Application of Dynamic Smagorinsky Model to the Finite Difference Lattice Boltzmann Method*

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Abstract
Three-dimensional turbulent flows past a square cylinder are simulated by the Finite Difference Lattice Boltzmann Method (FDLBM). We carry out the simulation in two approaches. First, we carry it out using fourth order numerical viscosity without any turbulent model, and second, we simulate using Dynamic Smagorinsky Model (DSM). These results are compared with those by the experiment conducted by Lyn et al. Numerical results by DSM agree with experimental ones and it is confirmed that this method is useful for numerical simulations of turbulent flows.

Key words: Computational Fluid Dynamics, Lattice Boltzmann Method, Turbulent Flow, Incompressible Flow, Large-Eddy Simulation

1. Introduction
The Finite Difference Lattice Boltzmann Method (FDLBM) is a novel method of the computational fluid dynamics. According to recent research, it has been confirmed that the FDLBM has an advantage that the acoustic field can be calculated with high accuracy and less calculation cost(1), (2). On the other hand, simulation of turbulence by the FDLBM is still under development. The estimation of turbulent flows is very important for industrial problem. However, direct numerical simulation of turbulence based on the Navier-Stokes equations is still hard task due to the complex and small structure of turbulence. The FDLBM is a suitable method for simulation of turbulence because it is the model based on microscopic equation. In addition, the FDLBM is adequate for parallel computation. Therefore, we consider that the FDLBM is useful method to simulate acoustic field with turbulence. It is important to simulate turbulent flow around bluff body correctly in order to estimate aerodynamic sound in a turbulent flow. In this report, we introduce a turbulence model into the FDLBM and carry out Large-Eddy Simulation (LES) of three-dimensional turbulent flow past a square cylinder. The result is compared with experimental one conducted by Lyn et al.(3) in order to confirm the efficacy of this model.

2. The Finite Difference Lattice Boltzmann Method
2.1. Discrete BGK equation
In the FDLBM, the evolution of the distribution function \( f_i \) for the particle velocity \( c_{i\alpha} \) is governed by the following equation (the discrete BGK equation)(4):

\[
\frac{\partial f_i}{\partial t} + c_{i\alpha} \frac{\partial f_i}{\partial x_\alpha} = \frac{f_i - f_i^{(0)}}{\phi} = -\frac{1}{\phi} (f_i - f_i^{(0)})
\]

where the subscript \( i \) indicates the particle’s direction and \( \alpha \) indicates spatial component \( x, y, \) and \( z \). \( f_i^{(0)} \) is the local equilibrium distribution function and \( \phi \) is the relaxation parameter. The
constant $a(>0)$ is the coefficient of negative viscosity and computational time can be shortened by this coefficient in high Reynolds number flows\(^5\). The macroscopic variables, the density and the momentum are defined as velocity moments of the distribution function:

\[
\rho = \sum_i f_i \\
\rho u_\alpha = \sum_i f_i c_i \alpha
\]  

(2.a) \hspace{1cm} (2.b)

2.2. BGK Lattice Boltzmann model in 3D

In this report, we use three-dimensional 15 velocity model (D3Q15) as the incompressible fluid model of the FDLBM. Figure 1 shows the discrete velocity set in D3Q15 model. In incompressible fluid model, the local equilibrium distribution function is defined as second order polynomial of velocity as follows:

\[
f^{(0)}_i = \rho \left[ A_n + B_n c_i u_\alpha + C_n (c_i u_\alpha)^2 + D_n u_\alpha^2 \right]
\]  

(3)

where the subscript $n$ is set to 0, 1, and 2 corresponding to the 0, 2, and $\sqrt{3}$ particle velocity, respectively.

The Navier-Stokes equations can be derived from the discrete BGK equation through the Chapman-Enskog expansion procedure and then the model coefficient $A_n$, $B_n$, $C_n$, and $D_n$ in Eq. 3 are determined as follows:

\[
A_0 = \frac{1}{23}, B_0 = 0, C_0 = 0, D_0 = -\frac{7}{24} \hspace{1cm} (4.a)
\]

\[
A_1 = \frac{1}{23}, B_1 = \frac{1}{24}, C_1 = \frac{1}{32}, D_1 = -\frac{1}{48} \hspace{1cm} (4.b)
\]

\[
A_2 = \frac{2}{23}, B_2 = \frac{1}{12}, C_2 = \frac{1}{16}, D_2 = -\frac{1}{24} \hspace{1cm} (4.c)
\]

The pressure $P$ and kinetic viscosity coefficient $\nu$ have the following relations in D3Q15 model:

\[
P = \frac{24}{23} \rho \hspace{1cm} (5.a)
\]

\[
\nu = \frac{2}{3}(\phi - a) \hspace{1cm} (5.b)
\]

3. Large-Eddy Simulation

3.1. FDLBM subgrid model

The basic principle in LES is that large scale motions are resolved and only unresolved small scale motions are modeled. To realize this, one needs a scale separation decomposing the unknowns into a local average (large scale) and a subgrid scale component (small scale) by applying the filtering operation\(^6\):

\[
\bar{u}(t,x) = \int_{-\infty}^{\infty} G(t,y) u(x-y) dy
\]  

(6)
where \( u \) is the filtered variable such as the density, the flow velocity, and the pressure. The function \( G(t, y) \) is filter function which has the characteristic length of the grid width. In order to transform the governing equation into one depending only on local averages, the filter operator is operated to the discrete BGK equation:

\[
\frac{\partial f_i}{\partial t} + c_i \frac{\partial f_i}{\partial x_i} - a c_i \frac{\partial f_i}{\partial x_i} \phi_{\text{total}} = -\frac{1}{\phi_{\text{total}}} (f_i - f_i^{(0)})
\]

(7)

where the relaxation parameter \( \phi_{\text{total}} \) is the spatiotemporal variable and shows the effects of the subgrid scale component. If the relaxation time depends on time and space, we can derive the filtered Navier-Stokes equations from the filtered discrete BGK equation (7) – (11):

\[
\frac{\partial \bar{u}_\alpha}{\partial x_\alpha} = 0 \quad (8.a)
\]

\[
\frac{\partial \bar{u}_\alpha}{\partial t} + \bar{u}_\beta \frac{\partial \bar{u}_\alpha}{\partial x_\beta} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_\alpha} + \frac{\partial}{\partial x_\beta} \{ 2(\nu + \nu_t) \bar{S}_{\alpha\beta} \}
\]

(8.b)

where \( \nu_t \) is the eddy viscosity coefficient and \( \bar{S}_{\alpha\beta} \) is the strain rate tensor of the filtered velocity determined by:

\[
\bar{S}_{\alpha\beta} = \frac{1}{2} \left( \frac{\partial \bar{u}_\alpha}{\partial x_\beta} + \frac{\partial \bar{u}_\beta}{\partial x_\alpha} \right)
\]

(9)

3.2. Dynamic Smagorinsky Model

We use the Dynamic Smagorinsky Model (DSM) proposed by Germano(12). Smagorinsky proposed the first subgrid scale stress model. In this model, the anisotropic part of the subgrid scale stress tensor takes the Boussinesq eddy viscosity form:

\[
\tau_{\alpha\beta} = -\frac{\delta_{\alpha\beta}}{3} \tau_{kk} = -2\nu_t \bar{S}_{\alpha\beta}
\]

(10)

where \( \delta_{\alpha\beta} \) is the Kronecker delta. In the Smagorinsky model, the subgrid eddy-viscosity is defined as follows:

\[
\nu_t = C \Delta^2 | \bar{S} |
\]

(11.a)

\[
| \bar{S} | = \sqrt{2\bar{S}_{\alpha\beta} \bar{S}_{\alpha\beta}}
\]

(11.b)

Here, \( C \) is the square of the Smagorinsky constant \( C_s \) and \( \Delta \) is the filter width (grid size) and is taken to be the geometric average of the grid spacing in three directions, \( \Delta = (\Delta x \Delta y \Delta z)^{1/3} \). The major defect of the Smagorinsky model is the freedom of choice of constant \( C_s \), because there is no standard way of choosing this constant. Ad hoc adjustments have to be made for different flow type and non-equilibrium flows to account properly for energy dissipation. To desire to eliminate these problems, the dynamic model was proposed by Germano et al. In this model, the new filter operator \( \tilde{G}(t, y) \) is introduced as the test filter function in order to determine the coefficient \( C \) from the instantaneous values of the flow velocity. In this report, the characteristic length of the test filter is set to twice of the grid width and the test filter function is approximated using the finite difference scheme as follows:

\[
\tilde{u}_i = \frac{\bar{u}_{i-1} + 4\bar{u}_i + \bar{u}_{i+1}}{6}
\]

(12)

In this model, along with Lilly’s modification(13), \( C \) is determined from:

\[
C = -\frac{\lambda_{\alpha\beta} \overline{M_{\alpha\beta} M_{\alpha\beta}}}{M_{\alpha\beta} M_{\alpha\beta}}
\]

(13.a)

\[
\lambda_{\alpha\beta} = \overline{\bar{u}_\alpha \bar{u}_\beta} - \overline{\bar{u}_\alpha} \overline{\bar{u}_\beta}
\]

(13.b)
\[
M_{\alpha \beta} = \tilde{\Delta}^2 |\tilde{\Delta}^2 |\tilde{S}| - \tilde{\Delta}^2 |\tilde{\Delta}^2 |\tilde{S}| \alpha \beta
\]  
(13.c)

When we employ subgrid model for the FDLBM, we determine the relaxation time \( \phi_{\text{total}} \) in D3Q15 model as follows:

\[
\nu_{\text{total}} = \nu + \nu_t = \frac{2}{3} (\phi_{\text{total}} - a)
\]  
(14)

\[
\phi_{\text{total}} = \frac{3}{2} (\nu + C \Delta^2 |\tilde{S}|) + a = \phi + \frac{3}{2} C \Delta^2 |\tilde{S}|
\]  
(15)

4. Flow past a square cylinder

4.1. Problem description

Three-dimensional turbulent flows past a square cylinder are simulated in order to confirm efficacy of the FDLBM subgrid model. We carry out in two approaches. First, we carry out the calculation without any turbulent model in Case1, and second, we simulate using DSM in Case2.

The geometry of the computational domain is shown in Fig. 3. As seen, a square cylinder with a side \( D \) is exposed to a constant free stream velocity \( U_0 \). The flow is described in a Cartesian coordinate system \( x, y, \) and \( z \) in which the \( x \)-axis is aligned with the inflow direction, the \( z \)-axis is parallel with the cylinder axis and the \( y \)-axis is perpendicular to both \( x \) and \( z \). The origin of the coordinate system is located at the center of the cylinder. The computational domain covers \( 31 \times 21 \times 2 \) in the streamwise(\( x/D \)), lateral(\( y/D \)), and spanwise(\( z/D \)) directions. The number of grid points is set to \( 160 \times 121 \times 41 \). The minimum distance from the cylinder surface to the nearest grid point is \( 0.025D \). The mesh is nonuniform on the \( x \) and \( y \) directions but uniform in \( z \) direction. The scheme is explicit in time and the second order Runge-Kutta scheme is used for the time integration. The convective term is discretized using the fourth order central differencing scheme added artificial viscosity term:

\[
\frac{\partial u_i}{\partial x} = \frac{u_{i-2} - 8u_{i-1} + 8u_{i+1} - u_{i+2}}{12\Delta x} + \frac{W}{12} \frac{\Delta x^3}{\Delta x^2} u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}
\]  
(16)

where the coefficient \( W \) is the parameter determining the weight of the artificial viscosity term. Central differencing schemes are often unstable and oscillatory. Therefore, the artificial viscosity term is added in order to stabilize the simulation. The weight of the numerical viscosity term should be as small as possible to simulate stably and set to 0.4 and 0.2 for Case1 and Case2, respectively. In Case1, the numerical viscosity is large enough to stabilize the calculation, and in Case2, on the other hand, the numerical viscosity is negligible comparing with the eddy viscosity. A uniform flow is prescribed at the inlet and the outlet as a boundary condition. No-slip condition is prescribed at the cylinder surface. In the spanwise direction, a periodic boundary condition is employed. The Reynolds number based on the cylinder width \( D \), and the velocity at the inflow boundary \( U_0 \), is \( 2.14 \times 10^4 \). No velocity fluctuation is imposed at the inflow boundary.
4.2. Numerical results

Figure 4 represents the distribution of the time-averaged streamwise velocity \( \overline{u} \) at \( y/D = 0.0 \). The distribution of the mean square of the streamwise and lateral velocity fluctuations \( u'^2 \), \( v'^2 \) at \( y/D = 0.0 \) are shown in Fig. 5 and 6. The circular symbol represents experimental result conducted by Lyn et al. The broken and solid lines show numerical results by the FDLBM (Case1) and the FDLBM subgrid model (Case2), respectively. Additionally, the time-averaged streamwise velocity calculated by LES based on the Navier-Stokes equations \(^{(14)} \) is shown in Fig. 4 by dashed-dotted line for reference. Results of the velocity fluctuations calculated by the Navier-Stokes equations are indicated in Ref. (15) and the tendency of our result is similar to those results. We can confirm that Case2 has improvement compared with Case1 and good agreement with experimental one. In the region of \( x/D > 4.0 \), the streamwise velocity is overestimated. It is discussed that the velocity distribution in far wake region is
affected by the disturbance of inflow condition and the velocity field tends to be overestimated in Ref. (16). However, this reason is not clear. It should be clarified in the future. The distribution of the time-averaged streamwise velocity and the mean square of the streamwise velocity fluctuation in the side of the square cylinder (Fig. 7) are shown in Fig. 8 and 9. The distributions of the time-averaged streamwise velocity agree with experimental one in both cases. These results show that the backflow caused by the separation of the flow is reproduced by these simulations. While, Case1 hardly catches the velocity fluctuation due to numerical viscosity. Thought result of Case2 has the better agreement with experimental one than Case1, the velocity fluctuation is underestimated. Thus, numerical viscosity must be suppressed by simulating with a finer mesh.

4.3. Flow field around the cylinder

Figures 10(a)-(c) and 11(a)-(c) represent the vorticity distribution in the x-y plane at t=140, 150, and 160. Behind the square cylinder, Karman vortex shedding occurs and the
Fig. 9 Distribution of $u''$ in the side of square cylinder

(a) $x = -0.5$
(b) $x = -0.25$
(c) $x = 0$
(d) $x = 0.25$
(e) $x = 0.5$

Fig. 10 Vorticity distribution in $x - y$ plane (a)(b)(c) and isosurface of vorticity at $t = 150$ (d) for calculation by numerical viscosity

(a) $t = 140$(Case1)  (b) $t = 150$(Case1)  (c) $t = 160$(Case1)  (d) $t = 150$(Case1)

(a) $t = 140$(Case2)  (b) $t = 150$(Case2)  (c) $t = 160$(Case2)  (d) $t = 150$(Case2)

Fig. 11 Vorticity distribution in $x - y$ plane (a)(b)(c) and isosurface of vorticity at $t = 150$ (d) for calculation by LES

(a)(b)(c) and isosurface of vorticity at $t = 150$ (d) for calculation by LES
complex structure of small vortices can be observed beside the cylinder in both cases. Additionally, we can confirm that the vortex structure of Case2 is more complicated than one of Case1. Isosurface of vorticity when the cylinder is looked diagonally downward at \( t=150 \) are shown in Fig. 10(d) and 11(d). Even a quasi-two-dimensional flow around a cylindrical body has three-dimensional fluctuation at high Reynolds number.

### 4.4. Dependence on calculation mesh

We simulate the same problem by the FDLBM subgrid model with coarser mesh than Case1 and Case2 in order to confirm the dependence on calculation mesh. The grid number is \( 140 \times 101 \times 26 \) and the minimum distance from the cylinder surface to the nearest grid point is \( 0.033D \). Other calculation conditions are the same in Case2. The distribution of the time-averaged streamwise velocity and the mean square of the streamwise and lateral velocity fluctuations at \( y/D = 0.0 \) are compared with Case2 as shown Fig. 12, 13, and 14, respectively. In this case, the streamwise velocity near by the cylinder agrees with that of Case2 and experimental one despite using the coarse mesh. However, the streamwise velocity in the region of \( x/D > 4.0 \) is overestimated more than Case2. It is considered that the recovering of the velocity loss behind the cylinder is prompted by numerical viscosity\(^{(17)}\). In addition, we can confirm that the velocity fluctuations are underestimated in the case with coarser mesh. As a result, the effect of numerical viscosity should be suppressed as small as possible in the simulation of turbulent flow by the FDLBM.

![Fig. 12 Distribution of mean-velocity in the streamwise direction](image)

![Fig. 13 Distribution of \( u'^2 \) in the streamwise direction](image)
5. Conclusion

We introduce the subgrid model to the FDLBM incompressible fluid model as the FDLBM turbulence model. We carry out two simulations. First, we carry it out using fourth order numerical viscosity without any turbulent model, and second, we simulate using DSM. Numerical results of the turbulence statistic quantities, such as the time-averaged velocity and the mean square of the velocity fluctuations, by the FDLBM subgrid model agree with experimental ones conducted by Lyn et al. In addition, complex and small structure of vortex beside the cylinder is observed from vorticity contours. It is confirmed that this method is useful for numerical simulations of turbulent flows.

References


