Secondary Flow near an Undulatory Surface Induced by Wall Blocking Effect

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Abstract
Secondary mean motions of Prandtl’s second kind near an undulatory surface are explained in terms of the turbulent blocking effect and kinematic boundary conditions at the surface, and the order of magnitude is estimated. The isotropic turbulence is distorted by the undulatory surface with a low slope $h/\lambda$, where $\lambda$ is the wavelength and $h$ is the amplitude. The prime mechanism for generating the mean flow is that the far-field isotropic turbulence is distorted by the nonlocal blocking effect of the surface to become the anisotropic axisymmetric turbulence near the surface with a principal axis that is not aligned with the local curvature (slope) of the undulation. Then, the local analysis can be applied and the mechanism is similar to that of the mean flow generation by the impingement of homogeneous axisymmetric turbulence over a planar surface, i.e. the gradient of the Reynolds stress caused by the turbulent blocking effect generates the mean motion. The results from this simple analysis are consistent with the previous exact analysis, in which the effects of the curvature are taken into account. The results also qualitatively agree with the flow visualization over an undulatory surface in a mixing box.

Key words: Boundary Layer, Secondary Flow, Turbulent Flow, Linear Theory, Undulatory Surface

1. Introduction
Turbulent flows over an undulatory surface can be seen in many environmental and industrial flows, and therefore, extensive studies have been performed on these turbulent flows(1)–(4). The previous results show that strong recirculating flows are generated in the trough region due to the separation and reattachment of the mean flow when the mean flow traverses the crests. On the other hand, in the case where the mean flow is parallel to the crests, Sajjadi et al.(5) have shown that the two pairs of circulations can be generated in the cross plane of the trough region instead of the strong recirculating flow due to the separation. The latter flow is classified as the secondary flow of Prandtl’s second kind(6) and is caused by streamwise vorticity generated by the gradients of the Reynolds stresses $u_i u_j$ (including the case $i = j$) near the undulatory surface(5). In this study, we focus on the wall blocking effects(7),(8) for the generation mechanism of the Reynolds-stress gradient near the undulatory surface. The blocking effects are derived from the kinematic condition that the parallel and normal components of the fluctuating velocity should be zero at the surface. The blocking effects become important in flows such as (i) high-Reynolds-number turbulent boundary layer(9), and (ii) flows in which the turbulent motions are generated away from the surface and transported there by the mean flow and then impacts onto a rigid surface, whereas the turbulent generation by the local mean shear at the surface is small(10)–(13). Due to the boundary
condition at the surface, the normal component of the velocity fluctuation to the surface decreases towards the surface, and its energy is redistributed to the velocity components parallel to the surface, which induces the splat effect (i.e. the turbulent components parallel to the surface are amplified near the surface)(7).

In this study, we analytically obtain the Reynolds stress due to the blocking effect near the undulatory surface. Then, the overall profile and amplitude of the secondary flow are obtained by considering the mechanism of the secondary-flow generation induced by the anisotropy of turbulence in the free stream(9). The detailed analysis of homogeneous isotropic turbulence over the undulatory surface, including the curvature of the undulation, has been reported(5), (14). The present analysis is for the case wherein the amplitude $h$ of the undulation is much smaller than the integral length scale $L_{\infty}$ away from the surface, and the wavelength $\lambda$ is much larger than $L_{\infty}$. In this case, the turbulence near the undulatory surface can be regarded as that near the planar surface. Therefore, the present analysis shows the principal mechanism of the generation of secondary flow over the undulatory surface, including the discussion of the effect of curvature on the secondary flow. Flow visualization is conducted in the mixing box with the undulatory surface, and the result is compared with the analysis.

2. Analysis of secondary flow based on the wall blocking effect

Consider a rigid undulatory surface with a low slope, amplitude $h$ and long wavelength $\lambda$, defined by

$$y_o = h \cos \frac{2\pi x_o}{\lambda}$$

(Fig. 1). It is assumed that the turbulence above the surface wave (where $y \geq L_{\infty}$) is homogeneous and isotropic. Note that the turbulence is not periodic. Its integral length scale $L_{\infty}$ ($= \int_0^\infty u_\infty(x + r)u_\infty(x)/v'_\infty dr$) is smaller than $\lambda$, where $u_\infty$ is the fluctuating velocity and $v'_\infty$ is the rms velocity away from the surface. It is also assumed that $h$ is much smaller than $L_{\infty}$, i.e.

$$h \ll L_{\infty} < \lambda.$$  

For simplicity and for considering the mechanism of the generation of secondary flow due to the pure blocking effect, it is also assumed that the free-stream mean velocity is zero. Note that the analysis can also be applied to turbulence with the mean flow parallel to the wave crests and troughs when the blocking effect is predominant(7),(14). On the other hand, with the mean flow traversing the crests, flow separation and recirculation occur in the trough region and therefore the present analysis can not be applied to these flows.

The initial and boundary conditions are as follows.

\begin{align*}
  u &= u_\infty \quad \text{at } -\infty < (x_0, y_0, z_0) < \infty, \quad \text{for } t < 0 \\
  u &= 0 \quad \text{at } y = 0, -\infty < (x_0, z_0) < \infty, \quad \text{for } t \geq 0 \\
  u &= u_\infty \quad \text{as } y_o \to \infty, -\infty < (x_0, z_0) < \infty, \quad \text{for } t \geq 0
\end{align*}
Two separate regions with differing dynamics determine the distortion of the fluctuating velocity field\(^7\). The outer, source region \(B^{(s)}\) has a thickness of the order of \(L_\infty\). The inner, viscous region \(B^{(v)}\) has a thickness of the order of \(\delta^{(v)} \sim 4(\nu t)^{1/2}\) (\(\nu\) is the kinematic viscosity of the fluid), which grows with time \(t\) according to the linear theory and within this region the turbulent velocity decays to zero at the surface, i.e. \(y = 0\). In this analysis, we ignore \(B^{(v)}\) because of the large \(Re_L = \nu' L_\infty / \nu\), i.e. we assume \(\delta^{(v)} \ll \delta^{(s)}\), where \(\delta^{(v)}\) and \(\delta^{(s)}\) are the viscous-region thickness and source-region thickness, respectively.

Under the condition of Eq. (2), the turbulent field near the undulatory surface can be approximated by the turbulent field over the inclined flat surface with the angle \(\gamma\) from the \(x_0\) axis (Fig. 2). From Eq. (1),

\[
\tan \gamma = \frac{2\pi h}{\lambda} \sin \frac{2\pi x}{\lambda}.
\]

Here, only the analysis for \(0 \leq x_0 \leq \lambda/2 (0 \leq \gamma \ll \pi/2)\) is shown. The same discussion is applied to the region \(\lambda/2 \leq x_0 \leq \lambda\), where only the angle of the slope changes. Due to the blocking effect, the turbulent component normal to the surface decreases, whereas the turbulent components parallel to the surface are amplified\(^7\). This leads to a change in the turbulence from isotropic to axisymmetric near the surface, i.e.

\[
u_{2a} < \nu_{1a} = \nu_{3a},
\]

followed by Hunt\(^7\). Here, the subscripts \(1a\) and \(3a\) denote the radial components and the subscript \(2a\) denotes the axial component of the axisymmetric turbulence. On the other hand, the turbulence away from the surface remains isotropic. This implies that the axis of the turbulence is not perfectly perpendicular to the surface, instead, it has a small angle \(\beta (\ll \gamma)\) (see Fig. 2). It is assumed that

\[
\alpha = \gamma - \beta = f \gamma,
\]

where \(f\) is a function. Then, the limiting values of \(f\) may be expressed as

\[
\begin{cases}
  f = 0 & \text{as } L_\infty / \lambda \to 0, \\
  f \sim 1 & \text{as } L_\infty / \lambda \sim 1,
\end{cases}
\]

which implies that when the wavelength \(\lambda\) is much larger than the integral length scale \(L_\infty\) (i.e. \(L_\infty / \lambda \to 0\)), the turbulent field is similar to that over the flat plate, in which the axis of symmetry lies perpendicular to the \(x\) axis\(^7\), and that when \(\lambda\) has the same order as \(L_\infty\) (i.e. \(L_\infty / \lambda \sim 1\)), the blocking is an average effect, and therefore the axis of symmetry lies perpendicular to the \(x_0\) axis, where \(x_0\) is the global axis, and \(x\) is the local axis aligned with the surface (see Fig. 2). Therefore, we assume that \(f \sim L_\infty / \lambda\). The angle \(\alpha\) is given by

\[
\alpha = \frac{L_\infty}{\lambda} \arctan \left( \frac{2\pi h}{\lambda} \sin \frac{2\pi x}{\lambda} \right).
\]
Fig. 3 Sketch of mean flow (secondary flow of Prandtl’s second kind) near the undulatory surface

Under the above assumptions, the turbulent field near the undulatory surface is similar to that over a planar surface, in which an axisymmetric turbulence whose axis of symmetry lies at an angle $\alpha$ from the surface impacts onto the surface (8). Thus, the time derivative of the secondary flow $U_S$ is given for $t < T_L = L_\infty / v'_\infty$ as

$$\frac{\partial U_S}{\partial t} = \frac{\partial}{\partial y} (\mu') \sim -v'_\infty \frac{\bar{v}'^2}{2L_\infty} \sin 2\alpha \left(1 - \frac{1}{R}\right),$$

where $R$ is the ratio of the largest to the smallest variances of the velocity components. In short, the Reynolds stress originating from the blocking effect decreases towards the surface due to the boundary condition at the surface, i.e. the velocity fluctuation should be zero at the surface, leading to the large gradient of the Reynolds stress near the surface. From Eq. (9), for the short time $t$ (i.e. much lesser than the eddy turnover time $T_L$), we obtain

$$U_S(t < T_L) \sim -v'_\infty \frac{\bar{v}'^2}{L_\infty} (R \gg 1 \text{ near } y = 0).$$

Equation (10) implies that the secondary flow is the strongest at $x_0 = \lambda/4$, and its magnitude is given by

$$U_S(t < T_L) \sim \frac{2\pi h v'_\infty}{\lambda^2}.$$ 

Similarly, for $\lambda/2 \leq x_0 \leq \lambda$, we obtain

$$U_S(t < T_L) \sim \frac{2\pi h v'_\infty}{\lambda^2}.$$ 

The results show that the secondary flow of Prandtl’s second kind occurs from trough to crest as sketched in Fig. 3. Since the Reynolds stress away from the surface is zero, the sign of the Reynolds stress gradient must change in the source layer $B^{(3)}$. Therefore, the secondary flow should circulate in the source layer $B^{(3)}$.

In the quasi-steady state where $t \sim T_L$, it can be approximated that

$$\overline{\left(\mathbf{U} \cdot \Delta \mathbf{U}_S\right)} \sim \frac{2\pi h v'_\infty}{\lambda^2}.$$ 

Therefore, the mean velocity of the secondary flow is given by

$$\overline{|U_S|}(t \sim T_L) = v'_\infty \sqrt{\frac{2\pi h}{\lambda}}.$$ 

In most cases, $h/\lambda = 0.05 \sim 0.1^{(2)-(4)}$. Therefore, the secondary flow induced by the blocking effect has a order of $v'_\infty$.

3. Discussion

3.1. Comparison with the analysis considering the effect of curvature

The predictions of the secondary flow at $t < T_L$ (Eqs. (11) and (12)) and in the quasi-steady state at ($t \sim T_L$) (Eq. (14)) agree with the exact analysis that considers the curvature.
of the undulation\(^{(5),(14)}\). This implies that in the following vorticity equation for the two-dimensional undulatory surface
\[
\frac{\partial \Omega}{\partial t} = \frac{\partial^2}{\partial x_0^2} \left( \frac{\partial^2}{\partial y_0^2} \right) (\overline{t^2 - v^2}) + \left( \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial y_0^2} \right) (-\overline{w}) + \nu \Delta^2 \Omega_z + O \left( \frac{L_{\infty}^4}{L_{\infty}^4} \right), \tag{15}
\]
the leading term is the second derivative of the Reynolds stress \(\frac{\partial^2}{\partial y_0^2} \overline{w}^2\) and the \(x_0\)-derivative arising from the curvature of the undulatory surface can be neglected. It should be noted that if we take into account the effect of the curvature under the condition of \(L_{\infty}/\lambda \ll 1\), the exact analysis has shown that the ‘wave layer \(B^{(w)}\)’ is set up outside the source layer \(B^{(s)}\), with a length scale of \(\lambda/\pi\)\(^{(5),(14)}\), leading to the weak recirculating mean flow \(U_w\) in \(B^{(w)}\). However, the magnitude of the secondary flow in the outer wave layer \(B^{(w)}\) is
\[
|U_w| = \frac{L_{\infty}}{\lambda} |U_S| \ll |U_S|, \tag{16}
\]
and therefore it is negligibly small as compared to the secondary flow \(|U_S|\) in \(B^{(s)}\) obtained in this analysis.

![Fig. 4 (a) Schematic of experimental apparatus (b) Sketch of the dispersion of dye on the undulatory surface](image)

### 3.2. Experimental observation of secondary flow over the undulatory surface

A qualitative laboratory experiment was performed to test the theoretical prediction. The experiment was conducted inside a mixing tank with quasi-homogeneous isotropic turbulence generated by an oscillating grid, as shown in Fig. 4(a). The installation used was based on that constructed by Thompson & Turner (1975). The mixing tank was a box made of a 1 cm perspex sheet and internally measuring 25.4 cm\(^2\) with a depth of 45.7 cm. An undulatory surface with wavelength \(\lambda = 7\) cm and amplitude \(h = 0.7\) cm was fixed at the bottom of the tank. The grid comprised square bars with a mesh size of \(M = 5\) cm and the bar width \(d = 1\) cm, i.e. the solidity is 0.36. The grid was placed approximately 11 cm above the bottom of the tank. The grid was oscillated vertically with a frequency of 11 Hz and stroke of 2.5 mm, which produces a nearly isotropic turbulence with sufficiently large \(Re_L\). The integral length scale was \(L_{\infty} \sim 1\) cm and the rms velocity was approximately 0.5 cm/s, so that \(h/L_{\infty} = 0.01\), \(L_{\infty}/\lambda = 1/7\), \(h/\lambda = 1/10\) and \(Re_L = 50\). The amplitude of the undulatory surface was larger than that assumed in the theory since the secondary flow over a low-amplitude surface undulation (where \(h \ll L_{\infty}\)) would be too weak to be visualized.

Figure 4(b) illustrates a typical result showing the manner in which dye released in the trough moves up the slopes and is elongated, whereas over a flat surface it spreads isotropically. The direction of the mean flow near the crests is clearly upward and away from the
crests. Note that the spread of the dye is mainly caused by the advection by the secondary flow, since the dye is released and spread in the vicinity of the surface. In fact, the mean spread rate of the dye is approximately 0.5 cm/s, which is close to $U_S \sim 0.4$ cm/s predicted by Eq. (14). The results also confirm that the present analysis is also valid for a wide range of wave slopes (i.e. amplitude/wavelength) and that the effect of curvature on the formulation of secondary flow over the undulatory surface is negligibly small.

4. Conclusions

Secondary mean motions of Prandtl’s second kind near an undulatory surface are explained in terms of the wall blocking effect of homogeneous isotropic turbulence and kinematic boundary conditions at the surface. The results of the simplified local analysis proposed in this study are consistent with the previous analysis performed by taking into account of the effects of curvature. The results also qualitatively agree with the flow visualization over an undulatory surface in a mixing box. The results show that the derivative of the Reynolds stress originating from the wall blocking of initially isotropic turbulence, and not the curvature, is the leading cause of the generation of the secondary flow over the undulatory surface.

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