right)^{3/2} - 3 \left( \frac{\xi^2 + \eta^2}{\rho^2} \right)^{3/2}$$

$\rho = 6$ is used in this paper and as usual, $-1 \leq \xi \leq 1, -1 \leq \eta \leq 1$.

The matrices $\mathbf{B}$ and $\mathbf{A}$ are no longer dependent on the dimensions of $\Omega_x$, then the shape function $\mathbf{N}(\xi)$ is also no longer dependent on the dimensions of $\Omega_x$. Because $q(\xi_1) = 0$, the Kronecker-delta property $\mathbf{N}(\xi_1) = [1 \ 0 \ 0 \ \cdots \ 0]$ for $\Omega_x$ of some different dimensions can be established, too. The approximate function is also expressed in terms of the local coordinate as

$$u^h(\xi) = \mathbf{N}(\xi) \hat{\mathbf{u}}$$

$\mathbf{A}^{-1}\mathbf{B}$ in Eq. (30) can be defined as $\mathbf{C}$:

$$\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$$

Then, from Eq. (30), entries of $\mathbf{N}(\mathbf{x})$ for the quadratic basis $(m=6)$ can be written as:

$$N_1(\mathbf{x}) = 1 - \left[ (x - x_1) \sum_{i=1}^{n-1} C_{3i} + (y - y_1) \sum_{i=1}^{n-1} C_{2i} + (x^2 - x_1^2) \sum_{i=1}^{n-1} C_{3i} \right]$$

$$+(xy - x_1y_1) \sum_{i=1}^{n-1} C_{4i} + (y^2 - y_1^2) \sum_{i=1}^{n-1} C_{5i}$$

$$N_{i+1}(\mathbf{x}) = \left( x - x_1 \right) C_{i1} + \left( y - y_1 \right) C_{2i} + \left( x^2 - x_1^2 \right) C_{3i} + \left( xy - x_1 y_1 \right) C_{4i} + \left( y^2 - y_1^2 \right) C_{5i} \quad (i = 1, \ 2, \ \cdots, \ n - 1)$$

where $C_{ji}$ $(j = 1, 2, \ \cdots, 5; \ i = 1, 2, \ \cdots, \ n - 1)$ are entries of $\mathbf{C}$. 
At the point $x_1$, because $\xi_1 = 0, \eta_1 = 0$, then the first derivatives of the shape function with the local coordinates can be obtained from Eqs. (38) and (39):

$$\frac{\partial N(\xi_1)}{\partial \xi} = \begin{bmatrix} -\sum_{i=1}^{n-1} C_{1i} & C_{11} & C_{12} & \cdots & C_{1(n-1)} \end{bmatrix}$$ \hspace{1cm} (40)

$$\frac{\partial N(\eta_1)}{\partial \eta} = \begin{bmatrix} -\sum_{i=1}^{n-1} C_{2i} & C_{21} & C_{22} & \cdots & C_{2(n-1)} \end{bmatrix}$$ \hspace{1cm} (41)

From Eqs. (40) and (41), we may see that formulas of the shape function derivatives with the local coordinates are very simple, and in fact, it is a merit of the ORCM using the local coordinates.

### 2.3 Collocation Scheme

The positivity conditions $^{(1), (10)}$ on the approximation function $N_i(x)$ of Eq. (30) and its second derivatives are stated as,

$$N_i(x_j) \geq 0$$ \hspace{1cm} (42)

$$\nabla^2 N_i(x_j) \geq 0, j \neq i$$ \hspace{1cm} (43)

$$\nabla^2 N_i(x_i) < 0$$ \hspace{1cm} (44)

where $N_i(x_j)$ is the approximation function of a point $i$ evaluated at a point $j$.

It has been shown that the significance of the positivity conditions in meshless collocation approaches, and violation of the positivity conditions can significantly result in a large error in the numerical solution $^{(10)}$.

In the classic collocation methods, for a point $x_1$ on $\Gamma$, because no over-range point is used in its $\Omega_x$, the positivity conditions on the boundary point cannot be satisfied. To show the data, we have calculated the positivity conditions of 2-D problem in this case. In Figs. 1 and 2, nodes of $n=3 \times 3=9$ is used for each small domain $\Omega_x$ without over-range points, and the red point on $\Gamma$ is node $x_1$. The local coordinate system is used, then the value of shape function $N_i(\xi_j), (j = 1, 2, \cdots, 9)$ and the values of $\nabla^2 N_i(\xi_j), (j = 1, 2, \cdots, 9)$ are shown in Tables 1 and 2 ( where subscript symbol 1 means node $x_1$ in $\Omega_x$ ). From Tables 1 and 2, it is seen that the positivity conditions of these nodes are not satisfied.

To overcome the demerit of the classic collocation methods, we have proposed the ORCM. Besides the collocation points over $\Omega$, let us assume other collocation points located at outside of $\Omega$ and call them over-range points, at which no satisfaction of any governing partial differential equation or boundary condition is needed. By introducing some over-range points of $\Omega$ in the $\Omega_x$, the unsatisfactory issue of the positivity conditions of the boundary point can be avoided in the ORCM. To show the data, we have also calculated the positivity conditions of regular nodal model in this case. Figure 3 shows $\Omega_x$.
with over-range points of \( \mathbf{x}_1 \) (the red node) on \( \Gamma \) and over-range nodes (the blue nodes) in \( \Omega_x \). The value of shape function \( N_i(\xi_j) \), \( (j = 1, 2, \ldots, 9) \) and the values of \( \nabla^2 N_i(\xi_j) \), \( (j = 1, 2, \ldots, 9) \) in this case are shown in Table 3. From Table 3, it is seen that the positivity conditions of these nodes are satisfied. Because the over-range points are used only in interpolating calculation in \( \Omega_x \) of \( \mathbf{x}_1 \), and themselves never become any node \( \mathbf{x}_1 \), the over-range points are not used in physics sense. It can be understood that no over-constrained condition is imposed into the solved problems.

### Table 1

The values of \( N_i(\xi_j) \) and \( \nabla^2 N_i(\xi_j) \) without using over-range points in case of line boundary

<table>
<thead>
<tr>
<th>( j )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_i(\xi_j) )</td>
<td>1.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>( \nabla^2 N_i(\xi_j) )</td>
<td>1.285</td>
<td>1.357</td>
<td>-2.064</td>
<td>1.707</td>
<td>-3.872</td>
<td>0.587</td>
<td>1.357</td>
<td>-2.064</td>
<td>1.707</td>
</tr>
</tbody>
</table>

### Table 2

The values of \( N_i(\xi_j) \) and \( \nabla^2 N_i(\xi_j) \) without using over-range points in case of corner boundary

<table>
<thead>
<tr>
<th>( j )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_i(\xi_j) )</td>
<td>1.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>( \nabla^2 N_i(\xi_j) )</td>
<td>3.439</td>
<td>-2.177</td>
<td>2.738</td>
<td>-2.177</td>
<td>-5.047</td>
<td>-0.776</td>
<td>2.738</td>
<td>-0.776</td>
<td>2.038</td>
</tr>
</tbody>
</table>

In order to give universality, let us assume a scalar problem governed by a partial differential equation:
\[ D(u) = b, \quad \text{in } \Omega \]  

with boundary conditions

\[ T(u) = t, \quad \text{on } \Gamma_t \]
\[ u - u_c = 0, \quad \text{on } \Gamma_u \]

\[ \text{to be satisfied in a domain } \Omega \text{ with boundary } \Gamma = \Gamma_t \cup \Gamma_u, \text{ where } D \text{ and } T \text{ are appropriate differential operators, } u \text{ is the problem unknown function, } b \text{ and } t \text{ are external forces or sources acting over } \Omega \text{ and along } \Gamma_t, \text{ respectively. } u_c \text{ is the assigned value of } u \text{ over } \Gamma_u. \]

As the MLS approximation \( u^b \) of \( u \), Eq. (29) can be substituted into Eqs. (45), (46) and (47). Therefore, the algebraic equations on \( \hat{u}_i \) (where \( f \) is the total number of freedoms over \( \Omega \)) can be obtained, and the number of the equations can be selected as \( N_f \) by using the collocation scheme, too.

Consider taking some collocation points in \( \Omega \), at which Eq. (45) is satisfied, and some collocation points on \( \Gamma_t \), at which both Eq. (45) and Eq. (46) are satisfied, as well as some collocation points on \( \Gamma_u \), at which both Eq. (45) and Eq. (47) are satisfied. Besides the collocation points over \( \Omega \), let us assume other collocation points located at outside of \( \Omega \). The over-range points can be used in interpolating calculation of boundary points, so that the unsatisfactory issue of the positivity conditions of boundary points in the classic collocation methods can be avoided.

Let us assume that the number of points in domain is \( K_d \), the number of boundary points is \( K_b \) and the number of over-range points is \( K_o \). Because the number of equations of the ORCM is \( (K_d + K_b + K_o) \) (Eq. (45)) \( + K_b \) (Eq. (46) or Eq. (47)), by taking the same number of the equations with that of the unknown variables: \( (K_d + K_b + K_o) = (K_d + K_b) + K_b \), we obtain that the number of the over-range points \( K_o \) must be equal to the number of boundary points \( K_b \).

3. Results of Analyses

3.1 Method of Error Estimation

The linear elastic cantilever beam problem with a parabolic-shear end load is analyzed by using the ORCM, and its numerical solutions are compared with the exact solutions. For the purpose of error estimation and convergence studies, the displacement norm \( \|u\| \) and energy norm \( \|\varepsilon\| \) are calculated. These norms are defined as

\[ \|u\| = \left( \int_{\Omega} u^T \cdot u \, d\Omega \right)^{1/2} \]  
\[ \|\varepsilon\| = \left( \frac{1}{2} \int_{\Omega} \sigma^T \cdot \sigma \, d\Omega \right)^{1/2} \]

where \( u = [u \ v]^T \) is the displacement vector, and \( \varepsilon = [\varepsilon_{11} \ \varepsilon_{22} \ \gamma_{12}]^T \) and \( \sigma = [\sigma_{11} \ \sigma_{22} \ \sigma_{12}]^T \) are the strain vector and stress vector, respectively.

The relative errors for \( \|u\| \) and \( \|\varepsilon\| \) are defined as

\[ R_u = \frac{\|u_{num} - u_{exa}\|}{\|u_{exa}\|} \]  
\[ R_\varepsilon = \frac{\|\varepsilon_{num} - \varepsilon_{exa}\|}{\|\varepsilon_{exa}\|} \]

3.2 Results of the Linear Elastic Cantilever Beam Problem

Some boundary value problems have been analyzed by using the ORCM, and it has been shown that the ORCM works well for those boundary value problems \((18), (20)\).
In this paper, to study the convergence of the ORCM using regular nodal models and irregular nodal models, the linear elastic cantilever beam problem (see Fig. 4) is solved, for which the following exact solution of the displacement is given as

\begin{align}
    u &= -\frac{P}{6EI}\left(3x(2L-x) + (2+v)y(y-D)\right) \\
    v &= -\frac{P}{6EI}\left[x^2(3L-x) + 3v(L-x)\left(y - \frac{D}{2}\right)^2 + \frac{4+5v}{4}D^2x\right] \\
    l &= \frac{P^3}{12}
\end{align}

(52)  \hspace{1cm} (53)  \hspace{1cm} (54)

The problem is solved for the plane stress case with height \(D=10\), length \(L=30\). Young’s modulus \(E=1\), Poisson ratio \(\nu=0.25\), and upward end load \(P=1\) are used. Boundary conditions of nodes on the left and the right boundaries (including the corner nodes) are chosen as \(u=u_{\text{exa}}, v=v_{\text{exa}}\), and boundary conditions of nodes on the top and the bottom boundaries are chosen as \(\sigma_{12}=0, \sigma_{22}=0\).

To meet the condition of \(K_o = K_b\), 8 nodes are added to the boundary points, which are located on the boundary near the four corners of the domain, for all nodal models in this paper. Regular nodal models of 173 (15 × 11 + 8) \((K_d = 11 \times 7, K_b = K_o = 48)\) nodes, 229 (17 × 13 + 8) \((K_d = 13 \times 9, K_b = K_o = 56)\) nodes and 293 (19 × 15 + 8) \((K_d = 15 \times 11, K_b = K_o = 64)\) nodes are used to study the convergence with nodal model refinement of the ORCM. Over-range points of one layer, which are located at outside of the four sides of the domain, are used. Figure 5(a) shows nodal distribution of the regular nodal model of 173 nodes. Irregular nodal models of 173 \((K_d = 77, K_b = K_o = 48)\) nodes, 229 \((K_d = 117, K_b = K_o = 56)\) nodes and 293 \((K_d = 165, K_b = K_o = 64)\) nodes are also used. Figure 5(b) shows nodal distribution of the irregular nodal model of 173 nodes. For the irregular nodal models, in \(\Omega_x\), 9 nodes are used as the same as the regular nodal models.
and other nodes within the $\Omega_x$ are not used in the $\Omega_x$. For the center node $\xi_1$ of $\Omega_x$, because $q(\xi_1) = 0$, the Kronecker-delta property $N(\xi_1) = [1 \ 0 \ 0 \ \cdots \ 0]$ can be established, too.

The results of the relative errors and convergences for the three regular nodal models are shown in Fig. 6. This figure shows that the ORCM works quite well, and the relative errors of both displacement norm and energy norm are very small. In addition, even so small number of nodes (173 nodes) is used, quite accurate numerical results are obtained, too.

The results of the relative errors and convergences for the three irregular nodal models are shown in Fig. 7. This figure shows that the ORCM using irregular nodal models does not quite work well as the ORCM using regular nodal models, while convergences on the relative errors $R_u$ and $R_e$ can be seen, too.

![Figure 6](image6.png)

**Fig. 6** Relative errors and convergences of regular nodal model ($n$ is total number of the nodes)

![Figure 7](image7.png)

**Fig. 7** Relative errors and convergences of irregular nodal model ($n$ is total number of the nodes)

Figure 8 shows values of $u$ at $x=0.5L$ by regular nodal model of 173 nodes. Figure 9 shows values of $v$ at $y=0.5D$ by regular nodal model of 173 nodes. Figures 10 and 11 show values of $\sigma_{11}$ and $\sigma_{12}$ at $x=0.5L$ by regular nodal model of 173 nodes, respectively. It can be seen that some accurate results for the displacements and the stresses, which do not need to use more nodes, are obtained by using the ORCM.
Fig. 8  Values of $u$ of regular nodal model of 173 nodes

Fig. 9  Values of $v$ of regular nodal model of 173 nodes

Fig. 10  Values of $\sigma_{11}$ for regular nodal model of 173 nodes
Figure 12 shows values of $u$ at $x=0.5L$ by the irregular nodal model of 173 nodes. Figure 13 shows values of $v$ at $y=0.5D$ by irregular nodal model of 173 nodes. Figures 14 and 15 show values of $\sigma_{11}$ and $\sigma_{12}$ at $x=0.5L$ by irregular nodal model of 173 nodes, respectively. Figure 16 shows values of $\sigma_{12}$ at $x=0.5L$ by irregular nodal model of 293 nodes. It can be seen that some accurate results (excepting some values of $\sigma_{12}$ for irregular nodal model (see Fig. 15)) for the displacements and stresses are also obtained, and larger numbers of nodes should be used in order to get smaller relative errors when using irregular nodal models.

Figures 17 and 18 show distributions of numerical values (by using regular nodal model of 293 nodes) and exact values of $\sigma_{11}$, respectively. Figures 19 and 20 show distributions of numerical values (by using regular nodal model of 293 nodes) and exact values of $\sigma_{12}$, respectively. By comparing the numerical solutions with the exact solutions, it can be seen that some more accurate distribution figures of the stresses are obtained by using the ORCM.
Fig. 13 Values of $v$ of irregular nodal model of 173 nodes

Fig. 14 Values of $\sigma_{11}$ for irregular nodal model of 173 nodes

Fig. 15 Values of $\sigma_{12}$ for irregular nodal model of 173 nodes
Fig. 16  Values of $\sigma_{12}$ for irregular nodal model of 293 nodes

Fig. 17  Distribution of numerical values of $\sigma_{11}$ for regular nodal model of 293 nodes

Fig. 18  Distribution of exact values of $\sigma_{11}$

Fig. 19  Distribution of numerical values of $\sigma_{12}$ for regular nodal model of 293 nodes
4. Conclusions

The linear elastic cantilever beam problem is analyzed by using the ORCM. By introducing some collocation points, which are located at outside of domain of the analyzed body, unsatisfactory issue of the positivity conditions of boundary points in collocation methods can be avoided. The convergence studies show that the ORCM possesses good convergence for both the displacement and deformation energy. Quite accurate numerical results of both regular nodal model and irregular nodal model have been obtained.

References


