Lattice Boltzmann Method for Fluid Flows in Anisotropic Porous Media with Brinkman Equation∗

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Abstract
The lattice Boltzmann method (LBM) is applied to simulation of fluid flows in anisotropic porous media with the Brinkman equation. The Brinkman equation is recovered from a kinetic equation for the distribution function that has a forcing term to introduce anisotropy of the permeability of the porous media. Since the forcing term contains the drag force proportional to the fluid velocity, the LBM needs matrix calculation to obtain the fluid velocity through the definition of the momentum. The velocity profiles of the LBM show good agreement with analytical solutions for the Poiseuille flow and for the Couette flow filled with anisotropic porous media. The contour lines of the stream functions obtained by the LBM show good agreement with those of the finite difference method (FDM) in the numerical simulation of a lid-driven cavity flow for different fundamental parameters, e.g., Darcy number, inclination of the principal permeability direction, and permeability ratio. By increasing grid size, the LBM is able to decrease the compressibility effect, and reduces the deviation of the maximum values of the stream function between the LBM and the FDM. On the same grid size, the LBM takes less time than the conventional FDM to get the steady solutions. This paper leads to the conclusion that the LBM can simulate incompressible flow in anisotropic porous media at the representative elementary volume scale.

Key words: Anisotropic Porous Media, Lattice Boltzmann Method, Computational Fluid Dynamics, Lid-Driven Cavity Flow

1. Introduction

The mass transfer in an anisotropic porous media is significant area of research due to its wide range of natural phenomena and engineering applications, such as flow through tissues of human body, contaminant transport in groundwater, and alloy solidification. Most of the studies have been concerned with homogeneous isotropic porous structures. Recently, however, a lot of researchers investigated the effects of inhomogeneity and anisotropy in porous media(1). Since the grain or the fibers have asymmetric geometry or preferential orientation, the anisotropy is the general property of a lot of porous media in industry and nature. The numerical studies of natural convective flow in anisotropic porous media were conducted by use of Brinkman equation(2) or Brinkman-Forchheimer equation with permeability tensor(3). They demonstrated that their formulations were accurate in predicting the flow and heat transfer for various inclinations of the principal permeability direction, permeability ratios, and Darcy numbers.

In recent years, the lattice Boltzmann method (LBM) has received considerable attention as an alternative numerical scheme for simulating complex phenomena(4), (5). In the LBM, the system evolves according to the updating rules that particle velocity distribution function moves along lattices with iterating propagation and collision. Accumulation of these particle motions reproduces macroscopic flow dynamics. Such a simple model substantially reduces
computer load. The LBM has the following advantages: 1) The convection operator of the kinetic equation is linear; 2) The LBM does not require special treatment, such as iteration or relaxation to satisfy the continuity equation; 3) The independent calculation of the individual particles motions makes easy implementation of complicated boundary possible. On the other hand, unfortunately the LBM has the disadvantage that researchers must carefully determine the relaxation time to make it work adequately in the near-incompressible regime, because the kinetic equation recovers the compressible Navier Stokes equation through the Chapman-Enskog procedure.

The LBM is successfully applied to study of the isothermal flows in porous media at the pore scale. The simple bounce-back rule for no-slip boundary condition enables the LBM to obtain the detailed local information of the flow in porous media at the pore scale. However, the calculation at the pore scale needs detailed pore geometries, and the size of computation domain has its limit due to the computer resources. D. Zang proved that the LBM required more than $240^3$ lattices to get the constant macroscopic quantities, such as the porosity, from numerical results on the fluid flow in packed beds filled with crushed glass beads at pore scale. Namely, the approach at the pore scale is perhaps unrealistic to simulate the fluid flow in porous media at the representative elementary volume (REV) scale that is the minimum size to determine the macroscopic quantity such as permeability. The LB models for porous media flow at the REV scale have already been proposed and succeeded to accurately recover Darcy’s law or Brinkman correction. M. A. A. Spaid completely recovered the Brinkman equation by modifying the particle equilibrium distribution function and incorporating a forcing term to the momentum. The natural convection in isotropic porous media was also studied with two distribution functions that one calculates fluid dynamics through porous media, and the other does the energy equation.

The objective of this paper is to confirm the applicability and the reliability of the LBM in simulation of fluid flow in anisotropic porous media. In the following section, I explain the LB model for incompressible flow in anisotropic porous media in detail. In §3, this LBM is validated by the numerical simulations including the Poiseuille flow and Couette flow, and lid-driven cavity flow in anisotropic porous media.

**nomenclature**

- $c$: lattice spacing
- $D_a$: Darcy number
- $e_i$: discrete velocity
- $f_i$: distribution function
- $f_i^{eq}$: equilibrium distribution function
- $F$: body force
- $F_i$: forcing term
- $g$: acceleration due to gravity
- $H$: characteristic length
- $k^*$: ratio of principal permeabilities
- $K_x, K_y$: principal permeability values
- $p$: pressure
- $\bar{\rho}$: mean density
- $\rho$: density
- $\psi$: stream function
- $\theta$: direction of principal permeability
- $U_0$: characteristic velocity
- $\Delta$: mean variation of density
- $\nu$: kinematic viscosity
- $\nu_e$: effective viscosity
- $\tau$: relaxation time
- $\delta_t$: time step
- $\delta_t$: time step
- $\theta$: direction of principal permeability
- $\psi$: stream function
- $\psi$: stream function
- Re: Reynolds number
- $u$: velocity
- $\delta_1$: time step

**2. The lattice Boltzmann model**

The LBM solves the following kinetic equations for the distribution function $f_i$,

$$f_i(x + e_i \delta_t, t + \delta_t) - f_i(x, t) = \frac{f_i(x, t) - f_i^{eq}(x, t)}{\tau} + \delta_t F_i. \quad (1)$$

For the $D2Q9$ model, the discrete velocities are defined by

$$e_0 = (0, 0),$$

$$e_i = c \cos((i - 1) \pi/2), \sin((i - 1) \pi/2), \quad i = 1 - 4,$$

$$e_i = \sqrt{2} c \cos((i - 5) \pi/2 + \pi/4), \sin((i - 5) \pi/2 + \pi/4), \quad i = 5 - 8.$$
Here, \( c \) is the lattice spacing. The equilibrium function \( f_{eq}^i \) for the D2Q9 model is given by

\[
f_{eq}^i = \omega_i \rho \left[ 1 + \frac{3e_i \cdot \mathbf{u}}{c^2} + \frac{9(e_i \cdot \mathbf{u})^2}{2c^4} - \frac{3\mathbf{u}^2}{2c^2} \right],
\]

where \( \omega_i \) is weight. The weights are \( \omega_0 = \frac{4}{9} \), \( \omega_i = \frac{1}{9} \) for \( i = 1-4 \), and \( \omega_i = \frac{1}{36} \) for \( i = 5-8 \).

Through the Chapman-Enskog procedure, Eq.(1) recovers the continuity equation,

\[
\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0,
\]

and the Brinkman equation(13) for anisotropic porous media,

\[
\begin{align*}
\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} &= -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right) + F_x, \\
\frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} &= -\frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \left( \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \right) + F_y,
\end{align*}
\]

in the limit of small Mach number. Here, the pressure \( p = c^2 \rho / 3 \), and the effective viscosity \( \nu_e = c^2 (\tau - 0.5) \delta t / 3 \). The body forces in Eqs.(5) and (6) are expressed as

\[
F_x = -\nu (k_{xx} u_x + k_{xy} u_y) + g_x, \quad F_y = -\nu (k_{yx} u_x + k_{yy} u_y) + g_y,
\]

where

\[
\begin{pmatrix}
    k_{xx} & k_{xy} \\
    k_{yx} & k_{yy}
\end{pmatrix} = \begin{pmatrix}
    \frac{1}{K_x} \cos^2 \theta + \frac{1}{K_y} \sin^2 \theta & \frac{1}{K_x} \sin \theta \cos \theta \\
    \frac{1}{K_x} \sin \theta \cos \theta & \frac{1}{K_y} \sin^2 \theta + \frac{\cos^2 \theta}{K_y}
\end{pmatrix},
\]

Here, \( \nu \) is the kinematic viscosity of the fluid. In the following simulations, \( \nu \) is assumed to be equal to \( \nu_e \) as Z. Guo did(8). As shown in Fig.1, \( K_x \) and \( K_y \) are the principal permeability values, and \( \theta (0 \leq \theta \leq \pi/2) \) is the angle between \( K_x \) and x-axis, and \( g_a \) is the acceleration due to gravity. The Darcy number is defined by

\[
Da = K_y / H^2,
\]

where \( H \) is the characteristic length.

It is proved that the most suitable choice for the forcing term \( F_i \) in Eq.(1) to obtain correct equations of hydrodynamics is taking(15)

\[
F_i = \omega_i \rho \left( 1 - \frac{1}{2\tau_i} \right) \left[ 3e_i \cdot \mathbf{F} + \frac{9(e_i \cdot \mathbf{u})^2}{c^4} - \frac{3\mathbf{u} \cdot \mathbf{F}}{c^2} \right],
\]

Fig. 1  Schematic of an anisotropic porous medium. It shows the velocity direction of the D2Q9 model on the bottom wall and at corner points.
The forcing term \( F_i \) shown in Eq.(10) defines the fluid velocity \( u_\alpha \) as follows:

\[
   u_\alpha = \sum_i E_{\alpha i} f_i / \rho + \frac{\delta_i}{2} F_\alpha .
\]  

(11)

The fluid density is defined as the sum of the distribution function.

\[
   \rho = \sum_i f_i .
\]  

(12)

Substitution of Eq.(7) into Eq.(11) gives

\[
   \left( \begin{array}{c} u_x \\ u_y \\ \end{array} \right) = \left( \frac{1}{\rho} \sum_i E_{ix} f_i \right) - \frac{\nu \delta_i}{2} \left( \begin{array}{cc} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{array} \right) \left( \begin{array}{c} u_x \\ u_y \end{array} \right) + \frac{\delta_i}{2} \left( \begin{array}{c} g_x \\ g_y \end{array} \right).
\]  

(13)

Since Eq.(13) is a linear equation for the velocity \( u_\alpha \), one can simply solve this linear problem with inverse matrix as follows:

\[
   \left( \begin{array}{c} u_x \\ u_y \end{array} \right) = \left( \frac{1}{\rho} \sum_i E_{ix} f_i + \frac{\delta_i}{2} g_x \right) \left( \begin{array}{cc} k_{xx} + \nu \delta_i k_{xy} & 0 \\ 0 & k_{yy} + \nu \delta_i k_{yx} \end{array} \right)^{-1} \left( \begin{array}{c} \frac{1}{\rho} \sum_i E_{iy} f_i + \frac{\delta_i}{2} g_y \\ \end{array} \right).
\]  

(14)

For the wall boundary, I use the second-order bounce back boundary rule for the non-equilibrium distribution function proposed by Q. Zou(15). For example, after streaming, \( f_0, f_1, f_3, f_4, f_6, f_8 \) are known on the bottom wall as shown in Fig.1. If \( u_\alpha \) is specified, and \( \sum_i E_{ix} f_i = \rho \) and \( \sum_i E_{iy} f_i = \rho u_\alpha \) are satisfied on the wall, the following equations are obtained to determine the unknown \( f_2, f_5, f_6, \) and \( \rho \).

\[
   f_2 + f_5 + f_6 = \rho - (f_0 + f_1 + f_3 + f_4 + f_5 + f_8),
\]

\[
   c(f_5 - f_6) = \rho u_x - c(f_1 - f_3 - f_7 + f_8),
\]

\[
   c(f_2 + f_5 + f_6) = \rho u_y - c(f_4 + f_7 + f_8).
\]  

(15)

With assumption of the bounce back rule for non-equilibrium part of the particle distribution normal to the boundary:

\[
   f_2 - f_2^{eq} = f_4 - f_4^{eq},
\]  

(16)

unknown \( f_2, f_5, f_6, \) and \( \rho \) can be found on the wall.

\[
   \rho = \frac{c}{c - u_y} \left[ f_0 + f_1 + f_3 + 2(f_4 + f_5 + f_8) \right],
\]

\[
   f_2 = f_4 + \frac{2}{3c} \rho u_y ,
\]

\[
   f_5 = f_7 - \frac{1}{2}(f_1 - f_3) + \frac{1}{2c} \rho u_x + \frac{1}{6c} \rho u_y,
\]

\[
   f_6 = f_8 + \frac{1}{2}(f_1 - f_3) - \frac{1}{2c} \rho u_x + \frac{1}{6c} \rho u_y.
\]  

(17)

For other boundaries such as the top wall, the velocity is specified in a similar manner. Although the equation \( \sum_i E_{ix} f_i = \rho u_\alpha \) is used in place of Eq.(11) to get this boundary rule for the distribution function, it will be shown in the following simulations that the velocity is adequately set on the static and moving walls.

The algorithm of computation is summarized.

Step 1. Set \( f_i^{eq(1)} \) corresponding to the initial condition, \( \rho^{(1)} \) and \( u_\alpha^{(1)} \). The initial values for the distribution function are given by \( f_i^{(1)} = f_i^{eq(1)} \).

Step 2. Equations (1) and (10) compute \( f_i^{eq(1)} \), and Eqs.(12) and (14) give \( \rho^{(1)} \) and \( u_\alpha^{(1)} \).

Step 3. The boundary values for the distribution function are given by Eq.(17).

Step 4. Compute \( f_i^{eq(1)} \) and \( u_\alpha^{eq(1)} \) by Eqs.(3) and (7).

Step 5. Advance one time step and return to step 2.
3. Results and Discussion

The lattice Boltzmann method for incompressible flows in anisotropic porous media is applied to the Poiseuille flow in a 2D channel of width \( H \) filled with an anisotropic porous medium. When the flow is driven by a constant force \( g_y \), the stream wise velocity \( u_y \) satisfies the following equation:

\[
\nu e \frac{\partial^2 u_y}{\partial x^2} - \nu k_{yy} u_y + g_y = 0, \tag{18}
\]

with \( u_y(0,y) = u_y(H,y) = 0 \), and the lateral velocity \( u_x \) is zero everywhere. The analytical solution of Eq.(18) can be written as

\[
u_k \frac{g_y}{\nu k_{yy}} \left[ 1 - \frac{\cosh[r(x-H/2)]}{\cosh(r H/2)} \right], \tag{19}\]

where \( r = \sqrt{\nu k_{yy}}/\nu e \). In all the following simulations, \( \delta_t = 1 \), and \( c = 1 \). The ratio of the principal permeabilities is defined by \( k^* = K_y/K_x \). In this simulation, \( k^* \) changes from 0.1 to 10, and \( \theta \) does from 0 to \( \pi/2 \). Darcy number \( Da \) is set to be \( 10^{-3} \) or \( 10^{-2} \). The lattice size is \( 101 \times 4 \) with a square mesh, and the relaxation time \( \tau \) is 0.8. Therefore, \( H = 100 \), and \( \nu = 0.1 \). The Reynolds number \( Re \) of the Poiseuille flow is defined by \( Re = Hu_0/\nu \), where \( u_0 \) is the peak velocity of the flow along the centerline. I use \( u_0 = 0.1 \) to calculate the velocity field not beyond the incompressible limit, and set the gravity force \( g_y = u_0 \nu K_{yy}/[1 - 1/\cosh(r H/2)] \) to make \( Re \) equal to 10. Periodic boundary conditions are applied to the inlet and the outlet, and bounce back boundary rule for the non-equilibrium distribution function(15) is applied at the walls. The velocity field is initialized to be zero at each lattice node with a constant density \( \rho = 1.0 \), and the distribution functions are set to be its equilibrium at \( t = 0 \). The convergence criterion is set to

\[ \max |(u^{(n+1)} - u^{(n)})/u^{(n)}| \leq 10^{-6}, \tag{20} \]

Fig. 2  Velocity profiles of the generalized Poiseuille flow for different directions of principal permeabilities \( \theta \) and for different ratios of principal permeabilities \( k^* \).
where \((n)\) and \((n + 1)\) represent the old and new time levels, respectively. All the calculations are done on the PC Dell 3.2GHz. I test the velocity profiles for two kinds of Darcy numbers, \(10^{-2}\) and \(10^{-3}\). In Fig.2, the numerical results of the present lattice Boltzmann model are compared with the analytical solutions written as Eq.(19). Symbols and solid lines represent the numerical and the analytical solutions, respectively. Excellent agreement can be observed between the numerical and the analytical solutions. This result confirms the validity of the present LBM to simulate fluid flows through anisotropic porous media under body force.

Secondly I take the Couette flow problem as a test case for the moving boundary condition. The problem is a channel flow where the right plate moves with a constant velocity \((u_y(H, y) = u_0)\), and the left plate remains stationary. Periodic boundary conditions are applied to the \(y\)-direction. The analytical solution of the velocity field in steady state is given by

\[
  u_y = u_0 \frac{\sinh(rx)}{\sinh(rH)},
\]

where \(r = \sqrt{\nu/\kappa_{yy}}\). The lattice size is \(121 \times 4\), \(\tau = 0.8\), \(H = 120\), \(\nu = 0.1\), and \(\text{Re} = 100\). The velocity of the right plate is given by \(u_0 = \text{Rev}/H\). The boundary conditions and the convergence criterion are the same as ones for the Poiseuille flow problem. Figure 3 shows the velocity profiles for different Darcy numbers. Symbols and solid lines are LBE and analytical solutions, respectively. They agree very well with each other for any Darcy number.

The value of velocity is calculated at the location \((x/H = 0.75)\) indicated by the dashed line in Fig.3 for a variety of ratios of the principal permeabilities \(k^*\) and directions \(\theta\). Figure 4 (a) shows that the value of flow velocity at the location \((x/H = 0.75)\) depends on \(k^*\). \(Da = 10^{-2}\), and \(\theta = \pi/4\). The numerical and the analytical solutions are indicated by the symbols and the solid line, respectively. The value of velocity at the same location is shown as a function of \(\theta\) in Fig.4 (b). The symbols (•) and (×) represent the LBE solutions for \(k^* = 10^{-1}\) and for \(k^* = 10\), respectively. Figures 4 (a) and 4 (b) show good agreement between the numerical and analytical solutions in a wide range of the principal permeability ratio \(k^*\) and of the angle \(\theta\). Since the wall velocity is not zero in the Couette flow simulation, it can be seen from Eq.(7) that body force \(F_\alpha\) is not zero on the moving wall. Due to the nonzero body force \(F_\alpha\), the velocity definition \(\rho u_\alpha = \sum_i e_{i\alpha} f_i + \delta_t \rho F_\alpha/2\) for the present LBM is not equal to the equation \(\rho u_\alpha = \sum_i e_{i\alpha} f_i\) used to implement the boundary condition. Figures 3 and 4, however, show that this difference of the velocity definitions does not influence the numerical results, and the boundary method proposed by Q. Zou(15) can be applied to the present LBM without modification.

Finally, I verify the applicability of the LBM for the calculation of lid-driven cavity flow in anisotropic porous media. The simulation is restricted to two-dimensional fluid dynamics in a square cavity. The cavity is covered by lattices \(101 \times 101\), and the top boundary moves from left to right with velocity \(u_0\). Two upper corners are singular points which are considered
as part of the moving lid in the simulations. As shown in Fig.1, at the particle directions 6 and 8 for right upper and the left bottom corner points, and at the particle directions 5 and 7 for the left upper and right bottom corner points, the distribution function cannot be determined from the bounce-back rule for the non-equilibrium distribution functions(15). Since any information is not transported into these particle directions from their evolution equations (1), the two distribution functions are given by the equilibrium functions(16). For example, at right upper corner point, the first-order boundary rule, \( f_6 = f_{eq}^6 \) and \( f_8 = f_{eq}^8 \), is applied. In the simulation of the cavity flow, the steady state is reached if

\[
\max \mid |u^{(n+1)}| - |u^{(n)}| \mid \leq 10^{-8}. \tag{22}
\]

In Figs.5 and 6, the stream functions calculated by the LBM are compared with those by the finite difference method (FDM) for different directions of the principal permeabilities \( \theta \). The solid and dashed lines represent the LBM and FDM solutions, respectively. In the calculation by the FDM, Eqs.(4)-(8) are solved numerically by using the well-known MAC (Maker-And-Cell) method(17). The spatial derivatives are approximated using the familiar second-order accurate centered difference schemes with a staggered mesh. I use the forward finite difference approximation for temporal derivatives and the multi-grid accelerator for the fast computation of the steady state solution. For any values of Darcy number and for all directions \( \theta \), the numerical results obtained by the present LB model agree well with those by the FDM. Figures 7 and 8 show the stream functions for various ratios of the principal permeabilities \( k^* \). Although the results approximately agree with each other, Fig.8 (d) indicates a little deviation between the LBM and FDM solutions.

To quantify these results, the maximum values of the stream functions and locations of the center of the vortex are listed in Table 1. Also tabulated are the results obtained by the FDM. Table 1 shows that the locations of the vortex center predicted by the LBM agree well with those by the FDM. The maximum values of the stream function obtained by the LBM also agree well with those by the FDM. However, they are slightly smaller than the FDM ones due to compressibility property of the LBM. Since the LBE derives the Navier Stokes equation in the incompressible limit, the density cannot be a constant in the simulation. It is important to find the effect of compressibility on the present solution. One quantity that represents compressibility is the mean variation of density(18). The mean density \( \bar{\rho} \) is defined by

\[
\bar{\rho} = \frac{\sum_i \rho(x_i, t)}{N}, \tag{23}
\]

where \( N \) is the total number of nodes. The mean variation of density is given by

\[
\Delta = \frac{1}{\bar{\rho}} \sqrt{\sum (\rho - \bar{\rho})^2} / N. \tag{24}
\]
Fig. 5 Stream function for Re = 100, k* = 0.1, and Da = 10^{-1} (101 × 101).

(a) θ = 0

(b) θ = π/6

(c) θ = π/3

(d) θ = π/2

Fig. 6 Stream function for Re = 100, k* = 0.1, and Da = 10^{-2} (101 × 101).

(a) θ = 0

(b) θ = π/6

(c) θ = π/3

(d) θ = π/2
Fig. 7  Stream function for Re = 100, θ = π/4, and Da = 10^{-1} (101 × 101).

(a) $k^* = 0.01$
(b) $k^* = 0.1$
(c) $k^* = 1$
(d) $k^* = 10$

Fig. 8  Stream function for Re = 100, θ = π/4, and Da = 10^{-2} (101 × 101).

(a) $k^* = 0.01$
(b) $k^* = 0.1$
(c) $k^* = 1$
(d) $k^* = 10$
Figure 9 shows stream functions obtained by the LBM for the lattice sizes, 201 and 401. The error is introduced into the RHS of Eq.(25), and induces the error in the stream function. Apparently the reason is that the compressible error decreases. Reducing the deviation of maximum values of the stream function between the LBM and FDM comes closer to the one of the FDM. As shown in the last row of Table 2, the large lattice size (d). In Fig.9, as the lattice size increases, the contour line of the stream function of the LBM is round. The numerical calculations are conducted under the same condition shown in Fig.8. In the actual computations, the stream function at the end wall does not equal zero because of the incompressible continuity equation given by $\nabla \cdot \mathbf{u} = 0$. Due to a non-constant $\rho$, the velocity $\mathbf{u}$ does not exactly satisfy the incompressible continuity equation given by $\nabla \cdot \mathbf{u} = 0$. When the stream function $\psi$ is calculated by using the definition written as $\psi = \int -u_x dx + u_y dy$, the different results are obtained not only by the different numerical integration rules (rectangular, trapezoidal, etc.), but also by the different directions for integration, namely integrating $u_x$ along $y$ from top to bottom or from bottom to top and integrating $u_y$ along $x$ from left to right or from right to left. In the actual computations, the stream function at the end wall does not equal zero because of round off and integration errors. To determine the stream function uniquely, I calculated the stream function using the following Poisson equation,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}. \quad (25)$$

To approximate the stream function for the incompressible flow, $\mathbf{u}$ has to satisfy $\nabla \cdot \mathbf{u} = 0$. Due to $\nabla \cdot \mathbf{u} \neq 0$ in a discrete velocity field obtained by the LBE calculation, the compressible error is introduced into the RHS of Eq.(25), and induces the error in the stream function. Figure 9 shows stream functions obtained by the LBM for the lattice sizes, 201 × 201 and 401 × 401. The numerical calculations are conducted under the same condition shown in Fig.8 (d). In Fig.9, as the lattice size increases, the contour line of the stream function of the LBM comes closer to the one of the FDM. As shown in the last row of Table 2, the large lattice size reduces the deviation of maximum values of the stream function between the LBM and FDM solutions. Apparently the reason is that the compressible error decreases.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Vortex centers: stream function and location</th>
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<tbody>
<tr>
<td>$\text{Da}$</td>
<td>$k^*$</td>
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<tr>
<td>$10^{-1}$</td>
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<td>10</td>
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</tbody>
</table>

Table 2 tabulates the mean density fluctuations $\Delta$ calculated by the present model for the lattice sizes, 101 × 101, 201 × 201, and 401 × 401. The values obtained by the FDM are listed in the rightmost column of Table 2 for the purpose of reference. Table 2 shows that $\Delta$ is proportional to $u_0^2$. The results agree with the known relation that the density variation by small compressibility is estimated as $\Delta \rho \approx u_0^2 / 2c_s^2(18)$. The sound speed $c_s = c / \sqrt{3}$ in the LBM. In the steady case satisfying $\nabla \cdot \mathbf{u} = 0$, the continuity equation represented by the LBM is $\nabla \cdot (\rho \mathbf{u}) = 0(5)$. Due to a non-constant $\rho$, the velocity $\mathbf{u}$ does not exactly satisfy the incompressible continuity equation given by $\nabla \cdot \mathbf{u} = 0$. When the stream function $\psi$ is calculated by using the following Poisson equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}. \quad (25)$$

To approximate the stream function for the incompressible flow, $\mathbf{u}$ has to satisfy $\nabla \cdot \mathbf{u} = 0$. Due to $\nabla \cdot \mathbf{u} \neq 0$ in a discrete velocity field obtained by the LBE calculation, the compressible error is introduced into the RHS of Eq.(25), and induces the error in the stream function. Figure 9 shows stream functions obtained by the LBM for the lattice sizes, 201 × 201 and 401 × 401. The numerical calculations are conducted under the same condition shown in Fig.8 (d). In Fig.9, as the lattice size increases, the contour line of the stream function of the LBM comes closer to the one of the FDM. As shown in the last row of Table 2, the large lattice size reduces the deviation of maximum values of the stream function between the LBM and FDM solutions. Apparently the reason is that the compressible error decreases.

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Mean density fluctuation, CPU time, and the maximum values of stream function (Da=10^{-7}, k^* = 10, \theta = \pi/4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lattice size</td>
<td>$101 \times 101$</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>$2.15 \times 10^{-3}$</td>
</tr>
<tr>
<td>$u_0$</td>
<td>0.1</td>
</tr>
<tr>
<td>$H$</td>
<td>100</td>
</tr>
<tr>
<td>CPU time (s)</td>
<td>118</td>
</tr>
<tr>
<td>$\psi_{\text{max}}$</td>
<td>0.03330</td>
</tr>
</tbody>
</table>
Although the FDM utilizes the multi-grid accelerator and the SOR method to shorten the calculation time, it takes more time than the LBM for the same grid size $101 \times 101$ in Table 2. While the FDM needs to solve the Poisson equation for the pressure to satisfy the continuity equation, the LBM satisfies the continuity equation by making the characteristic velocity not beyond the incompressible limit without the iterative calculation. On the other hand, as is obvious from Fig.9 (b), the LBM needs the grid size $401 \times 401$ to get the same converged solution as the FDM on the grid size, $101 \times 101$. Namely, in the exceptional case shown in Fig.8 (d) and Table 2, the LBM needs larger grid size and takes more time than the FDM due to the compressibility error. To reduce the calculation time, it is useful to apply the LBM for the incompressible flows which reduces the effects of the compressibility, or to use the technique of the preconditioned LBM which accelerates the convergence rate. On the whole, Figs. 5, 6, 7, and 8 show that the LBM is able to obtain the steady solution almost equal to the FDM with same grid size, and in regard to the computational time the LBM is more efficient than the conventional FDM. It is reasonable to be concluded that the LBM is able to adequately calculate the Brinkman equation with anisotropic permeability.

4. Conclusions

The LBM was adapted to simulation of fluid flow in anisotropic porous media at the REV scale with the Brinkman equation. The applicability of the LBE model was validated by the numerical simulations including the Poiseuille flow, the Couette flow, and the lid-driven cavity flow in two dimensions. The simulation results of the Poiseuille flow and of the Couette flow confirmed the validity of the present LBM to calculate fluid dynamics through anisotropic porous media in a wide range of the ratio of the principal permeabilities and of the direction of the principal permeability. The stream function patterns of cavity flow showed that the LB model was able to keep the almost same accuracy with the FDM with a few exceptions. In some cases, the compressible property of the LBM affected the steady solutions, and the compressible error was reduced by use of larger grid size. Taken all together, the LBM took less time than the conventional FDM to get the steady solutions on the same grid size, because it didn’t need to solve the Poisson equation to satisfy the continuity equation.

In this paper I restricted the fluid dynamics in porous media to the Brinkman model under the isothermal conditions. In the future, I will investigate the influence of the nonlinear drag in the Brinkman-Forchheimer equation, and simulate natural convection in anisotropic porous media by coupling with the thermal LBM to compare to the earlier study by P. Nithiarasu. (3)

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References


