On Tumbling of a Two-Dimensional Plate under Free Flight*

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Abstract

In this study, we experimentally deal with the tumbling, which is a rotating motion of a plate with its axis perpendicular to the falling direction. As the plate, we consider a rectangular-cross-section prism with a depth-to-width ratio $\lambda = 0.3$, an aspect ratio $AR = 10$, and inertia moment ratios $I^* = 0.75 - 43$, together with a wide range of a non-dimensional control parameter $C = 7.2 \times 10^1 - 1.6 \times 10^3$. As a result, we specify fundamental aerodynamic characteristics in terminal condition, like the reduced rotating rate Ω^* , the lift coefficient C_L , the drag coefficient C_D and the lift-to-drag ratio C_L/C_D as functions of both C and I^* . Moreover, the relation between the actual Reynolds number Re and the Reynolds number $Re(V_d)$ based on V_d can be approximately determined, being independent of C and I^* . Eventually, we successfully propose a series of empirical formulae to predict Ω^* , C_L , C_D and C_L/C_D as functions of $Re(V_d)$ alone, which is available for a wide parameter range except for large C and small I^* .

Key words: Autorotation, Flat Plate, Free Flight, Free Fall, Tumbling

1. Introduction

When a solid matter falls in fluid, its motion is usually complicated and sometimes experiences the autorotations, namely, various kinds of self-induced rotational motions sustained by fluid force (see Lugt (1983)⁽¹⁾ and Yoshinaga & Tate (1985)⁽²⁾). Among the autorotations such as the tumbling, the flat spin and the corning, we consider the tumbling, which is a rotational motion of a solid matter in fluid with the axis perpendicular to the falling direction. The tumbling is important not only for pure academic interests in leaves' and cards' falling motions, but also for aeronautics, space engineering, ballistics and meteorology.

So, there have existed some previous researches^{(3) - (14)} on the tumbling (for further details, see our previous work (2009)⁽¹⁵⁾). However, by these researches, we cannot know even fundamental relations between the important physical quantities representing aerodynamic characteristics and the governing parameters. Here, as the aerodynamic-characteristic quantities, we usually consider the reduced rotating rate Ω^* , the lift coefficient C_L , the drag coefficient C_D , the lift-to-drag ratio C_L/C_D and so on. —The relations reported in those previous researches are not accurate enough, because they are based on the experiments at small aspect ratio $AR = 0.25 - 6.0$, and because all the governing parameters are not specified completely. — For example, though Ω^* is considerably effective upon the vicinal flow field and then upon the other aerodynamic characteristics, it is difficult to determine the precise value of Ω^* in enough terminal condition by both wind-tunnel experiments and computations. The difficulties by the former and the latter are owing to supporting systems, and owing to the strong non-linearity and

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three-dimensionality related with flow separations, respectively.

Recently, we have carried out a series of free-flight tests in air, and revealed the relations of Ω^* , C_L , C_D and C_L/C_D in terminal condition with a pair of non-dimensional governing parameters C and I^* , a pair of C and $Re(V)$, or a pair of I^* and $Re(V)$, where C , I^* and $Re(V)$ will be denoted later. However, in this pioneering work, the value of C is kept to be constant to 3.9×10^2 . Moreover, the work concerns relatively high Reynolds number Re , to reveal the whole features over wide ranges of parameters including Re .⁽¹⁵⁾

So, in the present study, we methodically conduct many free-flight test in air and water, in order to determine Ω^* , C_L , C_D and C_L/C_D in terminal condition, and attempt to reveal the relation of them with a non-dimensional governing parameter C , in addition to I^* or $Re(V_d)$. —About the definition of $Re(V_d)$, see later.— In particular, we focus our attention upon a low- Re range, in order to specify more precise relations. Because, in such a low- Re range, the Re effects upon the tumbling characteristics become larger.⁽¹⁵⁾

2. Nomenclature

AR	: Aspect ratio, $\equiv l/w$	[-]
C	: Non-dimensional control parameter, $\equiv (gw^3)^{1/2}/\nu$	[-]
C_D	: Drag coefficient, $\equiv D/(0.5\rho_f U_\infty^2 lw)$	[-]
C_L	: Lift coefficient, $\equiv L/(0.5\rho_f U_\infty^2 lw)$	[-]
d	: Plate depth	[m]
D	: Drag	[N]
g	: Gravity acceleration	[m/s ²]
I	: Inertia moment	[kg m ²]
I^*	: Inertia moment ratio, $\equiv 32I/(\rho_f w^4 l)$	[-]
l	: Plate span	[m]
L	: Lift	[N]
n	: Rotating rate	[s ⁻¹]
Re	: Reynolds number, $\equiv U_\infty w/\nu$	[-]
$Re(V)$, $Re(V_d)$	: Reynolds number based on V , or V_d	[-]
t	: Time	[s]
U_∞	: Mean translation velocity	[m/s]
V , V_d	: Theoretical velocity	[m/s]
w	: Plate width	[m]
x	: Horizontal coordinate perpendicular to the rotation axis	[m]
y	: Vertical coordinate	[m]
z	: Horizontal coordinate parallel to the rotation axis	[m]
λ	: Depth-to-width ratio, $\equiv d/w$	[-]
ν	: Kinematic viscosity	[m ² /s]
ρ_f	: Fluid density	[kg/m ³]
ρ_s	: Plate density	[kg/m ³]
Ω^*	: Reduced rotating rate, $\equiv \pi n w/U_\infty$	[-]
Ω_{AOR}^*	: Ω^* averaged over one plate's rotation	[-]

3. Experimental Method

3.1 Model

Figure 1 shows a test plate; namely, a prism with a rectangular cross section. Here, l , w and d denote span, width and depth of the plate, respectively. Figure 2 shows a schematic diagram of the tumbling motion of the plate and acting forces, together with the present coordinate system.

The acting forces W , L and D are the weight, lift and drag on the plate, respectively. The resultant force R of both L and D is the same as $-W$ in the terminal condition. Mean translation velocity U_∞ is parallel to the mean falling direction of the plate. We specify W from the model mass considering the buoyancy force, and both L and D from the glide angle and W .

x and z represent horizontal coordinates perpendicular and parallel to the plate's rotation axis, respectively. And, y represents a vertical coordinate.

3.2 Governing parameters

Governing parameters are classified into two; namely, geometric and kinematic ones. At first, we consider the former. Referring to Fig. 1, we choose the depth-to-width ratio λ and the aspect ratio AR as the geometric governing parameters. The definitions of λ and AR are given by the following equations.

$$\lambda \equiv \frac{d}{w}. \quad (1)$$

$$AR \equiv \frac{l}{w}. \quad (2)$$

In all the present experiments in both air and water, λ and AR are fixed to 0.3 and 10, respectively.

Complementally speaking, we have to consider the condition of the plate's ends, because it influences the three-dimensionality of flow. In the present study, we do not put any end plates which ensure the two-dimensionality of flow. Instead, we keep an enough large value of AR ($= 10$) through all the tests, to ignore the AR effects. —About the AR effects, see Hirata et al. (2009).⁽¹⁵⁾—

Second, we consider the kinematic governing parameters. We need two non-dimensional parameters to identify the concerning phenomenon. Conventionally, we select the inertia moment ratio I^* and the Reynolds number Re (for example, see Smith (1971)⁽⁶⁾). The definitions of I^* and Re are as follows:

$$I^* \equiv \frac{32I}{\pi\rho_f w^4 l}, \quad (3)$$

and

$$Re \equiv \frac{U_\infty w}{\nu}, \quad (4)$$

where I denotes the inertia moment of the plate given by

$$I \equiv \frac{\rho_s w d l (w^2 + d^2)}{12}. \quad (5)$$

Here, ρ_f , ρ_s and ν denote fluid density, plate density and kinematic viscosity, respectively.

Fig. 1 Tumbling plate; a prism with a rectangular cross section.

Fig. 2 Schematic diagram of the tumbling of a plate and acting forces, together with the coordinate system.

From a theoretical point of view, we have to independently investigate each effect of such two governing parameters as I^* and Re . However, in actual free-flight experiments, individual estimations to such kinematic governing parameters are technically difficult. This has been just one main reason to prevent the understanding of the concerning phenomenon. —For example, in Iversen (1979), each parameter effect does not estimated separately from the other parameter effect, under the condition that the other parameter is not kept to be constant.—

In the present study, as well as Hirata et al. (2009), for the individual estimations to kinematic governing parameters avoiding such a difficulty, we choose the following non-dimensional control parameter C , instead of Re .

$$C \equiv \frac{\sqrt{gw^3}}{\nu} . \quad (6)$$

From a dimensional analysis, C represents the ratio of (inertia force $\times$ gravity force)^{1/2} to viscous force. And, from a practical point of view, C can be easily controlled in experiments. Then, in the present study, we fix C to one among six constant values ranging from 7.2×10^1 to 1.6×10^3 , and investigate the pure effects of both C and the other kinematic governing parameter I^* .

We may consider another Reynolds number $Re(V)$ or $Re(V_d)$, instead of C , I^* or Re . $Re(V)$ and $Re(V_d)$ are the Reynolds numbers based on theoretical velocity V and V_d instead of U_∞ , respectively. In the present study, we consider not $Re(V)$, but $Re(V_d)$ as well as Andersen et al. (2005)⁽¹³⁾. — $Re(V_d)$ has an advantage in such a direct interpretation of V_d as supposing $C_D = 1$, while $Re(V)$ has an advantage in simplicity of the formulation where we throughout use a sole characteristic length scale w . The relation of both is given by $Re(V_d) = \sqrt{\lambda} Re(V)$.— The definition of $Re(V_d)$ is given by

$$Re(V_d) \equiv \frac{V_d w}{\nu} . \quad (7)$$

And, the theoretical terminal velocity V_d is as follows:

$$V_d \equiv \sqrt{2 \left(\frac{\rho_s}{\rho_f} - 1 \right) dg} , \quad (8)$$

where g denotes the gravity acceleration.

In many practical aspects, we can easily find advantages in the use of V_d , instead of the use of U_∞ ; namely, the use of $Re(V_d)$ instead of the use of Re . For example, we can calculate the value of V_d , once physical properties and geometrical dimensions are given. On the other hand, we need experiments or numerical simulations to determine U_∞ . From another viewpoint, the use of V_d inevitably causes some lost information. As shown later, it will be experimentally confirmed that the uniqueness in the relation between V_d and U_∞ are approximately satisfied. Then, V_d could be sufficient to understand the concerning phenomenon in perspective, and more convenient than U_∞ . We will also discuss the lost information, later.

Hereafter, as two kinematic governing parameters, we consider a pair of C and I^* , a pair of C and $Re(V_d)$ or a pair of I^* and $Re(V_d)$, instead of a pair of I^* and Re . In the following section, supposing technical difficulties in actual experiments, we will first attempt a pair of C and I^* . Second, we will attempt a pair of C and $Re(V_d)$ or a pair of I^* and $Re(V_d)$.

Incidentally, the relation among C , I^* and $Re(V_d)$ is given by

$$\frac{Re(V_d)}{C} = \sqrt{2\lambda \left(\frac{3\pi}{8} \cdot \frac{I^*}{\lambda + \lambda^3} - 1 \right)} , \quad (9)$$

where $\lambda = 0.3$.

In Table 1, we summarise the values of governing parameters by the authors and other researchers, together with the corresponding symbols used in figures. And, Table 2 shows

Table 1 Parameters and the corresponding symbols in Figs. 5 – 15.

Reference	Symbol	λ	AR	C	I^*	Experimental method
Present	●, ■, ▲, ◆	0.3	10	7.2×10^1	4.2 – 42	free flight in air
Present	●, ■, ▲, ◆	0.3	10	1.1×10^2	4.7 – 43	free flight in air
Present	●, ■, ▲, ◆	0.3	10	2.0×10^2	4.2 – 36	free flight in air
Present	●, ■, ▲, ◆	0.3	10	3.7×10^2	3.5 – 43	free flight in air
Present	○, □	0.3	10	4.7×10^2	0.75 – 2.5	free flight in water
Present	●, ■	0.3	10	$1.3 \times 10^3 - 1.6 \times 10^3$	0.75 – 2.5	free flight in water
Hirata et al. (2009)	×	0.3	2.0 – 20	3.9×10^2	3.3 – 141	free flight in air
Dupleich (1949) quoted by Poreh & Wray (1979)	+	unknown (0.003 - 0.01)	1.0 – 5.0	$1.4 \times 10^3 - 6.5 \times 10^3$	1.60 – 4.50	free flight in air
Bustamante & Stone (1969)		unknown (thin)	1.7 – 6.0	unknown	unknown	wind tunnel
Iversen (1979)		unknown (0.0054 - 0.5)	0.25 – 4.0	unknown	more than about 10	wind tunnel (Glaser & Northup, 1971) free flight in air (Bustamante & Stone, 1969)

Table 2 Details of C , I^* and $Re(V_d)$, together with corresponding symbols in Figs. 5 – 15. In all the present experiments, $\lambda = 0.3$ and $AR = 10$.

Reference	Symbol	C	I^*	$Re(V_d)$	Re
Present	●	7.2×10^1	4.2	220	$1.9 \times 10^2 - 2.2 \times 10^2$
Present	■	7.2×10^1	6.1	250	$2.2 \times 10^2 - 2.5 \times 10^2$
Present	▲	7.2×10^1	10	320	$2.4 \times 10^2 - 2.7 \times 10^2$
Present	◆	7.2×10^1	42	700	$5.3 \times 10^2 - 5.8 \times 10^2$
Present	●	1.1×10^2	4.7	350	$2.7 \times 10^2 - 2.9 \times 10^2$
Present	■	1.1×10^2	7.3	450	$3.3 \times 10^2 - 3.4 \times 10^2$
Present	▲	1.1×10^2	13	530	$4.0 \times 10^2 - 4.8 \times 10^2$
Present	◆	1.1×10^2	43	1090	$8.8 \times 10^2 - 9.2 \times 10^2$
Present	●	2.0×10^2	4.2	600	$4.3 \times 10^2 - 4.5 \times 10^2$
Present	■	2.0×10^2	7.4	730	$5.7 \times 10^2 - 5.9 \times 10^2$
Present	▲	2.0×10^2	12	970	$6.8 \times 10^2 - 7.2 \times 10^2$
Present	◆	2.0×10^2	36	1830	$1.2 \times 10^3 - 1.3 \times 10^3$
Present	●	3.7×10^2	3.5	1000	$7.0 \times 10^2 - 7.8 \times 10^2$
Present	■	3.7×10^2	6.3	1300	$9.9 \times 10^2 - 1.0 \times 10^3$
Present	▲	3.7×10^2	22	2530	$1.8 \times 10^3 - 1.9 \times 10^3$
Present	◆	3.7×10^2	43	3560	$2.6 \times 10^3 - 2.9 \times 10^3$
Present	○	4.7×10^2	0.75	480	$3.3 \times 10^2 - 3.4 \times 10^2$
Present	□	4.7×10^2	2.5	1040	$7.3 \times 10^2 - 7.4 \times 10^2$
Present	●	1.6×10^3	0.75	1650	$1.1 \times 10^3 - 1.3 \times 10^3$
Present	■	1.3×10^3	2.5	3030	$2.2 \times 10^3 - 2.3 \times 10^3$
Hirata et al. (2009)	×	3.9×10^2	3.3 – 141	980 – 12300	$1.8 \times 10^3 - 6.4 \times 10^3$

the details of the present kinematic governing parameters, together with the corresponding symbols.

3.3 Aerodynamic Characteristics

As dependent variables of the above governing parameters, we consider physical quantities describing the tumbling characteristics in terminal condition, such as a rotating rate n and the lift L and the drag D of the plate. They are non-dimensionalised as a reduced rotating rate Ω^* , a lift coefficient C_L and a drag coefficient C_D , respectively. Strictly speaking, they are all the ensemble-means of the time-averaged values over several rotations in terminal condition; except for Ω_{AOR}^* which will be shown only in Fig. 3. Their

definitions are as follows.

$$\Omega^* \equiv \frac{\pi \omega l}{U_\infty} \quad (10)$$

$$C_L \equiv \frac{L}{\frac{1}{2} \rho_f U_\infty^2 w l} \quad (11)$$

$$C_D \equiv \frac{D}{\frac{1}{2} \rho_f U_\infty^2 w l} \quad (12)$$

From Eq. (10), we see that Ω^* represents the ratio of the plate's tip velocity to the plate's translation velocity U_∞ in terminal condition. Besides, we consider another important aerodynamic-characteristics quantity C_L/C_D which is referred to as the lift-to-drag ratio, in addition to Ω^* , C_L and C_D .

Hereafter, we investigate physical quantities like Ω^* , C_L , C_D and C_L/C_D as functions of a pair of C and I^* , a pair of C and $Re(V_d)$ or a pair of I^* and $Re(V_d)$.

3.4 Experimental apparatus

The three-dimensional motion of a plate is detected by a synchronised set of two or three cameras, whose foci are fixed near the prospected orbit of the plate. We use one of the cameras only to judge the exact two-dimensionality of the motion. The camera is placed on the prospected-orbit plane. By this operation, we can ensure the two-dimensionality on a higher level than the previous work⁽¹⁵⁾.

In order to record the motion just in terminal condition, we choose the space enough below the launched position as a test space. On the other hand, the test space is enough above the bottom floor to avoid the ground effect. Of course, we check whether or not the recorded motion is terminal, prior to their analyses.

Figures 3 and 4 show the schematics of experimental apparatus in air and in water, respectively.

In Fig. 3, a launched flat plate begins to fall tumbling. We take photographs by two high-speed video cameras synchronised with each other at 500 frames/s, and get the angle and the position of the plate.

In Fig. 4, a flat plate falls in a water tank with a water level of 7.0×10^{-1} m. The water tank has such dimensions as 9.7×10^{-1} m in span, 1.3 m in width and 7.5×10^{-1} m in height. The maximum blockage ratio BR through all the experiments varies from 0.016 % to 0.065 %, where BR is defined by the maximum projected area of the model divided by the minimum vertical-cross-section area of water. The interval between experiments is kept to

Fig. 3 Experimental apparatus in air.

Fig. 4 Experimental apparatus in water.

be enough long to settle the disturbances in the water tank. We take synchronised photographs by two or three high-speed video cameras at 500 frames/s, and get the angle and the position of the plate as well.

Specifically speaking, as targets, we mark three white dots on a longer cross-section centre line on one plate's end surface. One dot is on the centre, and the other two dots are oppositely outside the centre. The former is for the position measurements, and the latter two are for the angle measurements.

The plate has such dimensions as $l = 5.0 \times 10^{-2} - 1.5 \times 10^{-1}$ m, $w = 5.0 \times 10^{-3} - 1.5 \times 10^{-2}$ m and $d = 1.5 \times 10^{-3} - 4.5 \times 10^{-3}$ m in the air experiments, or as $l = 3.3 \times 10^{-2} - 6.6 \times 10^{-2}$ m, $w = 3.3 \times 10^{-3} - 6.6 \times 10^{-3}$ m and $d = 1.0 \times 10^{-3} - 2.0 \times 10^{-3}$ m in the water experiments.

The plates are made of foamed polystyrene, balsa wood, aluminum or copper. We can control the plate's inertia moment I by choosing such materials. —For the other details in experimental apparatus, see ref.⁽¹⁵⁾—

4. Results and Discussion

4.1 Ω_{AOR}^*

Among the four aerodynamic-characteristics quantities, Ω^* is supposed to be primarily influential to the tumbling. So, we first consider Ω^* . Figure 5 shows Ω_{AOR}^* plotted against Re , for several combinations of C and I^* whose values are described in Table 2, together with a low- Re part of the data in Hirata et al. (2009).⁽¹⁵⁾ Here, Ω_{AOR}^* denotes the reduced rotating rate averaged not over enough long time but over only one plate's rotation.

From a qualitative viewpoint, the present results are similar with Hirata et al. That is, as Re increases from zero, Ω_{AOR}^* monotonously increases approaching a constant value in a range of 0.4 – 0.6. Of course, the range is the same as Hirata et al. where the range is independent of Re at high $Re \gtrsim 4 \times 10^3$ as shown in Fig.5. Such a tendency can be also confirmed by a thick solid line described by a formula, which will be discussed later. Of course, such a tendency is also the same as Hirata et al. whose experimental data are denoted by cross symbols.

When we carefully see the data with the same C and the same I^* , we notice that the data collapse near the thick solid line, not with a random scattering manner, but with a coherent scattering manner on a certain curve which has such a trend as Ω_{AOR}^* decreases with increasing Re .

Fig. 5 Reduced rotating rate Ω_{AOR}^* averaged over one rotation versus Reynolds number Re , for $\lambda = 0.3$, $AR = 10$, $C = 7.2 \times 10^1 - 4.7 \times 10^2$ and $I^* = 0.75 - 43$.

To conclude, it is considered that the scattering manner is reflected by a close interaction between C_L and Ω_{AOR}^* . In Hirata et al., it has been found out that the scattering manner is well approximated by a semi-theoretical formula, which is based on the dynamical equilibrium on a free-flight plate supposing that the lift force L is produced mainly by the Magnus effect. However, the formula is merely for a single value of $C = 3.9 \times 10^2$. So, we expand it, considering the effect of C in addition to that of I^* . Now, we consider the dynamical equilibrium on a free-flight plate, and we get

$$L^2 + D^2 = (\rho_s l d w g)^2, \quad (13)$$

where $\rho_s - \rho_f$ is approximated by ρ_s . We suppose that the lift force L is produced mainly by the Magnus effect, then

$$L \propto \rho_f U_\infty n w^2 l. \quad (14)$$

And,

$$D \propto \frac{1}{2} \rho_f U_\infty^2 w l. \quad (15)$$

From Eqs. (13) – (15), we get

$$\frac{n^2 w^2}{U_\infty^2} + \frac{1}{4} \frac{C_2^2}{C_1^2} = \frac{1}{C_1^2} \frac{g^2 d^2 (\rho_s / \rho_f)^2}{U_\infty^4}, \quad (16)$$

where C_1 and C_2 are the coefficients which are related with Eqs. (14) and (15), respectively. From Eq. (16) together with Eqs. (1), (3) – (6) and (10), we get such a non-dimensionalised equation as

$$\left(1 + \frac{\pi^2}{4} \left(\frac{C_2}{C_1} \right)^2 \frac{1}{\Omega^{*2}} \right) \Omega^{*2} = \frac{9\pi^4}{64 C_1^2} \frac{1}{(1 + \lambda^2)^2} \frac{C^4 I^{*2}}{Re^4}. \quad (17)$$

If we suppose $C_1 = \pi^2$ and $C_2 = 1$, then $\pi^2/4(C_2/C_1)^2 = 0.025$. On the other hand, we can conform by experiments that Ω^* usually varies from 0.3 to 0.6. Therefore, $1 + \pi^2/4(C_1/C_2)^2 1/\Omega^{*2}$ has a variable range of 19 %, while Ω^{*2} has a variable range of 400 %. So, we simply approximate the left hand of Eq. (17) to Ω^{*2} , and get

$$\Omega^* = A \frac{C^2 I^*}{Re^2} \quad (18)$$

with a coefficient

$$A = \frac{3\pi^2}{8} \frac{1}{C_1(1 + \lambda^2)}. \quad (19)$$

We experimentally find $\sqrt{0.5}$ as A . Of course, Eq. (18) with $C = 3.9 \times 10^2$ well agrees with Hirata et al. In Fig. 5, we can confirm that thin solid lines representing Eq. (18) are in good agreements with the present data, over a wide range of $C = 7.2 \times 10^1 - 4.7 \times 10^2$.

From a quantitative viewpoint, we see a discrepancy in the value of Ω_{AOR}^* between the present experiments and Hirata et al. The main reason is considered to be owing to less accuracy in the low- Re -range experiments in Hirata et al., where the plate's rotation axis could not be perfectly horizontal with the trajectory on an imperfect plane, because a wider Re range including much higher Re 's is their original concern.

4.2 Influence of C and I^*

Now, we have confirmed the validity of a formula Eq. (18). However, we have to know the value of Re in advance, if we use Eq. (18). In other words, only by Eq. (18), we can predict not the mean value Ω^* of Ω_{AOR}^* but merely the scattering manner of Ω_{AOR}^* on the Re - Ω^* plane. —As will be discussed in Fig. 15, Eq. (18) with the empirically-determined value of Re gives us less accurate Ω^* . — Then, in this subsection, we investigate both the C effects and the I^* effects upon Ω^* , C_L , C_D and C_L/C_D in wide parameter ranges of C and I^* . And, we attempt to find out an empirical formula to precisely predict the value of Ω^* together with those for C_L , C_D and C_L/C_D , considering C and I^* as two kinematic governing

parameters.

Figures 6, 7, 8 and 9 show the I^* effects upon Ω^* , C_L , C_D and C_L/C_D for several values of C , respectively, in the ranges of $C = 7.2 \times 10^1 - 1.6 \times 10^3$ and $I^* = 0.75 - 43$. And, Table 2 summarises the symbols and their corresponding parameters used in these figures.

In Fig. 6, we first see the data for $C = 7.2 \times 10^1$. Ω^* monotonously increases with increasing I^* , asymptoting to a constant value between 0.4 and 0.6, while there exist small data scatterings. We can observe such similar effects of I^* upon Ω^* for the other five values of C as that for $C = 7.2 \times 10^1$. Strictly Speaking, Ω^* for $C = 2.0 \times 10^2$ and 3.7×10^2 tends to have a constant value being independent of I^* , and does not increase with increasing I^* . Because, there are no data at enough small I^* due to the technical difficulty to find out appropriate materials for the model. However, if we consider the results for $C = 7.2 \times 10^1$ and 1.1×10^2 , we might suppose the similar effects of I^* upon Ω^* .

Referring to all the data with six values of C , we propose an empirical formula given by

$$\Omega^* = \alpha_{\Omega} \left(1 - \beta_{\Omega} e^{-\gamma_{\Omega} I^*} \right), \quad (20)$$

with $\alpha_{\Omega} = 4/7$, $\beta_{\Omega} = 0.27C^{0.32}$ and $\gamma_{\Omega} = 0.0006C^{1.4}$. Thin solid lines in Fig. 6 represent this empirical formula Eq. (20). We can confirm that the lines are in good agreement with the experimental data. —More specifically, the agreement for $C = 7.2 \times 10^1$ seems inferior to those for other C 's. For this reason, we can suggest that the measurements for $C = 7.2 \times 10^1$ are conducted at the smallest values of Re , where the tumbling tends to become unstable due to viscous effects.—

In Fig. 7, we first see the data for $C = 7.2 \times 10^1$. C_L monotonously increases with increasing I^* , asymptoting to a constant value of about 7/5, while there exist some data scatterings. We can observe such similar effects of I^* upon C_L for $C = 1.1 \times 10^2$ and 2.0×10^2 as that for $C = 7.2 \times 10^1$. However, for $C \geq 3.7 \times 10^2$, C_L monotonously decreases with increasing I^* , asymptoting to about 7/5. Referring to all the data with six values of C , we propose an empirical formula given by

$$C_L = \alpha_L \left(1 - \beta_L e^{-\gamma_L I^*} \right), \quad (21)$$

with $\alpha_L = 7/5$, $\beta_L = 1800C^{-1.4} - 0.5$ and $\gamma_L = 0.5$. Thin solid lines in Fig. 7 represent this empirical formula Eq. (21). We can confirm that the lines are in good agreement with the experimental data, as well. —More specifically, the agreement for $C = 7.2 \times 10^1$ seems inferior to those for other C 's, as well as Fig. 6.—

Fig. 6 Reduced rotating rate Ω^* of versus inertia moment ratio I^* , for $\lambda = 0.3$, $AR = 10$, $C = 7.2 \times 10^1 - 1.6 \times 10^3$ and $I^* = 0.75 - 43$. Empirical formula is described as Eq. (20).

Fig. 7 Lift coefficient C_L versus inertia moment ratio I^* , for $\lambda = 0.3$, $AR = 10$, $C = 7.2 \times 10^1 - 1.6 \times 10^3$ and $I^* = 0.75 - 43$. Empirical formula is described as Eq. (21).

Fig. 8 Drag coefficient C_D versus inertia moment ratio I^* , for $\lambda = 0.3$, $AR = 10$, $C = 7.2 \times 10^1 - 1.6 \times 10^3$ and $I^* = 0.75 - 43$. Empirical formula is described as Eq. (22).

Fig. 9 Lift-drag ratio C_L/C_D versus inertia moment ratio I^* , for $\lambda = 0.3$, $AR = 10$, $C = 7.2 \times 10^1 - 1.6 \times 10^3$ and $I^* = 0.75 - 43$. Empirical formula is described as Eq. (23).

In Fig. 8, C_D tends to scatter in a range centering about $7/6$, while there exist some data scatterings, being independent of both C and I^* . So, we propose an empirical formula given by

$$C_D = 7/6. \quad (22)$$

Incidentally speaking, in Fig. 8, we also draw the experimental data by Dupleich⁽³⁾ which are quoted by Poreh & Wrey⁽⁸⁾. We can see a qualitative discrepancy between the present and Dupleich where C_D rapidly decreases with increasing I^* at $I^* < 3$. As one possible explanation for this inconsistency, we might suggest low values of AR in his experiment. However, it is not easy to discuss precisely in the present stage, because of his insufficient descriptions of governing parameters and because of the lack of our enough knowledge on the AR effects. Specifically speaking, concerning the former reason, we cannot know which of Dupleich's data Poreh & Wray use. And thus, such governing parameters as λ , AR and C are possibly not constant, but vary within some ranges. Concern the latter reason, further investigations are required in future.

In Fig. 9, we first see the data for $C = 7.2 \times 10^1$. According to both Figs. 7 and 8, C_L/C_D monotonously increases with increasing I^* , asymptoting to a constant value of about $6/5$, while there exist small data scatterings. Then, as well as Fig. 7, we can observe such similar effects of I^* upon C_L/C_D for $C = 1.1 \times 10^2$ and 2.0×10^2 as that for $C = 7.2 \times 10^1$. However, for $C \geq 3.7 \times 10^2$, C_L/C_D monotonously decreases with increasing I^* , asymptoting to about $6/5$. From Eqs. (21) and (22), we get an empirical formula given by

$$C_L/C_D = 6/7 \alpha_L \left(1 - \beta_L e^{-\gamma_L I^*} \right), \quad (23)$$

with $\alpha_L = 7/5$, $\beta_L = 1800C^{-1.4} - 0.5$ and $\gamma_L = 0.5$. Thin solid lines in Fig. 9 represent this empirical formula Eq. (23). We can confirm that the lines are in good agreement with the experimental data, as well.

4.3 Influence of $Re(V_d)$

In the last subsection, we regard C and I^* as two kinematic governing parameters, and investigated the effects of I^* upon such quantities as Ω^* , C_L , C_D , and C_L/C_D under the condition that C keeps constant to one of six values. As a result, we have successfully proposed simple empirical formulae to predict those quantities with enough accuracies.

On the other hand, it is possible to choose other sets of two kinematic governing parameters. Now, we attempt another set, namely, a pair of C and $Re(V_d)$ or a pair of I^* and $Re(V_d)$, instead of a pair of C and I^* .

As is mentioned before, the relation among C , I^* and $Re(V_d)$ is given by Eq. (9). Figure 10 shows the relation between I^* and $Re(V_d)$ for six constant values of C , on the $I^*-Re(V_d)$ plane. That is, each dashed line in the figure represents Eq. (9) for one of six values of C . —Each solid line will be explained later.—

Figure 10 also shows experimental values of Re plotted against I^* for the six C 's, where the symbols as is described in Table 2. All the experimental Re 's are always somewhat smaller than $Re(V_d)$. This suggests that the actual value of C_D is larger than unity which is assumed in Eq. (8).

Figure 11 shows Re plotted against $Re(V_d)$, for $C = 7.2 \times 10^1 - 1.6 \times 10^3$ and $I^* = 0.75 - 43$. All the symbols in the figure correspond to those in Table 2, as well. In general, we should consider Re as a function of both C and $Re(V_d)$, or both I^* and $Re(V_d)$, in this context. However, we can suppose that the relation between Re and $Re(V_d)$ is approximately determined as a function with uniqueness. Figure 11 proves this, although there exist small dispersions. In spite of wide variations of C and I^* in the experimental data, the relation can be given by

$$Re = 0.78 Re(V_d), \quad (24)$$

with a good approximation.

Again, we see Fig. 10. Each solid line in the figure represents the relation Eq. (24) for one of six values of C . We can confirm that Eq. (24) shows a good approximation.

The fact that $Re(V_d)$ is approximately a unique governing parameter of Re implies such a possibility as $Re(V_d)$ could be a unique governing parameter of aerodynamic-characteristics quantities such as Ω^* , C_L , C_D and C_L/C_D . As two examples, Figs. 12 and 13 show the $Re(V_d)$ effects upon Ω^* and C_L/C_D , respectively. Again, Table 2 summarises the symbols and their corresponding parameters used in these figures. And, a solid line in each figure denotes the a proposed empirical formula, which will be discussed below.

In Figs. 12 and 13, Ω^* and C_L/C_D increase with increasing $Re(V_d)$, asymptoting to constant values, respectively. We should note that data scatterings are rather small, in spite of wide variations of C and I^* . To conclude, Ω^* and C_L/C_D are well approximated as follows.

$$\Omega^* = 4/7(1 - e^{-0.0045 Re(V_d)}). \quad (25)$$

$$C_L/C_D = 6/5(1 - e^{-0.0045 Re(V_d)}). \quad (26)$$

Fig. 10 Relation between I^* and the Reynolds number $Re(V_d)$ based on V_d for six constant values of C , together with Reynolds number Re versus inertia moment ratio I^* , for $\lambda = 0.3$, $AR = 10$, $C = 7.2 \times 10^1 - 1.6 \times 10^3$ and $I^* = 0.75 - 43$.

Fig. 11 Reynolds number Re versus the Reynolds number $Re(V_d)$ based on V_d , for $\lambda = 0.3$, $AR = 10$, $C = 7.2 \times 10^1 - 1.6 \times 10^3$ and $I^* = 0.75 - 43$.

A solid line in each figure denotes these empirical formulae. And, according to Eq. (22), C_L is given by

$$C_L = 7/5 \left(1 - e^{-0.0045 Re(V_d)} \right), \quad (27)$$

which is also in good agreement with the experimental data.

Of course, $Re(V_d)$ might not always be a unique governing parameter, from a theoretical viewpoint. As an example, we consider the relation between I^* and C_L/C_D as shown in Fig. 9, again. Figure 14 shows C_L/C_D plotted against I^* , for $C = 7.2 \times 10^1 - 1.6 \times 10^3$ and $I^* = 0.75 - 43$, whose experimental data are the same as those in Fig. 9. However, each solid line in the figure represents not the relation Eq. (23), but the relation Eq. (26) with the strict relation Eq. (9) among C , I^* and $Re(V_d)$, for one of six values of C . As a result, we can confirm less accuracy of Eq. (26) for large C and small I^* .

Incidentally speaking, we can get the relation between Re and Ω^* or that between $Re(V_d)$ and Ω^* , using Eq. (18) together with the relation Eq. (24) between $Re(V_d)$ and Re in Fig. 11 and with the relation Eq. (9) among C , I^* and $Re(V_d)$. This also gives us another formula to predict Ω^* from two of C , I^* and $Re(V_d)$. In Fig. 15, this semi-theoretical formula is drawn by solid lines for four fixed values of I^* , and by dotted lines for four fixed values of C , together with the same empirical data in Fig. 5. For comparison, we also show the

Fig. 12 Reduced rotating rate Ω^* versus the Reynolds number $Re(V_d)$ based on V_d , for $\lambda = 0.3$, $AR = 10$, $C = 7.2 \times 10^1 - 1.6 \times 10^3$ and $I^* = 0.75 - 43$.

Fig. 13 Lift-drag ratio C_L/C_D versus the Reynolds number $Re(V_d)$ based on V_d , for $\lambda = 0.3$, $AR = 10$, $C = 7.2 \times 10^1 - 1.6 \times 10^3$ and $I^* = 0.75 - 43$.

empirical formula Eq. (25) with/without Eq. (24) which is drawn by a thick solid line. Unfortunately, the semi-theoretical formula based on Eq. (18) with Eqs. (24) and (9) seems less accurate than the empirical formula Eq. (25) with/without Eq. (24), and of course than Eq. (20).

5. Conclusions

We have experimentally dealt with the tumbling of a two-dimensional plate under free flight. Then, we have specified fundamental aerodynamic characteristics in terminal condition, like the reduced rotating rate Ω^* , the lift coefficient C_L , the drag coefficient C_D and the lift-to-drag ratio C_L/C_D . And, we have successfully proposed a series of empirical formulae to predict Ω^* , C_L , C_D and C_L/C_D as functions of both a control parameter C and the inertia moment ratio I^* , which can be available for wide governing-parameter ranges such as $C = 7.2 \times 10^1 - 1.6 \times 10^3$ and $I^* = 0.75 - 43$. Moreover, the relation between the actual Reynolds number Re and the Reynolds number $Re(V_d)$ based on a theoretical velocity V_d can be approximately determined, being independent of C and I^* . Eventually, we have

Fig. 14 Lift-drag ratio C_L/C_D versus inertia moment ratio I^* , for $\lambda = 0.3$, $AR = 10$, $C = 7.2 \times 10^1 - 1.6 \times 10^3$ and $I^* = 0.75 - 43$. Empirical formula is given by Eq. (26).

Fig. 15 Reduced rotating rate Ω_{AOR}^* averaged over one rotation versus Reynolds number Re , for $\lambda = 0.3$, $AR = 10$, $C = 7.2 \times 10^1 - 3.7 \times 10^2$ and $I^* = 4.2 - 43$.

successfully proposed another series of empirical formulae to predict Ω^* , C_L , C_D and C_L/C_D as functions of $Re(V_d)$ alone, which is available for a wide parameter range except for large C and small I^* . In addition, the scattering manner of Ω^* is well approximated by a semi-theoretical formula for in wide ranges of C and I^* . This semi-theoretical formula is reflected by a close interaction based on the dynamical equilibrium on a free-flight plate supposing that the lift force L is produced mainly by the Magnus effect. Unfortunately, the semi-theoretical formula is likely to be less accurate than the other two empirical formulae.

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