Dipole decomposition of two-dimensional incompressible flows

Kazuyuki UENO*, Yuko MATSUMOTO** and Keiichi ISHIKO***
* Iwate University
4-3-5 Ueda, Morioka 020-8551, Japan
E-mail: uenok@iwate-u.ac.jp
** National Institute of Technology, Numazu College
3600 Ooka, Numazu 410-8501, Japan
*** National Institute of Technology, Asahikawa College
2-2 Shunkodai, Asahikawa 071-8142, Japan

Received: 13 March 2017; Revised: 8 June 2017; Accepted: 22 June 2017

Abstract
A decomposition method of a given two-dimensional incompressible flow field into a dipole sequence is developed. Necessary condition for dipole sequence is revealed using a wavelet transform of di-vorticity of the given flow. Subsequently, a practical way to extract dipole sequence by a recurrence formula is proposed. Each obtained dipole is characterized not only by the dipole moment but also by its own length scale. The reconstructed flow fields always give divergence-free fields even if the successive correction with the recurrence formula is truncated at a finite number. Typical two-dimensional flows are decomposed into dipoles, and graphical representations of extracted dipoles are shown. Many columns of dipoles and isolated dipoles in various length scale are found in a two-dimensional turbulent flow.

Keywords: Dipole flow, Decomposition, Divergence-free, Graphical representation

1. Introduction

There are many observation methods for flow fields: dye, smoke, oil flow, particle image velocimetry, laser-induced fluorescence, shadowgraph method, Mach–Zehnder interferometer, and so on. Physical insights into complicated flows considerably depend on each observation method. Similar dependency is also found in graphical representations of results of theoretical analysis and numerical simulations as well as experiments. Consequently, a variety of methods for observation, visualization and graphical representation are necessary for understanding complicated flow fields. Contour surfaces related to vortices are one of the most important graphical representations for flow fields. A suitable scalar field for each problem is selected, e.g. the magnitude of the vorticity vector, one of the components of the vorticity vector, or the second invariant of the velocity-gradient tensor (Jeong and Hussain, 1995). Such vortex contours show impressive and detailed information on disturbances.

A dipole is one of the simplest vortical elements of incompressible flows. It is a point element (or a spherical/spheroidal element) in three-dimensional space in contrast to curvilinear geometry of vortices. Consequently, it is easy to grasp the position of dipoles, and interaction between dipoles is rather clear (Matsumoto et al., 2014). A dipole is characterized by its dipole moment which is not a vector field but a vector quantity related to each dipole. Dipole moments are not restricted by a divergence-free condition, though a velocity field induced by dipoles satisfies the divergence-free condition.

Attempts to express complicated flow fields by a superposition of dipole flows were given by Hashimoto (1988), Buttke (1993), Cortez (1996) and Summers (2001). In these studies, they discussed evolution of dipole moments and dipole positions. Nevertheless, identification of initial dipole positions and initial dipole moments from a given flow field were not discussed in detail in their studies. The main objective of the present study is to establish such an identification; i.e. a decomposition method from a given incompressible flow field into a sequence of dipoles is proposed. Using this method, decomposition examples of typical two-dimensional flows are illustrated.
The dipole decomposition in this study is applicable to any incompressible velocity (or vorticity) field obtained by theoretical analysis, numerical simulations and PIV (particle image velocimetry) measurements. Even if the given flow field is unsteady, this decomposition is applied to a snap-shot field at an instant.

2. Basis dipole flow

The vorticity field of the basis dipole flow of the present study is given by

\[ \omega_1(x, y) = -\varepsilon_{ij}\frac{\mu_j}{a^2} \frac{\partial f_0}{\partial x_i}, \]

where \(\varepsilon_{ij}\) denotes the Levi–Civita symbol, and summation convention is applied to subscript indexes. Notations \(x_1, x_2, \mu_1, \mu_2\) are identical to \(x, y, \mu, \nu\), respectively, in this paper. The vector \(\mu_1\) denotes the dipole moment. The basis dipole of the present study has a compact vortex pair around its center, and \(a\) denotes the length scale of the vortex pair. The function \(f_0(r_0)\) is a “smoothing function” as shown in Fig. 1. An arbitrary function satisfying the following requirements are acceptable for \(f_0\): twice differentiable, sufficiently well-localized in both physical space and Fourier space, and satisfying

\[ 2\pi \int_0^\infty f_0(r_0) r_0 \, dr_0 = 1. \]

A typical example of \(f_0\) is the Gaussian distribution function. Another choice adopted in this study is the following function:

\[ f_0(r_0) = \begin{cases} \frac{1}{\pi} \left(1 - \frac{J_0(\lambda_1 r_0)}{J_0(\lambda_1)}\right) & \text{for } r_0 < 1, \\ 0 & \text{for } r_0 \geq 1, \end{cases} \]

where \(J_0(r_0)\) is the Bessel function of 0th order. The constant \(\lambda_1 = 3.83170597\ldots\) is the first zero of \(J_1(\lambda) = 0\). Cartesian coordinates \(x, y\) are related to \(r_0\) as follows:

\[ r_0 = \sqrt{(x-X)^2 + (y-Y)^2} \frac{a}{a}, \]

where \(X\) and \(Y\) give “translation” and \(a\) gives “dilation” of the smoothing function. According to the dilation,

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_0 \, dx \, dy = a^2. \]

The function \(f_0/a^2\) tends to two-dimensional Dirac delta function as \(a \to 0\). Hence, Eq. (1) tends to the vorticity field of a point dipole (Saffman and Meiron, 1986; Hashimoto, 1998) as \(a \to 0\), though we discuss vorticity fields for finite \(a\) in this paper.

Left figure of Fig.2 shows typical vorticity distribution of a dipole flow when \(f_0\) is given by Eq. (3). This vortex pair is the well-known Lamb–Chaplygin dipole (Lamb, 1932; Meleshko and van Heijst, 1994). The vorticity distributes in the region of \((x-X)^2 + (y-Y)^2 < a^2\). The distance between the positive-vortex center and the negative-vortex center corresponds to the length scale \(a\). (In a three-dimensional dipole decomposition in a future study, the basis dipole has a compact vortex ring around its center. In that case, the diameter of the vortex center line of the vortex ring corresponds to \(a\).)
The flow velocity of the basis dipole is given by

\[ u_i(x, y) = \mu_j \left( \frac{\partial^2 g_0}{\partial x_i \partial x_j} - \delta_{ij} \frac{\partial^2 g_0}{\partial x_k \partial x_k} \right), \]  

where \( g_0 \) is the solution of the following Poisson equation:

\[ \frac{\partial^2 g_0}{\partial x_k \partial x_k} = -\frac{f_0}{a^2}. \]  

This equation is reduced to

\[ \frac{\partial^2 g_0}{\partial r_0^2} + \frac{1}{r_0} \frac{\partial g_0}{\partial r_0} = -f_0(r_0) \]  

by using the parameter \( r_0 \). The symbol \( \delta_{ij} \) in Eq.\((6)\) denotes Kronecker delta. Right figure of Fig.\(2\) shows streamlines of the basis dipole flow.

For large \( r_0 \), the function \( g_0(r_0) \) tends to the free-space Green function of the two-dimensional Poisson equation:

\[ g_0(r_0) = -\frac{1}{2\pi} \log r_0 + O(r_0^{-1}) \quad \text{for} \quad r_0 \gg 1. \]  

Therefore, the velocity \((6)\) for large \( r_0 \) is identical to the point dipole velocity. On the other hand, \( g_0(r_0) \) is a smooth and finite-value function without singularity even for \( r_0 \ll 1 \).

The dipoles in the study by Hashimoto (1998) are point dipoles with singularity. On the other hand, the dipoles of Buttke (1993), Cortez (1996) and Summers (2001) have a smoothed kernel which is introduced to avoid numerical instability. Thus, the diameter of the kernel of all dipoles is a same constant and it has no physical meaning. In contrast to their kernel diameter, the length-scale quantity \( a \) introduced in our study is an essential factor of the decomposition of flow fields.

Owing to the length scale \( a \), it is possible to identify the self-propelling velocity \( \mu_1/(2\pi a^2) \) (Matsumoto et al., 2009; Matsumoto and Ueno, 2014) and the circulation \( \Gamma = 2 |\mu| a^{-1} \int_0^\infty f_0 \, dr_0 \) of the positive vortex in the basis dipole (the circulation of the negative vortex is the same magnitude and opposite sign). These quantities are not necessary for the dipole decomposition, but they are useful when we study complicated flows by showing results of the dipole decomposition.

3. Superposition of the basis dipole flows

Superposing a number of the basis dipole flows with various \( a \) and \( \mu_j \), we obtain the following multiscale flow field.

\[ u_i(x, y) = U_{\infty} + \sum \frac{\mu_j}{\mu_i} \left( \frac{\partial^2 g_0^{(n)}}{\partial x_i \partial x_j} - \delta_{ij} \frac{\partial^2 g_0^{(n)}}{\partial x_k \partial x_k} \right), \]  

where \( g_0^{(n)} = g_0(r_0^{(n)}) \) and

\[ r_0^{(n)} = \sqrt{(x - X^{(n)})^2 + (y - Y^{(n)})^2} / a^{(n)}. \]
Here, \( \mu^{(n)} (X^{(n)}, Y^{(n)}) \), \( a^{(n)} \) denote the moment, the position of the center and the length scale, respectively, of the dipole with number \( n \). The constant \( U_{\infty} \) shows a uniform velocity at an infinite far distance.

If a set of \( \{ \mu^{(n)} (X^{(n)}, Y^{(n)}, a^{(n)}) \} \) is obtained, the above superposition is calculated straightforward. Nevertheless, any decomposition method to obtain \( \{ \mu^{(n)} (X^{(n)}, Y^{(n)}, a^{(n)}) \} \) from a given flow field \( u_i(x, y) \) has not been found yet. The subject of this paper is to establish a method of such a decomposition. We define “dipole sequence” as a sequence \( \{ \mu^{(n)} (X^{(n)}, Y^{(n)}, a^{(n)}) \} \) that satisfies Eq. (10) for a given \( u_i(x, y) \). It is noteworthy that dipole sequence \( \{ \mu^{(n)} (X^{(n)}, Y^{(n)}, a^{(n)}) \} \) is not unique.

Let us introduce the following vector field function at the first step of solving the problem:

\[
\alpha_i = e_i \frac{\partial \omega_{\text{d}}}{\partial x_j} = - \frac{\partial^2 u_i}{\partial x_j \partial x_k} \tag{12}
\]

This field is called ‘flexion vector’ (Truesdell, 1954; Lewalle, 2000; Lewalle, 2006) or ‘di-vorticity’ (Kida, 1985). Laplacian of the above gives

\[
\alpha_i(x, y) = \sum_n \frac{\mu^{(n)}_j}{(a^{(n)})^2} \psi^{(n)}_{ij}, \tag{13}
\]

where

\[
\psi_{ij} = \frac{\partial^2 f_0}{\partial x_i \partial x_j} - \delta_{ij} \frac{\partial^2 f_0}{\partial x_k \partial x_l}. \tag{14}
\]

and \( \psi^{(n)}_{ij} = \psi_{ij} (\alpha^{(n)}_j) \). The synthesis formulas (10) and (13) always give divergence-free fields satisfying

\[
\frac{\partial u_i}{\partial x_l} = 0 \quad \text{and} \quad \frac{\partial \alpha_i}{\partial x_l} = 0, \tag{15}
\]
even if the series are truncated at finite \( n \).

We introduce

\[
M_j(X, Y, \alpha) = \sum_n \mu^{(n)}_j \delta(X - X^{(n)}) \delta(Y - Y^{(n)}) \delta(\alpha - \alpha^{(n)}), \tag{16}
\]

where \( \delta(*) \) denotes Dirac’s delta function. We call \( M_j \) “dipole density” in this study. The relation (13) is extended to the following convolution integral by using \( M_j \):

\[
\alpha_i(x, y) = \iiint M_j \psi_{ij} \, dX \, dY \, d\alpha \tag{17}
\]

The kernel function \( \psi_{ij} \) guarantees \( \partial \alpha_i / \partial x_l = 0 \) regardless of \( M_j \).

### 4. Necessary condition for dipole sequence

Synthesis of complicated flows from dipole flows is given in form of the convolution integral (17). On the other hand, the decomposition of a given complicated flow into dipole density \( M_j(X, Y, \alpha) \) (or the dipole sequence \( \{ \mu^{(n)}_j (X^{(n)}, Y^{(n)}, a^{(n)}) \} \)) has not been obtained yet. It is the objective of this study. In this section, we show the necessary condition for obtaining a dipole sequence from arbitrary two-dimensional flow field without net circulation.

The Laplacian of the smoothing function \( -\partial^2 f_0 / \partial x_i \partial x_k \) is a sufficiently well-localized function and satisfies

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ - \frac{\partial^2 f_0}{\partial x_i \partial x_k} \right\} \, dx \, dy = 0. \tag{18}
\]

Using \( -\partial^2 f_0 / \partial x_i \partial x_k \) as a wavelet, the following convolution integrals give a continuous wavelet transform and the corresponding inverse wavelet transform (Daubechies, 1992; Farge, 1992; Antoine, 1999; Addison 2002):

\[
m_i(X, Y, \alpha) = \frac{1}{C_{\alpha \omega}} \int \int \alpha_i \left\{ - \frac{\partial^2 f_0}{\partial x_i \partial x_k} \right\} \, dx \, dy, \tag{19}
\]

\[
\alpha_i(x, y) = \iiint m_i \left\{ - \frac{\partial^2 f_0}{\partial x_i \partial x_k} \right\} \, dX \, dY \, d\alpha. \tag{20}
\]

The value of the admissible constant \( C_{\alpha \omega} \) is derived in Appendix A.
A sequential operation of the convolution integrals of Eqs. (17) and (19) gives
\[
\iiint K_{ij} M_j(X', Y', a') \frac{dX' \, dY' \, da'}{(a')^3} - m_i(X, Y, a) = 0,
\]
where
\[
K_{ij}(X, Y, a; X', Y', a') = \frac{a \, da}{C_{ad}} \iiint \left\{ -\frac{\partial^2 f_0}{\partial x_i \partial x_k} \right\}_{X,Y,a} \cdot \phi_{ij}(X', Y', a') \, dx \, dy.
\]
The function \( K_{ij}(X, Y, a; X', Y', a') \) is called “reproducing kernel” (Daubechies 1992; Farge, 1992; Antoine, 1999). The relation (21) is the necessary condition for \( M_j \). There exist a countless number of functions \( M_j \) which satisfy Eq. (21).
Substituting Eq. (16) into (21), we obtain
\[
\sum_n \frac{\mu_j^{(n)}}{(a^{(n)})^3} K_{ij}(X, Y, a; X^{(n)}, Y^{(n)}, a^{(n)}) - m_i(X, Y, a) = 0.
\]
The relation (23) is the necessary condition for the dipole sequence \( \mu_j^{(n)}, X^{(n)}, Y^{(n)}, a^{(n)} \). Values of the sequence are determined in the next section.

5. Extraction of dipoles

The dipole sequence \( \mu_j^{(n)}, X^{(n)}, Y^{(n)}, a^{(n)} \) is not unique. A practical way to extract dipole sequence \( \mu_j^{(n)}, X^{(n)}, Y^{(n)}, a^{(n)} \) from \( m_i(X, Y, a) \) is explained in this section.

We propose a recursive way. First, \( m_i^{(0)}(X, Y, a) \) is given by
\[
m_i^{(0)}(X, Y, a) = m_i(X, Y, a) = \frac{1}{C_{ad}} \iiint a_i \left\{ -\frac{\partial^2 f_0}{\partial x_i \partial x_k} \right\} a \, dx \, dy.
\]
Then, we define \( X^{(1)}, Y^{(1)}, a^{(1)} \) as the place where \( |m_i^{(0)}(X, Y, a)| \) takes its maximum value. Taking Eq. (23) into account, we require the moment \( \mu^{(1)} \) to satisfy
\[
\sum_n \frac{\mu_j^{(n)}}{(a^{(n)})^3} K_{ij}(X^{(1)}, Y^{(1)}, a^{(1)}; X^{(n)}, Y^{(n)}, a^{(n)}) \mu^{(1)} = \chi m_i^{(0)}(X^{(1)}, Y^{(1)}, a^{(1)}),
\]
where \( \chi \) is a constant in the range of \( 0 < \chi \leq 1 \). The reproducing kernel is reduced to a product of a scalar and the unit tensor when the first and the second sets of independent variables are the same. Thus, we obtain
\[
\mu^{(1)} = \frac{2\chi C_{ad} (a^{(1)})^3}{C_{tp}} m_i^{(0)}(X^{(1)}, Y^{(1)}, a^{(1)}),
\]
where
\[
C_{tp} = 2\pi \int_0^{\infty} [h_0(r_0)]^2 r_0 \, dr_0,
\]
and
\[
h_0(r_0) = \left( \frac{\partial^2 f_0}{\partial r^2} + \frac{1}{r_0} \frac{d f_0}{d r_0} \right).
\]
The dimensionless constant \( C_{tp} \) is calculated before the dipole decomposition (see Appendix B). The residual of \( a_i \) is given by
\[
Ra^{(1)}(x, y) = a_i - \frac{\mu^{(1)}}{(a^{(1)})^3} \psi^{(1)}.
\]
The function \( m_i^{(1)}(X, Y, a) \) corresponding to this residual is obtained by the following convolution integral:
\[
m_i^{(1)}(X, Y, a) = \frac{1}{C_{ad}} \iiint Ra^{(1)} \left\{ -\frac{\partial^2 f_0}{\partial x_i \partial x_k} \right\} a \, dx \, dy.
\]
The following procedure is repeated for \( n = 2, 3, 4, \cdots \). First, we define \((X^{(n)}, Y^{(n)}, a^{(n)})\) as the place where \(m_i^{(n-1)}(X, Y, a)\) takes its maximum value. Then, we obtain the following value and functions using \((X^{(n)}, Y^{(n)}, a^{(n)})\):

\[
\mu_i^{(n)} = \frac{2\chi C_{ad} (a^{(n)})^3}{C_{fp}} \mu_i^{(n-1)}(X^{(n)}, Y^{(n)}, a^{(n)}),
\]

\[
Ra_i^{(n)}(x, y) = \alpha_i - \sum_{q=1}^{n} \frac{\mu_j^{(q)}}{(a^{(q)})^2} \psi_j^{(q)},
\]

\[
m_{ij}^{(n)}(X, Y, a) = \frac{1}{C_{ad}} \int \int Ra_i^{(n)} \left\{ -\frac{\partial^2 f_0}{\partial x_k \partial x_k} \right\} a \, dx \, dy.
\]

Dipole decomposition of a given two-dimensional flow into a dipole sequence \(\mu_i^{(n)}, X^{(n)}, Y^{(n)}, a^{(n)}\) is given by recursive operation of Eqs. (24)–(33).

We as yet have no mathematical proof of convergence of \(\lim_{n \to \infty} Ra_i^{(n)} = 0\). However, the numerical experiments described in the next section show that this recurrence formula for \(0 < \chi \leq 1\) gives dipole decomposition within sufficiently small residuals (see Figs. 6 and 11).

Convergence of \(Ra_i^{(n)}\) by the recurrence formula (24)–(33) becomes slower and slower as \(n\) increases. Therefore, multiple dipoles are extracted per one recursive step in actual calculation. Details of simultaneous extraction of multiple dipoles are described in Appendix C.

### 6. Dipole decomposition of typical flows

The dipole decomposition is applicable to an arbitrary incompressible flow field without net circulation: \(\Gamma_\infty = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \omega_r \, dx \, dy = 0\). If one wants to decompose a flow field with finite net circulation \(\Gamma_\infty \neq 0\), a certain monopole vortex with \(\Gamma_\infty\) should be subtracted before the two-dimensional dipole decomposition.

The dipole decompositions of several flows were performed in this section. In the actual calculations, the summation in Eq. (32) and the convolution integral in Eq. (33) were performed in Fourier space \(k, k_q\) in order to reduce calculation cost and errors of the numerical integration. Details of Fourier transform and the calculation in Fourier space are explained in Appendix A. The Fourier transform was performed twice per a recursive step \(n\), because extraction of \(\mu_i^{(n)}, X^{(n)}, Y^{(n)}, a^{(n)}\) according to Eq. (31) is not able to perform in Fourier space. We used FFT (fast Fourier transform) algorithm in these Fourier transforms.

#### 6.1. Lamb–Chaplygin dipole

A Bessel function shown in Eq. (3) is adopted as a smoothing function \(f_0(r_0)\). Equation (6) derived from this \(f_0\) gives the well-known Lamb–Chaplygin dipole (Lamb, 1932; Meleshko and van Heijst, 1994). Details of \(f_0\) and \(g_0\) are explained in Appendix B.

The flow field of the Lamb–Chaplygin dipole was decomposed as a first example, because \(f_0(r_0)\) of this study is associated with it. We periodically divide \(xy\) plane into \(10 \times 10\) square region, and a Lamb–Chaplygin dipole is placed in each square. Grid number in computation is given in Table 1.
Fig. 3 Dipole decomposition of the flow field of the Lamb–Chaplygin dipole. An arrow head shows the dipole moment $\mu_i$ and a line segment shows the length scale $a$. Contour lines show vorticity distribution of the given flow field.

Figure 3 shows the result of the dipole decomposition of the given flow field of the Lamb–Chaplygin dipole. Contour lines show vorticity distribution of the given flow field. An arrow head shows the dipole moment $\mu_i$ of the result of the dipole decomposition. The line segment orthogonal to the arrow head shows the length scale $a$ obtained by the dipole decomposition. In the following part of this paper, two small circles at the end of the line segment are called “the right foot” and “the left foot” of the dipole, where “right” and “left” are associated with the direction of the dipole moment. In the case of the Lamb–Chaplygin dipole, we obtain the left foot placed at the center of the positive vortex (red color) and the right foot placed at the center of the negative vortex (blue color) as shown in Fig. 3.

Since the given flow field is just the same as Eq.(6), only one dipole was extracted by the dipole decomposition. This result is reasonable and shows one of validation evidences of the dipole decomposition.

6.2. Circular vortex

The second example for the dipole decomposition is the following two-dimensional circular vortex:

$$ u_x = -\left\{ v(r) - \frac{1}{25} v\left(\frac{1}{25} r\right) \right\} \sin \theta, \quad u_y = \left\{ v(r) - \frac{1}{25} v\left(\frac{1}{25} r\right) \right\} \cos \theta, $$

(34)
Fig. 5 Dipole decomposition of a separated vortex pair for various $\chi$. Contour lines show vorticity distribution of the given flow field. Arrow heads show dipole moments divided by the cube of each length scale $\mu_l/a^3$ and line segments show length scales $a$. Arrow heads in five figures are plotted on the same scale. Weak dipoles for $|\mu_l/a^3| < 0.015$ are omitted. The maximum values of $|\mu_l/a^3|$ are 1.063, 0.580, 0.290, 0.142 and 0.113 when $\chi = 1, 0.5, 0.25, 0.1$ and 0.01, respectively.

where $(r, \theta)$ denotes a local polar coordinates, and
\[
v(\rho) = \begin{cases} 
(4\rho - 6\rho^3 + 4\rho^5 - \rho^7) & \text{for } \rho \leq 1, \\
\rho^{-1} & \text{for } \rho > 1.
\end{cases}
\] (35)

The above circular vortex is placed in each periodic square region. Positive vorticity is concentrated around the center of the vortex, and weak negative vorticity widely distribute in the background of the vortex, so that the circulation over the entire two-dimensional space is equal to 0.

Figure 4 shows vorticity contours of the flow field given by Eq. (34) and the result of the dipole decomposition when $\chi = 0.1$ in Eq. (31). The arrow heads showing the result of the decomposition are aligned in the tangential direction of the circular vortex. The left feet of most dipoles are placed at the center of the vortex. Thus, the line segments showing $a$ are in the radial direction.

Results of dipole decomposition somewhat depend on the value of $\chi$. Large $\chi$ decreases the number of dipoles, but symmetry of arrangement of extracted dipoles is lost. On the other hand, small $\chi$ guarantees symmetry arrangement of extracted dipoles, though the number of dipoles increases. We choose $\chi = 0.1$ for all examples in this paper.

In Fig. 4, the area of the plotted arrow heads is proportional to $|\mu_l/a^3|$, because $\mu_l/a^3$ illustrates both large scale dipoles and small scale dipoles in moderate appearance in the same sheet. The dimension of $\mu_l/a^3$ is the same as that of the dipole density $M_j$ ($s^{-1}$ in the unit). In order to avoid deterioration of visibility, plotting of weak dipoles less than 10% of the maximum value in $|\mu_l/a^3|$ are omitted.

### 6.3. Separated vortex pair

The next example is a vortex pair which has an irrotational field between the positive vortex and the negative one:

\[
u_x = -v(2r_1) \sin \theta_1 + v(2r_2) \sin \theta_2, \quad u_y = v(2r_1) \cos \theta_1 - v(2r_2) \cos \theta_2,
\] (36)
Fig. 6 History of $L^2$ norms of residuals of the reconstructed flow fields for various $\chi$ shown in Fig. 5

Fig. 7 Dipole decomposition of a vortex pair with head-tail structure for $\chi = 0.1$: (a) vorticity contours of the given flow field, (b) dipoles illustrated by arrow heads of $\mu_i/a^3$ and line segments of $a$. Plotting of weak dipoles less than 10% of the maximum value in $|\mu_i/a^3|$ are omitted.

where $(r_1, \theta_1)$ denotes a local polar coordinates around the positive-vortex center, and $(r_2, \theta_2)$ denotes another local polar coordinates around the negative-vortex center. The function $v$ is given by Eq. (35).

Figure 5 shows vorticity contours of the flow field given by Eq. (36) and the results of the dipole decomposition for various $\chi$. There are large scale dipoles at the midpoint of two vortices. Furthermore, small scale dipoles are found around each vortex. The large scale dipoles at the midpoint place their both feet near the vortex center, while the small scale dipoles place their one foot at the center of the vortex. The positive vortex is a base of the left feet, while the negative vortex is a base of the right feet.

In case of $\chi = 1$, several dipoles with $|\mu_i/a^3| > 0.5$ are extracted in asymmetry arrangement. A decrease of $\chi$ results in a decrease of the maximum value of $|\mu_i/a^3|$ and a better symmetry of dipole arrangement. Graphical representations of dipole arrangements for $\chi = 0.1$ and $0.01$ are qualitatively analogous to each other.

Figure 6 shows history of $L^2$ norms of residuals of the di-vorticity $R\omega_i^{(0)}$ in Eq.(32), the residuals of the vorticity $R\omega_i^{(n)} = \omega_i + \sum_{q=1}^{n} e_{ij} f^{(q)} (-\mu_j^{(q)} \partial_{i}^{(q)} / \partial x_j)$ and the residuals of the velocity $R u_i^{(n)} = u_i - U_{\infty} - \sum_{q=1}^{n} \mu_i^{(q)} (\delta^2 g_j^{(q)} / \partial x'_i \partial x'_j) - \delta_j^{(q)} g_j^{(q)} / \partial x'_i \partial x'_j$ of the reconstructed flow fields. The reconstructed flow fields are asymptotically closing to the given flow field. Figure 6 suggests the mean convergence rate of the residuals is proportional to $\chi$. Extraction of dipoles in case of small $\chi$ results in a smooth and slow convergence by small increments of $|\mu_i/a^3|$.

6.4. Vortex pair with head-tail structure

A flow field of a vortex pair with “head-tail structure” (Kida, 1991) is decomposed into dipoles. The vorticity distribution was obtained from a numerical simulation by a vortex method (Matsumoto et al., 2009), and the velocity at 256 x 256 grid points was calculated by Biot–Savart law.

Figure 7 shows the result of the dipole decomposition. In this case, a short column of dipoles is found. The length scales of head dipoles are larger than those of tail dipoles. This arrangement corresponds to the width of “head” and “tail” in the left counters. The column of dipoles is one of the most important structures in turbulent flows as shown later.

6.5. Jet

The following jet flow is decomposed into dipoles:

$$u_i = \cosh^{-2} (2y - 0.6 \sin \frac{1}{2} \pi x).$$

(37)

Figure 8 shows the result of the dipole decomposition of the jet. A long column of many dipoles along the centerline of
Fig. 8  Dipole decomposition of a jet for $\chi = 0.1$. Arrow heads show dipole moments divided by the cube of each length scale $\mu_i/a^3$ and line segments show length scales $a$. Contour lines show vorticity distribution of the given flow field. Plotting of weak dipoles less than 10% of the maximum value in $|\mu_i/a^3|$ are omitted.

Fig. 9  Dipole decomposition of a rolled-up shear layer, $t = 8, \chi = 0.1$: (a) vorticity contours of the given flow field, (b) dipoles illustrated by arrow heads of $\mu_i/a^3$ and line segments of $a$. Plotting of weak dipoles less than 10% of the maximum value in $|\mu_i/a^3|$ are omitted.

the jet is found. The left feet of dipoles are placed in positive-vorticity region (red color) and the right feet are placed in negative-vorticity region (blue color) of the jet. Arrangement of dipoles is similar to that of head-tail structure of the vortex pair.

6.6. Double shear layers

Rolled up shear layers are decomposed into dipoles. The velocity distribution was obtained by a numerical simulation of Ishiko et al. (2006) in a finite-difference method, where $128 \times 128$ grid points were employed for a $2\pi \times 2\pi$ periodic square region. Figure 9 shows the close-up view of the dipole decomposition of a rolled up shear layer at $t = 8$. Several dipoles in $0.84 \leq a \leq 1.41$ with the clockwise direction place their right feet in the rolled-up negative vortex. This feature is similar to that of a circular vortex, though the positions of the feet are not concentrated at the center. On the other hand, antiparallel two columns of dipoles in $0.11 \leq a \leq 0.50$ are found in narrow spiral arms of the shear layers. Dipoles in the anticlockwise columns have strong moment at the roots of the arms. These are regarded as localized jets squeezed between the rolled-up vortex and the shear layer arms.

6.7. Two-dimensional turbulent flow

A two-dimensional turbulent flow is decomposed into dipoles. Reynolds number based on the root-mean-square (r.m.s.) value of the initial velocity and r.m.s. value of the initial vorticity is 256. Evolution of the flow field was obtained
Fig. 10  Dipole decomposition of a two-dimensional turbulent flow, \(Re = 256, \tau = 2, \chi = 0.1\); (a) vorticity contours, (b) dipole decomposition in which weak dipoles less than 1/16 of the maximum value in \(|\mu_s|/\alpha^2\) are omitted.
by a direct numerical simulation of Ishiko et al. (2009) in a finite-difference method.

Figure 10 shows a close-up view of the dipole decomposition of the two-dimensional turbulent flow at dimensionless time $t = 2$. We find many columns of dipoles illustrating jet-like streams or head-tail structures of vortex pairs. Furthermore, we find many antiparallel columns of dipoles illustrating shear layers. Isolated dipoles in various length scale are also found. Each isolated dipole shows a vortex pair, that is to say, a lump of fluid which moves together.

Figure 11 shows the number of extracted dipoles $N_d$ versus the product of $\chi$ and the number $n$ of the recursive steps. About 5000 dipoles per unit $n \chi$ are newly extracted in the case of the two-dimensional turbulent flow obtained at $1024 \times 1024$ grid points. Figure 11 also shows history of residuals $|R\mathbf{u}|$, $|R\omega|$, and $|R\alpha|$. Normalized values of $L^2$ norms $|R\mathbf{u}|$, $|R\omega|$ and $|R\alpha|$ are less than 0.1% when $n \chi = 500$. In this range of $n \chi$, the number of extracted dipoles $N_d$ is in the order of $10^6$. This number is roughly equal to the grid number of the given flow field.

7. Conclusions

A decomposition method of a given two-dimensional incompressible flow field $u(x, y)$ into a dipole sequence $\{\mu_j^{(n)}, X_j^{(n)}, Y_j^{(n)}, \alpha_j^{(n)}\}$ is developed. Necessary condition for dipole sequence is revealed using a wavelet transform $m_j(X, Y, a)$ of di-vorticity of the given $u(x, y)$. Subsequently, a practical way to extract dipole sequence $\{\mu_j^{(n)}, X_j^{(n)}, Y_j^{(n)}, \alpha_j^{(n)}\}$ from $m_j(X, Y, a)$ is proposed. In this procedure, the maximum value of magnitude of $m_j(X^{(n)}, Y^{(n)}, a^{(n)})$ gives the position $(X_j^{(n)}, Y_j^{(n)}, a_j^{(n)})$ and the dipole moments $\mu_j^{(n)}$ in accordance with a recurrence formula.

Each obtained dipole is characterized not only by the dipole moment $\mu_j^{(n)}$ but also by its own length scale $a_j^{(n)}$. Truncation errors between the given flow fields and the reconstructed flow fields discussed in this paper are sufficiently small at $n \chi \sim 500$ using multiple points extraction of dipoles. In addition, the syntheses always give divergence-free fields satisfying $\partial u_x / \partial x = 0$ and $\partial a_k / \partial x = 0$ even if the dipole sequence is truncated at finite $n$.

Typical two-dimensional flows are decomposed into dipoles, and graphical representations of extracted dipoles are shown in Sec. 6. An isolated dipole shows a vortex pair, that is to say, a lump of fluid which moves together. Short column of dipoles illustrates head-tail structure of vortex pair, while long column of dipoles illustrates a jet-like streak. Many columns of dipoles and isolated dipoles in various length scale are contained in the two-dimensional turbulent flow.

Appendix A. Calculation of convolutions in Fourier space

Calculations of convolution integrals in the dipole decomposition shown in this paper were performed in Fourier space in order to decrease numerical errors and computation cost.

Let us define the Fourier transform of di-vorticity $\alpha_j$ as follows:

$$\hat{\alpha}_j(k_x, k_y) \equiv \frac{1}{(2\pi)^2} \int \alpha_j e^{-i(k_x x + k_y y)} \, dx \, dy,$$  \hspace{1cm} (A.1)

$$\alpha_j(x, y) \equiv \int \hat{\alpha}_j e^{i(k_x x + k_y y)} \, dk_x \, dk_y,$$  \hspace{1cm} (A.2)
where \( i \) is the imaginary unit. The Fourier transform of \( m_j \) is given as follows:

\[
\hat{m}_j(k_x, k_y, a) \equiv \frac{1}{(2\pi)^2} \int m_j e^{-i(k_x X + k_y Y)} dX dY,
\]

(A.3)

\[
m_j(X, Y, a) \equiv \int \hat{m}_j e^{i(k_x X + k_y Y)} dk_x dk_y.
\]

(A.4)

Furthermore, we introduce the following Fourier transform of dimensionless smoothing function \( f_0(r_0) \):

\[
\hat{f}_0(k_0) \equiv \frac{1}{2\pi} \int_0^\infty f_0(r_0) I_0(k_0 r_0) r_0 dr_0,
\]

(A.5)

where \( k_0 \) is the normalized wavenumber with respect to \( a \):

\[
k_0 = a k = a \sqrt{k_x^2 + k_y^2}.
\]

(A.6)

The convolution integrals (19) and (20) are reduced to the following algebraic products in Fourier space:

\[
\hat{m}_j(k_x, k_y, a) = \frac{(2\pi)^2 a^3}{C_{ad}} k^2 \hat{a}_r(k_x, k_y) \hat{f}_0(k_0),
\]

(A.7)

\[
\hat{a}_r(k_x, k_y) = (2\pi)^2 k^2 \int \hat{m}_j(k_x, k_y, a) f_0(k_0) da.
\]

(A.8)

It is possible to replace \( \hat{a}_r \) by \( \hat{u}_r \) as follows:

\[
\hat{m}_j(k_x, k_y, a) = \frac{(2\pi)^2 a^3}{C_{ad}} k^4 \hat{u}_r(k_x, k_y) \hat{f}_0(k_0),
\]

(A.9)

In the actual computation, this equation was used for the wavelet transform.

Substituting the wavelet transform (A.7) into the inverse wavelet transform (A.8), we obtain

\[
\hat{a}_r(k_x, k_y) = \frac{(2\pi)^4}{C_{ad}} k^4 \hat{a}_r(k_x, k_y) \int \left[ \hat{f}_0(k_0) \right]^2 a^3 da.
\]

Therefore, the admissible constant \( C_{ad} \) is determined by the following integral:

\[
C_{ad} = \frac{(2\pi)^4}{a^3} \int_0^\infty \left[ \hat{f}_0(k_0) \right]^2 k_0^3 dk_0.
\]

(A.10)

### Appendix B. Bessel type smoothing function

Bessel type smoothing function shown in Eq. (3) is adopted for dipole decomposition of this paper. The function \( g_0 \) satisfying Eq.(8) is derived from the Bessel type \( f_0 \):

\[
g_0(r_0) = \begin{cases} 
1 + J_0(\lambda_1 r_0) + \frac{1 - r_0^2}{4\pi} & \text{for } r_0 < 1, \\
-\frac{1}{2\pi} \log r_0 & \text{for } r_0 \geq 1.
\end{cases}
\]

(B.11)
Substituting this \( g_0 \) into (6), we obtain the velocity distribution of the Lamb–Chaplygin dipole (Lamb, 1932; Meleshko and van Heijst, 1994). The function \( h_0 \) introduced in (28) is also derived from \( f_0 \) as follows:

\[
h_0(r_0) = \begin{cases} 
\frac{\lambda_1^2 J_0(\lambda_1 r_0)}{\pi} & \text{for } r_0 < 1, \\
0 & \text{for } r_0 > 1.
\end{cases}
\]

(B.12)

Figure 12 shows distributions of \( g_0 \) and \( h_0 \). The function \( h_0 \) has discontinuity at \( r_0 = 1 \). Substituting \( h_0(r_0) \) into (27), we obtain

\[
C_{rp} = \frac{\lambda_1^4}{\pi} = 68.61496 \cdots.
\]

(B.13)

Substituting the smoothing function (3) into (A.5), we obtain the Fourier transform of the smoothing function:

\[
\hat{f}_0 = \frac{\lambda_1^2 J_1(k_0)}{2\pi^2 k_0 (\lambda_1^2 - k_0^2)}.
\]

(B.14)

Further, the Fourier transform of \( h_0 \) is obtained as follows:

\[
\hat{h}_0 = k_0^2 \hat{f}_0 = \frac{\lambda_1^2 k_0 J_1(k_0)}{2\pi^2 (\lambda_1^2 - k_0^2)}.
\]

(B.15)

Figure 13 shows distributions of \( \hat{f}_0 \) and \( \hat{h}_0 \) in Fourier space.

Substituting \( \hat{f}_0 \) into (A.10), we obtain

\[
C_{ad} = 4\lambda_1^4 \int_0^\infty \frac{k_0 |J_1(k_0)|^2}{(\lambda_1^2 - k_0^2)^2} dk_0 = 29.36394 \cdots.
\]

(B.16)

This value is obtained by a numerical integration.

Appendix C. Simultaneous extraction of multiple dipoles using dyadic cells

The wavelet space \( X Y a \) is divided into the following dyadic cells:

\[
2^{p-2} a_1 < a \leq 2^{p-1} a_1 \quad (p = 1, 2, 3, \cdots)
\]

(C.17)

\[
2^p a_1 (q_X - 1) \leq X < 2^p a_1 q_X \quad (q_X = 1, 2, 3, \cdots)
\]

(C.18)

\[
2^p a_1 (q_Y - 1) \leq Y < 2^p a_1 q_Y \quad (q_Y = 1, 2, 3, \cdots)
\]

(C.19)

The constant \( a_1 \) is the minimum value of \( a \) in our computation. Figure 14 shows a part of the dyadic cells.

We extract dipoles using the dyadic cells. First, we select the points \((X^{(n)}_{\nu\xi, \nu'\nu}, Y^{(n)}_{\nu\xi, \nu'\nu}, a^{(n)}_{\nu \xi, \nu'\nu})\) where \( |m^{(n-1)}(X, Y, a)| \) is equal to the maximum value in each dyadic cell. Next, we narrow down the selected points \((X^{(n)}_{\nu\xi, \nu'\nu}, Y^{(n)}_{\nu\xi, \nu'\nu}, a^{(n)}_{\nu \xi, \nu'\nu})\) to the points whose magnitude of \(|m^{(n-1)}|\) is greater than those of neighbor cells indicated in Fig. 14. Finally, dipole moments \( \mu_i^{(n)} \) of remaining points are determined according to Eq. (31).

Reproducing kernel between two arbitrary remaining points is sufficiently small. Consequently, the above simultaneous extraction of dipoles is almost equivalent to the one by one extraction described in Sec. 5.
Fig. 14 Dyadic cells in $XYa$ space. Gray cells show “neighbor cells” of the black cell $(q_x, q_y, p)$. 

References


