On the Choice of Adaptive Gains for the Inertial Matrix Identification Problem*

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Abstract
This paper studies the problem of inertia matrix identification using the adaptive tracking control proposed by Ahmed et al. The singular perturbation method and the averaging method are applied to analyze the linearized closed loop systems. Then a combination of periodic commands, PD gains and adaptive gains are established such that all estimates simultaneously converge to true values in local with a specified convergence rate.

Key words: Space Robot, Spacecraft, Inertia matrix, Adaptive Control, Tracking Control, Identification, Singular Perturbation Method, Averaging Method

1. Introduction

The demands for rapid acquisition, tracking and pointing capability of spacecraft lead to develop adaptive tracking control laws for the rigid body dynamics(1)–(5). The adaptive control law proposed by Ahmed et al.(3) consists of the sum of the feedforward control based on the estimate of the unknown inertia matrix and the PD feedback control. The estimate of the unknown inertia matrix does not always converge to a true value, but it converges to the true value if command signals satisfy the PE (Persistency Exciting) condition.

This paper studies the problem of identification of the unknown inertia matrix. One approach for identification is to input exciting signals to spacecraft. But it is desirable to specify the attitude of the spacecraft in order to keep the pipeline of communication. It is also desirable to predict the experimental period in order to save expenses. Therefore, we apply the adaptive tracking control law proposed by Ahmed et al.(3) to this problem. In this case, desired attitude of the spacecraft can be specified as command signals. But the convergence rate of estimation was not discussed in Ahmed et al.(3). In some choice of adaptive gains, all or a part of estimates converge very slowly in spite of the choice of persistency exciting command signals. This phenomenon also occurs in the adaptive control of robots(8).

Hence we apply the adaptive control law for the identification of the unknown inertia matrix and focus on the problem of selecting PD and adaptive gains to specify the convergence rates of the estimates.

The inertia matrix is precisely evaluated in a design procedure and corrected in a static testing. But it is impossible to perfectly evaluate it and it varies across the ages. Hence we suppose that its nominal value is located in the neighborhood of its true value. Since it is possible to slowly vibrate the spacecraft for all directions, we also suppose that the desired attitude of spacecraft is described by low frequency periodic command signals. Then we linearize the closed loop system and evaluate the convergence rate of the estimated inertia matrix.

A coefficient matrix of a linearized closed loop system contains the unknown inertia matrix and time-varying command signals. Firstly we apply the singular perturbation method and prove that the linearized closed loop system is approximately separated into the attitude control part and the estimation part. The attitude control part does not contain time-varying command signals and is time-invariant. We prove that the convergence rate of the attitude
control part is approximately specified faster than that of the estimation part using the nominal value of the inertia matrix. On the contrary, the estimation part does not contain the unknown inertia matrix but contains time-varying command signals. It is linear periodically time-varying and is still difficult to evaluate the convergence rate of the estimated inertia matrix. Hence we apply the averaging method to propose a method for selecting gains such that all estimates simultaneously converge to true values in local with a specified convergence rate.

2. On the Choice of Adaptive Gains

2.1. Adaptive Tracking Control Proposed by Ahmed et al.

Firstly, we summarize the adaptive tracking control proposed by Ahmed et al.\(^{(3)}\). The equations of motions of the spacecraft are given by

\[
\dot{\rho} = \frac{1}{2}(\rho^* \Omega + \zeta \Omega) \quad (1)
\]

\[
\dot{\zeta} = -\frac{1}{2} \rho^T \Omega \quad (2)
\]

\[
J \dot{\Omega} = -\Omega \times J \Omega + u, \quad (3)
\]

where \((\rho, \zeta) \in \mathbb{R}^3 \times \mathbb{R}\) are the Euler parameters\(^{(9)}\) representing the orientation of the body frame \(B\) with respect to an inertial frame \(I\), \(\Omega \in \mathbb{R}^3\) is the angular velocity of \(B\) with respect to \(I\) represented in \(B\). \((\rho, \zeta)\) and \(\Omega\) are supposed to be measurable.

\[
J = \begin{bmatrix}
J_{11} & J_{12} & J_{13} \\
J_{12} & J_{22} & J_{23} \\
J_{13} & J_{23} & J_{33}
\end{bmatrix} \in \mathbb{R}^{3 \times 3}
\]

is the inertial matrix represented in \(B\) and is supposed to be unknown. \(u \in \mathbb{R}^3\) is the control torque represented in \(B\). The notion \(a^x\) denotes

\[
a^x = \begin{bmatrix}
0 & -a_3 & a_2 \\
-a_3 & 0 & -a_1 \\
a_2 & a_1 & 0
\end{bmatrix}
\]

for a vector \(a = [a_1 \ a_2 \ a_3]^T \in \mathbb{R}^3\).

Let \((\xi, \mu) \in \mathbb{R}^3 \times \mathbb{R}\) denote command signals which are the Euler parameters representing the desired attitude \(D\). Let \(\nu \in \mathbb{R}^3\) denotes the angular velocity of \(D\) with respect to \(I\) represented in \(I\), and \(\dot{\nu} \in \mathbb{R}^3\) denotes the time derivative of \(\nu\). Then \(\nu\) and \(\dot{\nu}\) are given by

\[
\nu = 2(\mu \dot{\xi} - \dot{\mu} \xi) - 2\xi^x \dot{\xi} \quad (4)
\]

\[
\dot{\nu} = 2(\mu \ddot{\xi} - \ddot{\mu} \xi) - 2\xi^{x^2} \dot{\xi}. \quad (5)
\]

Let \((\varepsilon, \eta) \in \mathbb{R}^3 \times \mathbb{R}\) denote attitude tracking errors which are the Euler parameters representing the orientation of \(B\) with respect to \(D\) represented in \(B\). \(\omega \in \mathbb{R}^3\) is the angular velocity tracking error which is the angular velocity of \(B\) with respect to \(D\) represented in \(B\). Then \(\varepsilon, \eta, \omega\) are given by

\[
\varepsilon = \mu \rho - \zeta \xi + \rho^{x^2} \xi \quad (6)
\]

\[
\eta = \mu \zeta + \xi^T \rho \quad (7)
\]

\[
\omega = \Omega - C \nu \quad (8)
\]

\[
C = (\eta^2 - \varepsilon^T \varepsilon)I + 2\varepsilon \varepsilon^T - 2\eta \varepsilon^x,
\]

where \(I\) denotes the identity matrix.

Let \(a\) denotes the vectorization of \(J\) given by

\[
a := \begin{bmatrix}
J_{11} & J_{12} & J_{13} & J_{12} & J_{22} & J_{23} & J_{23} & J_{33}
\end{bmatrix}^T.
\]
The adaptive tracking control law proposed by Ahmed et al.\(^{(3)}\) is summarized as follows:

**Adaptive tracking control (Ahmed et al.\(^{(3)}\) Theorem 1):** Suppose that \(\nu, \dot{\nu}\) are bounded and \(K_1 \in \mathbb{R}^{3 \times 3}, K_2 \in \mathbb{R}^{3 \times 3}, Q \in \mathbb{R}^{6 \times 6}\) are positive definite. Consider the control law

\[
\begin{align*}
    u &= -Y\hat{\nu} - (K_2 K_1 + I)\epsilon - K_2\omega \\
    \dot{\hat{\nu}} &= Q^{-1}Y^T(\omega + K_1\epsilon)
\end{align*}
\]

(10) (11)

where \(Y \in \mathbb{R}^{3 \times 6}\) is defined by

\[
Y := -(\omega + C\nu)^T L(\omega + C\nu) + L(\omega^T C\nu - C\nu) + \frac{1}{2}L(K_1(\epsilon^T \omega + \eta \omega))
\]

and \(L : \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 6}\) denotes the linear operator which maps a vector \(a = [a_1 \ a_2 \ a_3] \in \mathbb{R}^3\) into

\[
L(a) := \begin{bmatrix}
    a_1 & 0 & 0 & a_3 & a_2 \\
    0 & a_2 & 0 & a_3 & 0 \\
    0 & 0 & a_3 & a_2 & a_1
\end{bmatrix}.
\]

Then \(\rho, \xi, \Omega\) asymptotically converge to \(\xi, \mu, \nu\) respectively (i.e. \(\epsilon \rightarrow 0, \eta \rightarrow 1, \omega \rightarrow 0\)), the estimate \(\hat{\alpha}\) is bounded and its time derivative \(\dot{\hat{\alpha}}\) asymptotically converge to 0 (i.e. \(\dot{\hat{\alpha}} \rightarrow 0\)). We note that \(Y\) depends on \(\omega, C, \epsilon, \eta, t\), but we omit this dependence for notational simplicity.

In general, the estimate \(\hat{\alpha}\) does not converge to the true value \(\alpha\) for arbitrarily choice of command signals \((\xi, \mu)\). However, it was shown that \(\hat{\alpha}\) converges to \(\alpha\) in the following case:

**PE condition (Ahmed et al.\(^{(3)}\) Proposition 2):** Suppose that \(\nu\) is periodic and there exist scalars \(t_1, \cdots, t_n \in \mathbb{R}\) with \(0 \leq t_1 \leq t_2 \leq \cdots \leq t_n\) such that

\[
\text{rank} \begin{bmatrix} W(t_1) \\ \vdots \\ W(t_n) \end{bmatrix} = 6, \quad (12)
\]

where \(W\) is given by

\[
W(t) := L(\nu(t)) + \nu(t)^T L(\nu(t)).
\]

(13)

Then \(\hat{\alpha}\) converges to \(\alpha\) (i.e. \(\hat{\alpha} \rightarrow \alpha\)).

**2.2. Linearized Analysis**

As summarized in the previous subsection, assuming the PE condition given by Eq. (12), the estimate of the inertia matrix converges to the true value. However, the convergence rate of estimation was not discussed in Ahemd et al.\(^{(3)}\). In the subsequent of this paper, we study a combination of command signals and gains that achieve a specified convergence rate of estimation.

Substituting Eqs. (6)–(10) into Eqs. (1)–(3) and (11), we obtain a 13-dimensional closed loop system

\[
\begin{align*}
    \dot{\epsilon} &= \frac{1}{2}(\epsilon^T \omega + \eta \omega) \\
    \dot{\eta} &= -\frac{1}{2} \epsilon^T \omega \\
    J\dot{\omega} &= -(\omega + C\nu)^T J(\omega + C\nu) + J(\omega^T C\nu - C\nu) - Y\hat{\nu} - (K_2 K_1 + I)\epsilon - K_2 \omega \\
    \dot{\hat{\alpha}} &= Q^{-1}Y^T(\omega + K_1\epsilon).
\end{align*}
\]

(14) (15) (16) (17)

Denote the estimation error by

\[
\beta := \alpha - \hat{\alpha},
\]

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apply the change of variables

\[ \gamma := \eta - 1 \]
\[ \sigma := \omega + K_1 \epsilon \]

and linearize around the equilibrium point \( \epsilon = 0, \gamma = 0, \sigma = 0, \beta = 0 \). Since \( \gamma \) does not affect the other variables, \( \gamma \) is negligible and we obtain a 12-dimensional linear time-varying system

\[
\begin{bmatrix}
\dot{\epsilon} \\
\dot{\sigma}
\end{bmatrix} = A \begin{bmatrix}
\epsilon \\
\sigma
\end{bmatrix} + \begin{bmatrix}
0 \\
-J^{-1} W
\end{bmatrix} \beta
\]

\[
\dot{\beta} = \begin{bmatrix}
0 \\
-Q^{-1} W^T
\end{bmatrix} \begin{bmatrix}
\epsilon \\
\sigma
\end{bmatrix}
\]

\[
A := \begin{bmatrix}
-\frac{1}{2} K_1 & \frac{1}{2} I \\
-J^{-1} & -J^{-2} K_2
\end{bmatrix}
\]

(18)

(19)

(20)

An unknown constant \( J \) and a time-varying term \( W(t) \) are included in the coefficients of Eqs. (18) and (19), and, in general, it is difficult to evaluate the convergence rate of estimation.

In the subsequent of this subsection, since it is possible to slowly vibrate the spacecraft, we suppose that command signals \( (\xi, \mu) \) are low-frequency periodic signals.

Then, using the singular perturbation technique, the linearized system given by Eqs. (18) and (19) can be approximately decomposed to a fast varying attitude control part and a slowly varying estimation part. Moreover, using the averaging technique, we obtain an averaged approximate of the estimation part which does not include the unknown inertia matrix and is time-invariant. Based on this approximated estimation part, we propose a method for selecting gains achieving a specified convergence rate.

2.3. Decomposition of the Linearized System Using the Singular Perturbation Method

The control input given by Eq. (10) is the sum of the feedforward control based on the estimate of the unknown inertia matrix \( J \) and the PD control. It is desirable to suppress the attitude tracking errors, and therefore we select the band of command signals to be lower than that of the PD control.

In order to represent slowly varying command signals \( (\xi, \mu) \), we introduce a sufficiently small scalar parameter \( \nu \) \((0 < \nu \ll 1)\) such that command signals are represented by

\[
\xi(t) = \tilde{\xi}(\nu t)
\]
\[
\mu(t) = \tilde{\mu}(\nu t),
\]

(21)

(22)

where \( (\tilde{\xi}, \tilde{\mu}) \) denote the Euler parameters which are supposed to be three times continuously differentiable. And a gain \( Q^{-1} \) is supposed to be represented by

\[
Q^{-1} = \nu^{-3} \tilde{Q}^{-1},
\]

(23)

where \( \tilde{Q} \in \mathbb{R}^{6 \times 6} \) is a positive definite matrix.

Then \( \nu, \nu, W \) satisfy the following equations

\[
\nu(t) = \nu \tilde{\nu}(\nu t)
\]
\[
\dot{\nu}(t) = \nu^2 \tilde{\nu}(\nu t)
\]
\[
W(t) = \nu^2 \tilde{W}(\nu t),
\]

(24)

(25)

(26)

where \( \tilde{\nu}, \tilde{\nu}, \tilde{W} \) are defined by

\[
\tilde{\nu} := 2(\tilde{\mu} \tilde{\xi} - \tilde{\mu} \tilde{\xi}) - 2 \tilde{\xi} \tilde{\xi}^\nu \tilde{\xi}
\]
\[
\dot{\nu} := 2(\tilde{\mu} \tilde{\xi} - \tilde{\mu} \tilde{\xi}) - 2 \tilde{\xi} \tilde{\xi}^\nu \tilde{\xi}
\]
\[
\tilde{W}(t) := L(\tilde{\nu}(t)) + \tilde{\nu}(t)^2 L(\tilde{\nu}(t)).
\]
Substituting Eqs. (23) and (26) into Eqs. (18) and (19), Eqs. (18) and (19) are transformed into

\[
\begin{bmatrix}
\dot{\varepsilon} \\
\dot{\sigma}
\end{bmatrix} =
A
\begin{bmatrix}
\varepsilon \\
\sigma
\end{bmatrix} +
\begin{bmatrix}
0 \\
-J^{-1}\tilde{W}(\nu t)
\end{bmatrix} \tilde{\beta}
\]

(27)

\[
\dot{\tilde{\beta}} =
\begin{bmatrix}
0 \\
-\nu \tilde{Q}^{-1}\tilde{W}(\nu t)^T
\end{bmatrix}
\begin{bmatrix}
\varepsilon \\
\sigma
\end{bmatrix}.
\]

(28)

where \(\tilde{\beta}\) is defined by

\[
\tilde{\beta} := \nu \beta.
\]

(29)

Apply the change of time scale given by

\[
\tau := \nu t,
\]

(30)

Eqs. (27) and (28) are transformed into

\[
u \begin{bmatrix}
\frac{d\varepsilon_f}{d\tau} \\
\frac{d\sigma_f}{d\tau}
\end{bmatrix} =
A
\begin{bmatrix}
\varepsilon_f \\
\sigma_f
\end{bmatrix} +
\begin{bmatrix}
0 \\
-J^{-1}\tilde{W}(\tau)
\end{bmatrix} \tilde{\beta}_s
\]

(31)

\[
\frac{d\tilde{\beta}_s}{d\tau} =
\begin{bmatrix}
0 \\
-\tilde{Q}^{-1}\tilde{W}(\tau)^T
\end{bmatrix}
\begin{bmatrix}
\varepsilon_f \\
\sigma_f
\end{bmatrix}.
\]

(32)

Now we have the standard structure of singular perturbation systems. Since the parameter \(\nu\) is sufficiently small, \(\varepsilon\) and \(\sigma\) correspond to fast variables and \(\tilde{\beta}\) corresponds to a slow variable. By the assumption that \((\tilde{\xi}, \tilde{\mu})\) are three times continuously differentiable, \(\tilde{W}(t)\) is continuously differentiable and \(\dot{\tilde{W}}(t)\) is bounded. Then it is possible to apply the singular perturbation technique for linear time-varying systems (see e.g. Section 5.3 in Ref. (7)) into Eqs. (31) and (32), and we derive approximate equations for fast and slow dynamics.

An approximate equation for fast dynamics is called a boundary layer system

\[
u \begin{bmatrix}
\frac{d\varepsilon_f}{d\tau} \\
\frac{d\sigma_f}{d\tau}
\end{bmatrix} =
A
\begin{bmatrix}
\varepsilon_f \\
\sigma_f
\end{bmatrix}.
\]

(33)

An approximate equation for slow dynamics is called a reduced system

\[
\frac{d\tilde{\beta}_s}{d\tau} =
\begin{bmatrix}
0 \\
-\tilde{Q}^{-1}\tilde{W}(\tau)^T
\end{bmatrix}
\begin{bmatrix}
\varepsilon_f \\
\sigma_f
\end{bmatrix}.
\]

(34)

Then \(\varepsilon, \sigma, \tilde{\beta}\) are approximated by \(\varepsilon_f, \sigma_f, \beta_s\) as follows (see Eq. (3.4) in Ref. (7)):

\[
\begin{bmatrix}
\varepsilon \\
\sigma
\end{bmatrix} =
(I + \nu GH + O(\nu^2))
\begin{bmatrix}
\varepsilon_f \\
\sigma_f
\end{bmatrix} + (-G + O(\nu))\tilde{\beta}_s
\]

\[
\tilde{\beta} = (-\nu H + O(\nu^2))
\begin{bmatrix}
\varepsilon_f \\
\sigma_f
\end{bmatrix} + \tilde{\beta}_s
\]

\[
G := A^{-1}
\begin{bmatrix}
0 \\
-J^{-1}\tilde{W}(\nu t)
\end{bmatrix}
\]
$$H := \begin{bmatrix} 0 & -\tilde{W}(ut)^T \end{bmatrix} A^{-1}.$$ 

$G$ and $H$ depend on $\nu$ and $t$, but we omit this dependence for notational simplicity.

Define $\beta_i := \nu^2 \beta_i$, similar to Eq. (29). Then $\beta_i$ is also a solution of the reduced system

$$\dot{\beta}_i = -\nu \tilde{W}(ut)^T (K_2 + K_1^{-1})^{-1} \tilde{W}(ut) \beta_i$$

which represents a slowly varying estimation part.

In summary, $\epsilon, \sigma, \beta$ are approximated by the solution $(\epsilon_f, \sigma_f)$ of (33) and the solution $\beta_i$ of (35) as follows:

$$\begin{bmatrix} \epsilon \\ \sigma \end{bmatrix} = (I + \nu GH + O(\nu^2)) \begin{bmatrix} \epsilon_f \\ \sigma_f \end{bmatrix} + (-\nu^2 G + O(\nu^3)) \beta_i$$

$$\beta = (-\nu^3 H + O(1)) \begin{bmatrix} \epsilon_f \\ \sigma_f \end{bmatrix} + \beta_i.$$  

(37)

We note that the first term of Eq. (37) represents the effect of fast variables $\epsilon_f$ and $\sigma_f$ to the estimation error $\beta$. We shall discuss the initial variation of the estimation error based on this equation.

**Remark:** The singular perturbation approach legitimizes ad hoc simplifications of dynamic models. We note that no general formula is not known for transforming a system which does not explicitly contain a small parameter $\nu$ into a standard form of singular perturbation systems given by Eqs. (31) and (32), and this is one of the crucial difficulties to apply the singular perturbation technique\(^{(7)}\). In this subsection, assuming command signals $(\xi, \mu)$ to be slowly varying, we have introduced a small parameter $\nu$ in Eqs. (21)–(23) such that the constraint $\xi(t)^T \xi(t) + \mu(t)^2 = 1$ is satisfied for all $\nu$.

### 2.4. Selection of Command Signals and Gains Based on the Averaging Method

In this subsection, we suppose that command signals $(\xi, \eta)$ are periodic. Then we propose a method for selecting command signals and gains which achieve the estimation error converging to 0 in local with a specified convergence rate.

Firstly we focus on the boundary layer system given by Eq. (33) and discuss a choice of $K_1$ and $K_2$. The matrix $A$ is factored by

$$A = \begin{bmatrix} I & 0 \\ \frac{1}{2} K_1 & I \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} I \\ -J^{-1}(K_2 K_1 + I) & -\frac{1}{2} K_1 - J^{-1} K_2 \end{bmatrix} \begin{bmatrix} I & 0 \end{bmatrix}^{-1}.$$ 

Since the eigenvalues of a matrix is invariant with respect to a similarity transformation, the characteristic polynomial of $A$ is given by

$$\det\left(s^2 I + \frac{1}{2} K_1 + J^{-1} K_2 + J^{-1} (K_2 K_1 + I)\right) = 0.$$ 

It follows that $A$ has eigenvalues near the imaginary axis only if the minimum singular values of $\frac{1}{2} K_1 + J^{-1} K_2$ or $J^{-1} (K_2 K_1 + I)$ are small. Choose positive definite matrices $K_1$ and $K_2$ such that

$$O(||J||^{-1}) \geq \|(K_2 K_1 + I)^{-1}\|$$

(38)

is satisfied and that

$$O(1) \geq ||K_1^{-1}||$$

(39)

or

$$O(||J||^{-1}) \geq ||K_2^{-1}||$$

(40)
is also satisfied. Then real parts of eigenvalues of $A$ are greater than $O(1)$-order, and the decay time constant of the boundary layer system is assigned to $O(1)$-order. We note that, for example, those conditions are approximately satisfied if we choose $K_1$ to be the identity matrix and $K_2$ to be the nominal value of the inertia matrix.

Next we focus on the reduced system given by Eq. (35) and discuss a choice of command signals $(\xi, \mu)$, a time decay constant $\lambda$ and a gain $Q^{-1}$. Choose signals $(\tilde{\xi}, \tilde{\mu})$ such that a matrix

$$
\tilde{Z} := \frac{1}{T} \int_0^T \tilde{W}(t)(K_2 + K_1^{-1})^{-1} \tilde{W}(t) dt
$$

is positive definite, where $\tilde{T}$ is a period of $(\tilde{\xi}, \tilde{\mu})$. Choose a sufficiently large constant $\lambda$ satisfying

$$
\lambda \geq O(\nu^{-1})
$$

as a time decay constant of the estimation part, and choose a constant matrix $\tilde{Q}$ by

$$
\tilde{Q}^{-1} = \lambda^{-1} \tilde{Z}^{-1}.
$$

Then, applying the averaging method (see e.g. Section 10.5 in Ref. (6)), the averaged reduced system is given by

$$
\beta_{ave} = -\nu \tilde{Q}^{-1} \beta_{ave}
$$

$$
= -\nu \lambda^{-1} \beta_{ave}, \tag{41}
$$

which is $O(\nu^{-1})$-order approximate of the reduced system given by Eq. (35). It follows that the time decay constant of (35) is approximately assigned to $\lambda^{-1}$ which is greater than $O(\nu^{-2})$-order.

Finally we focus on Eq. (37). Since we have chosen $\lambda$ such that $\lambda \geq O(\nu^{-1})$, the first term of Eq. (37) is $O(1)$-order. Hence the initial value of the reduced system $\beta(0)$ is not far from the initial value of the estimate $\hat{\beta}(0)$. $\beta$ is not strongly affected by $\sigma_f$ or $\sigma_T$ on $O(1) \geq t \geq 0$. Since the boundary layer system given by Eq. (33) converges to 0 sufficiently faster than the reduced system given by Eq. (35), $\sigma_f$ and $\sigma_T$ are negligible on $t \gg O(1)$. Hence the estimation error $\beta$ converges to 0 with a time decay constant $\lambda \nu^{-1}$.

In summary, we have derived a combination of command signals $(\xi, \mu)$ and gains $K_1, K_2, Q^{-1}$ so that the estimate $\hat{\alpha}$ converges to the true value $\alpha$ with a specified convergence rate $\lambda$. We have introduced a sufficiently small parameter $\nu$ to approximately derive the estimation part. However, once we choose a low-frequency command signals, we have the following result by formally choosing $\nu = 1$.

**Selection of command signals $(\xi, \mu)$, gains $K_1, K_2, Q^{-1}$ to achieve a time decay constant $\lambda$:**

- Choose positive definite matrices $K_1$ and $K_2$ such that Eq. (38) is satisfied and that each of Eq. (39) or Eq. (40) is also satisfied. For example, choose $K_1$ as the identity matrix and $K_2$ as the nominal value of the unknown inertia matrix $J$.
- Choose low frequency periodic command signals $(\xi, \mu)$ which are three times continuously differentiable and a matrix

$$
Z := \frac{1}{T} \int_0^T W(t)^T (K_2 + K_1^{-1})^{-1} W(t) dt \tag{42}
$$

is positive definite, where $T \gg 1$ denotes a period of $(\xi, \mu)$.
- Choose a time decay constant $\lambda$ satisfying $\lambda \geq O(T^2)$ and a positive matrix $Q^{-1}$ by

$$
Q^{-1} := \lambda^{-1} Z^{-1}. \tag{43}
$$

Then the estimate of $J$ converges to the true value (i.e. $\hat{\alpha} \rightarrow \alpha$) with a convergence rate $\lambda$.

**Remark:** A PE condition given by Eq. (12) and the positive definiteness of $Z$ is equivalent.
Remark: In this subsection, we have restricted command signals to be periodic for simplicity. However, applying the generalized averaging method (see e.g. Section 10.6 in Ref. (6)), it is possible to discuss nonperiodic command signals. Suppose that $W$ defined by Eq. (13) is two times continuously differentiable and $W(t)^T (K_2 + K_1^{-1})^{-1} W(t)$ is the sum of a periodic matrix $Z_{ss}(t)$ consisted of finite numbers of sinusoids with different frequencies and a matrix $Z_{tr}(t)$ exponentially converging to 0, i.e.

$$W(t)^T (K_2 + K_1^{-1})^{-1} W(t) = Z_{ss}(t) + Z_{tr}(t).$$

Then it follows from the generalized averaging method that the reduced system given by (35) is approximated by a linear time-invariant system

$$\dot{\beta} = -Q^{-1} \left( \frac{1}{T_{ss}} \int_0^{T_{ss}} Z_{ss}(t) dt \right) \beta,$$

where $T_{ss}$ is a period of $Z_{ss}(t)$. Hence the gain $Q^{-1}$ is selected in a similar way to Eq. (43). Although we have discussed the convergence of $\alpha$ in local, this extension to nonperiodic command signals gives a new insight from Ahmed et al.(3).

2.5. Numerical Example

In this subsection, we evaluate the effectiveness of the proposed method for selecting command signals and gains by simulations. Let

$$J = \begin{bmatrix} 250 & 40 & 20 \\ 40 & 250 & 60 \\ 20 & 60 & 250 \end{bmatrix} [\text{kg m}^2]$$

denotes the true value of the inertial matrix, and let

$$J_0 = \begin{bmatrix} 300 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 250 \end{bmatrix} [\text{kg m}^2]$$

denotes its nominal value.

Firstly, choose gains $K_1$ and $K_2$ as

$$K_1 = I [\text{rad/sec}], \quad K_2 = J_0 \phi, \quad \phi = 1 [1/\text{sec}]$$

such that Eqs. (38)-(40) are approximately satisfied. Then, substituting $J_0$ into $J$ in Eq. (20), eigenvalues of $A$ are approximately assigned to

$$\{-1.29, -1.18, -0.670, -0.509, -0.503, -0.502\}$$

and the time decay constant of the boundary layer system given by (33) is specified to $O(1)$-order. Indeed, eigenvalues of $A$ are given by

$$\{-0.996, -0.995, -0.994, -0.505, -0.504, -0.503\}$$

and the above estimates give valid approximates. Next, choose command signals as

$$\xi = \begin{bmatrix} \cos \frac{t}{100} \\ \sin \frac{t}{100} \\ \sin \frac{t}{120} \\ \sin \frac{t}{120} \sin \frac{t}{120} \end{bmatrix}, \mu = 0,$$

then $Z$ given by Eq. (42) is shown to be positive definite. Followed by Eq. (24), choose a parameter $\nu$ as $\nu = 10^{-2}$ and time decay constant of the estimation part as

$$\lambda = \nu^{-2} = 10^4 [\text{sec}],$$
which approximately corresponds to 3-periods ($10^4/1200\pi \approx 2.65$) of command signals. Finally, compute $Q^{-1}$ from Eq. (43).

The initial values of the attitude tracking errors $(e, \eta)$ and the angular velocity tracking error $\omega$ are

\[
e(0) = \begin{bmatrix} 0.01 & -0.01 & 0 \end{bmatrix}^T,
\eta(0) = \sqrt{0.9998},
\omega(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T [\text{rad/sec}],
\]

and the initial value of the estimates $\alpha$ is the vectorization of $J_0$ followed by Eq. (9).

The initial response of the estimate $\alpha$ is shown in Figs. 1 and 2. The estimate $\alpha$ (solid line) is affected by a state tracking error around $t = 0$ and this is expected from Eq. (37). Then it converges to the true value $\alpha$ along the trajectory of the averaged system (broken line) with the specified time decay constant $\lambda = 10^4$. The initial responses of the attitude tracking errors $(e, \eta)$ and the angular velocity tracking error $\omega$ are shown in Figs. 3–5. The time decay constant of the attitude control part is specified to $O(1)$-order, and the errors converge $(e \to 0, \eta \to 1, \omega \to 0)$ within $10 [\text{sec}]$ indeed.

Moreover, choose a time decay constant by

\[\lambda = 10^5 [\text{sec}]\]

and run a simulation with the same conditions.

The initial response of the estimate $\alpha$ is shown in Figs. 6 and 7. The estimate $\alpha$ (solid line) is weakly affected by a state tracking error around $t = 0$ and this is expected from Eq. (37), since the first term of Eq. (37) is proportional to $\lambda^{-1}$ for a fixed $\nu$. Then it converges to the true value $\alpha$ along the trajectory of the averaged system (broken line) with the specified time decay constant $\lambda = 10^5$. The initial responses of the attitude tracking errors $(e, \eta)$ and the angular velocity tracking error $\omega$ are similar to Figs. 3–5, and therefore omitted. We note that the time decay constant of the attitude control part is also specified to $O(1)$-order by the choice of $K_1$ and $K_2$. Hence, by choosing a larger time decay constant $\lambda$, the initial variation of the estimation error $\beta$ is suppressed without affecting the performance of the attitude control part.

The above simulation results illustrate the effectiveness of the proposed choice of commands and gains.

We note that the gain $Q^{-1}$ is chosen to be $10^5$-order for $\lambda = 10^4$, so that it might be regarded as high gain from its large value. However, $Q^{-1}Z$ which dominates the estimation speed is $10^{-4}$-order, and therefore the gain $Q^{-1}$ is regarded as low gain in the sense that the estimation speed is slow. In this way, the specification of time decay constant $\lambda$ is not obvious and this motivates the authors to study this problem.
Fig. 2 Detailed time history of Fig. 1

Fig. 3 Attitude tracking error $\varepsilon$ for $\lambda = 10^4[\text{sec}]$

Fig. 4 Attitude tracking error $\eta$ for $\lambda = 10^4[\text{sec}]$

Fig. 5 Angular velocity error $\omega$ for $\lambda = 10^4[\text{sec}]$
3. Conclusion

We have proposed a method for selecting command signals and gains in order to apply the adaptive tracking control proposed by Ahmed et al. to the mass property estimation problem, so that the estimate of the inertia matrix converges to the true value in local with a specified convergence rate. The analytical results coincide well with simulation results, and the effects of the proposed method are expressed quantitatively.

References


