Nonlinear Wave Equation for Ultrasound Beam in Nonuniform Bubbly Liquids

Tetsuya KANAGAWA∗,∗∗∗, Takeru YANO†, Masao WATANABE†† and Shigeo FUJIKAWA††

∗ Division of Mechanical and Space Engineering, Hokkaido University, Research Fellow of the Japan Society for the Promotion of Science, Kita 13 Nishi 8, Kita-ku, Sapporo, Hokkaido 060–8628, Japan
∗∗∗ Present address: Department of Mechanical Engineering, The University of Tokyo, 7–3–1, Hongo, Bunkyo-ku, Tokyo 113–8656, Japan
† Department of Mechanical Engineering, Osaka University, 2–1, Yamadaoka, Suita, Osaka 565–0871, Japan
†† Division of Mechanical and Space Engineering, Hokkaido University, Kita 13 Nishi 8, Kita-ku, Sapporo, Hokkaido 060–8628, Japan

Abstract

In our previous paper (Kanagawa et al., J. Fluid Sci. Tech., 5, 2010), we have proposed a systematic method for derivation of various types of nonlinear wave equations for plane waves in bubbly liquids. The method makes use of an asymptotic expansion with multiple scales in terms of a small wave amplitude as an expansion parameter and a set of scaling relations of physical parameters, based on basic equations of two-fluid model of bubbly flows. In this paper, we extend the method so as to handle a weakly diffracted ultrasound beam in a quiescent liquid containing a number of spherical gas bubbles distributed with a weak nonuniformity. Because of the high expandability of the original method, the extension can be accomplished by adding a scaling relation of the diameter of the beam to the original set of scaling relations. As a result, we derive a generalized Khokhlov–Zabolotskaya–Kuznetsov (KZK) equation [or a generalized Kadomtsev–Petviashvili (KP) equation] for a long wave and low frequency case.

Key words : Bubbly Liquid, Pressure Wave, Ultrasound, Beam, Diffraction, Nonuniformity, Nonlinear Wave Equation, KZK Equation, KP Equation, Multiple Scales

1. Introduction

As it is widely known, waves in bubbly liquids exhibit fairly complex behaviors as compared with those in single phase fluids as a result of complexity of two-phase fluids and variety of wave properties such as the dispersion, dissipation, and interference. Even if the wave amplitude is very small, as in most cases of engineering applications, nonlinear effects can appear in the process of long range propagation, and this sometimes leads to a complete change of wave behavior. A number of theoretical studies have therefore been carried out so far(1)–(11), and various types of nonlinear wave equations in bubbly liquids have been derived. The most famous nonlinear wave equations are the Korteweg–de Vries–Burgers (KdVB) equation(1), (2) for a long wave with a low frequency and the nonlinear Schrödinger (NLS) equation(7) for an envelope of a carrier wave with a high frequency. The existence of various types of nonlinear wave equations is an evidence of complexity of waves in bubbly liquids. Systematic studies of these complex phenomena are important for the understanding of physical mechanism and the enhancement of their applications.

In our previous paper(12), we have proposed a systematic method for derivation of various types of nonlinear wave equations, thereby unifying the derivation methods of the KdVB and
NLS equations. The essence of the method is choosing a set of scaling relations appropriate to a wave phenomenon concerned, which specifies the three ratios \( \frac{U^*}{c_{L0}^*}, \frac{R_0^*}{L^*}, \text{ and } \frac{\omega^*}{\omega_B^*} \) as functions of a typical dimensionless wave amplitude \( U^* \) and \( L^* \) are a typical propagation speed of the wave and a typical wavelength, respectively, \( R_0^* \) is a bubble radius in an initially unperturbed state, \( c_{L0}^* \) is a sound speed in the unperturbed liquid, \( \omega^*/(2\pi) \) is a frequency of the wave, and \( \omega_B^*/(2\pi) \) is an eigenfrequency of the bubble]. The superscript * denotes a dimensional quantity throughout this paper.

The wave motions treated in the previous paper are plane waves, i.e., spatially one-dimensional waves where the averaged physical quantities are functions of time and space coordinate in the direction of wave propagation. Furthermore, the initial void fraction is assumed to be constant, which means that the initial number density of bubbles is spatially uniform in the liquid. In this paper, we extend our method so as to handle an ultrasound beam in nonuniform bubbly liquids. The extension is important because most applications of pressure waves in bubbly liquids utilize ultrasound beams and the void fraction is actually nonuniform.

The extension of the method is completed by the following two steps: (i) adding a scaling relation of a typical diameter of sound beam to the original set of scaling relations and (ii) assuming that a nonuniformity of nondimensionalized initial number density of bubbles and its spatial variation are as small as a nondimensionalized wave amplitude. Using the scaling relations and weak nonuniformity assumption, in the same manner as that in the previous study, we apply the method of multiple scales\(^{(13)}\),\(^{(14)}\) to a set of two-fluid model equations for bubbly flows\(^{(15)}\),\(^{(16)}\). As a result, we can derive a nonlinear wave equation, which may be called a generalized Khokhlov–Zabolotskaya–Kuznetsov (KZK) equation\(^{(17)}\),\(^{(18)}\) (i.e., KZK equation with dispersion and nonuniform effects) or a generalized Kadomtsev–Petviashvili (KP) equation\(^{(19)}\) (i.e., KP equation with dissipation and nonuniform effects). The KZK equation has originally been derived for nonlinear sound beams in compressible Newtonian fluids and widely been employed in medical applications\(^{(20)}\), while the KP equation has been derived for the stability analysis of solitons of water surface waves. In the theory of two-phase flows, this is the first derivation of the generalized KZK (or KP) equation in a nonuniform bubbly liquid, although Khismatullin and Akhatov\(^{(9)}\) have derived the KP equation (without dissipation effect) in uniform bubbly liquids from a mixture (not two-fluid) model.

2. Problem statement and basic equations

2.1. Problem

We shall consider the nonlinear propagation of a sound beam in a nonuniform bubbly liquid. Let the sound beam be radiated from a circular sound source with a diameter \( D^* \) placed in the bubbly liquid. The sound source is not restricted to be planar, and it may be concave or convex; the former may yield a focused beam and the latter a spreading beam. We set the origin of the coordinate system at the center of the sound source, and adopt the central axis normal to the surface of the sound source as the \( x^* \) axis (see Fig. 1). We assume
that the averaged physical quantities defined everywhere in the bubbly liquid by volume and surface averages\(^{(15)}\) are symmetric around the \(x^*\) axis. Accordingly, all the averaged physical quantities are the functions of the time \(t^*\), the distance from the sound source, \(x^*\), and the distance from the \(x^*\) axis, \(r^*\), where
\[
r^* = \sqrt{y^2 + z^2}.
\] (1)

The wave motion treated here contains two extensions from the previous plane wave problem\(^{(12)}\). Firstly, we extend the one-dimensional wave (plane wave) to a three-dimensional wave, based on the conditions that the initial bubble radius \(R_0^*\) is sufficiently small compared with a typical wavelength \(L^*\), and the latter is sufficiently small compared with the diameter of the sound source \(D^*\), i.e.,
\[
R_0^* \ll L^* \ll D^*.
\] (2)

As demonstrated below, this condition yields the sound beam with weak diffraction (weak focusing or weak spreading). The second extension is that the spatial distribution of initial number density of bubbles, \(n_0^*(x^*, r^*)\), is not uniform. We assume that the nonuniformity is weak, i.e.,
\[
\left| \frac{n_0^*}{n_{00}} - 1 \right| \ll 1,
\] (3)
where \(n_{00}\) (constant) is the average of \(n_0^*\).

### 2.2. Basic equations

A set of averaged equations for bubbly flows is composed of the conservation equations of mass and momentum for the gas and liquid phases based on a two-fluid model\(^{(15)}\), the equation of motion for the bubble wall, the equations of state for the gas and liquid phases, the mass conservation equation inside the bubble, and the balance of normal stresses at the bubble–liquid interface\(^{(12),(15),(16)}\). The conservation equations are given by
\[
\frac{\partial \alpha_G \rho_G^*}{\partial t^*} + \nabla \cdot (\alpha_G \rho_G^* \mathbf{u}_G^*) = 0,
\] (4)
\[
\frac{\partial \alpha_L \rho_L^*}{\partial t^*} + \nabla \cdot (\alpha_L \rho_L^* \mathbf{u}_L^*) = 0,
\] (5)
\[
\frac{\partial \alpha_G \rho_G^* \mathbf{u}_G^*}{\partial t^*} + \nabla \cdot (\rho_G^* \mathbf{u}_G^* \mathbf{u}_G^*) + \alpha_G \nabla \rho_G^* + \mathbf{F}^* = \mathbf{0},
\] (6)
\[
\frac{\partial \alpha_L \rho_L^* \mathbf{u}_L^*}{\partial t^*} + \nabla \cdot (\rho_L^* \mathbf{u}_L^* \mathbf{u}_L^*) + \alpha_L \nabla \rho_L^* + \mathbf{F}^* + \mathbf{P}^* \nabla \alpha_G = \mathbf{0},
\] (7)
where \(t^*\) is the time, \(\alpha\) is the volume fraction, \(\rho^*\) is the density, \(\mathbf{u}^* = (u^*, v^*)\) is the fluid velocity \((u^*\) and \(v^*\) are the components in the \(x^*\) and \(r^*\) directions, respectively), \(p^*\) is the pressure, \(\mathbf{P}^*\) is the liquid pressure averaged on the bubble–liquid interface\(^{(21)}\), and the subscripts \(G\) and \(L\) denote volume-averaged variables in the gas and liquid phases, respectively.

In Eqs. (4)–(7), the compressibility of liquid phase is taken into account, which leads to wave attenuation due to the acoustic radiation from oscillating bubbles. The viscosity of gas phase, the phase change across the bubble–liquid interface, the bubble–bubble interaction, and the translation of bubbles are neglected for simplicity. We also ignore the attenuation of waves and bubble oscillations due to thermal effects\(^{(22)}–(25)\).

For the interfacial momentum transport \(\mathbf{F}^*\) in Eqs. (6) and (7), we adopt the following model of virtual mass force\(^{(16)}\)
\[
\mathbf{F}^* = -\beta_1 \alpha_G \rho_G^* \left( \frac{D_G \mathbf{u}_G^*}{Dt^*} - \frac{D_L \mathbf{u}_L^*}{Dt^*} \right) + \beta_2 \alpha_G \left( \mathbf{u}_G^* - \mathbf{u}_L^* \right) \frac{D_G \rho_G^*}{Dt^*} - \beta_3 \alpha_G \left( \mathbf{u}_G^* - \mathbf{u}_L^* \right) \frac{D_L \rho_L^*}{Dt^*},
\] (8)
where the values of coefficients $\beta_1$, $\beta_2$, and $\beta_3$ may be set as $1/2$ for the spherical bubble, although we proceed without explicitly showing these values to clarify the contribution of each term in the right-hand side of Eq. (8) to the final result. Equation (8) is suggested by the analysis of virtual mass force in a compressible liquid \cite{26,27}.

The Keller equation \cite{28} for spherical oscillations of a bubble in a compressible liquid is given by

$$\left(1 - \frac{1}{c_{L0}^2} \frac{DG}{Dt^*}\right) R^* \left( \frac{DG}{Dt^*} + \frac{3}{2} \left(1 - \frac{1}{3c_{L0}^2} \frac{DG}{Dt^*}\right) \right) \left( \frac{DG}{Dt^*} \right)^2 = \left(1 + \frac{1}{c_{L0}^2} \frac{DG}{Dt^*} \right) \left( \frac{DG}{Dt^*} + \frac{R^*}{c_{L0}^2} \frac{DG}{Dt^*} (p_L^* + P^*) \right).$$

(9)

where $R^*$ is the averaged bubble radius, and $c_{L0}^*$ and $\rho_{L0}^*$ are the sound speed and the density in the initial unperturbed liquid, respectively. The definitions of operators $DG/Dt^*$ and $DL/DT^*$ are

$$\frac{DG}{Dt^*} \equiv \frac{\partial}{\partial t^*} + u_G^* \cdot \nabla^*,$$

$$\frac{DL}{DT^*} \equiv \frac{\partial}{\partial t^*} + u_L^* \cdot \nabla^*.$$ (10)

The second term in the right-hand side of Eq. (9) represents a damping effect, which is mainly responsible for the wave attenuation due to the acoustic radiation from oscillating bubbles.

To close the system of Eqs. (4)–(9), the following equations are used: (i) Tait’s equation of state for liquid,

$$p_L^* = p_{L0}^* + \rho_{L0}^* c_{L0}^2 \left( \left( \frac{\rho_L^*}{\rho_{L0}^*} \right)^n - 1 \right),$$

(11)

where $n$ is the material constant (e.g., $n = 7.15$ for water). (ii) The polytropic equation of state for gas,

$$\frac{p_G^*}{p_{G0}^*} = \left( \frac{\rho_G^*}{\rho_{G0}^*} \right)^\gamma,$$

(12)

where $\gamma$ is the polytropic exponent. (iii) The conservation equation of mass inside the bubble,

$$\frac{\rho_G^*}{\rho_{G0}^*} = \left( \frac{R_0^*}{R^*} \right)^3.$$ (13)

(iv) The balance of normal stresses across the bubble–liquid interface,

$$p_G^* - (p_L^* + P^*) = \frac{2\sigma^*}{R^*} + \frac{4\mu^*_L DG}{R^*} \frac{DG}{Dt^*},$$

(14)

where $\sigma^*$ is the surface tension and $\mu^*_L$ is the viscosity of liquid phase; the effect of liquid viscosity $\mu^*$ is neglected except at the bubble–liquid interface. (v) The constraint of the volume fractions,

$$\alpha_G + \alpha_L = 1.$$ (15)

From now on, we put $\alpha_G \equiv \alpha$ and $\alpha_L = 1 - \alpha$ for simplicity. The physical quantities in the initial unperturbed state signified by the subscript 0 are all constants, except for $n_0^*$ in Eq. (38) in §3.3.

In addition, we employ the angular eigenfrequency of linear spherical symmetric oscillations of a single bubble, $\omega_B^*$,

$$\omega_B^* = \sqrt{\frac{3\gamma (p_{L0}^* + 2\sigma^*/R_0^*) - 2\sigma^*/R_0^*}{\rho_{L0}^* R_0^{*2}}},$$

(16)

where the effects of the liquid viscosity and liquid compressibility are not included in Eq. (16).
3. Derivation of generalized KZK equation

The derivation procedure employs the asymptotic expansion in a parameter sufficiently small compared with unity. We use the dimensionless wave amplitude as the expansion parameter, and represent it by $\epsilon$, where $0 < \epsilon \ll 1$.

3.1. Scaling relations

As in the previous paper, we introduce a set of scaling relations appropriate to the low frequency and long wavelength (12),

$$U^* = O\left(\sqrt{\epsilon}\right) \equiv V \sqrt{\epsilon}, \quad (17)$$

$$R^\ast_0 \equiv O\left(\sqrt{\epsilon}\right) \equiv \Delta \sqrt{\epsilon}, \quad (18)$$

$$\frac{\omega^*}{\omega_b^*} \equiv O\left(\sqrt{\epsilon}\right) \equiv \Omega \sqrt{\epsilon}, \quad (19)$$

where $V, \Delta,$ and $\Omega$ are quantities of $O(1)$ (where $\Omega$ denotes a normalized angular frequency of the wave). The forms of typical scales $U^*$ and $L^*$ are explicitly determined in §3.4, and $U^*$ and $L^*$ are related by $U^* \equiv L^* \omega^* = L^*/T^*$, where a typical period of the wave $T^*$ is defined by the inverse of the angular frequency of the wave, $\omega^*$. Furthermore, the nondimensional liquid viscosity $\mu$ is defined as

$$\frac{\mu^*}{\rho L^0 U^* L^*} \equiv O(\epsilon) \equiv \mu \epsilon. \quad (20)$$

Let us introduce an additional scaling relation for the sound beam propagation with the weak diffraction, which is defined by the ratio of the wavelength $L^*$ and the diameter of the sound source $D^*$:

$$\frac{L^*}{D^*} \equiv O\left(\sqrt{\epsilon}\right) \equiv \Gamma \sqrt{\epsilon}, \quad (21)$$

where $\Gamma$ is a quantity of $O(1)$ and stands for the effect of the weak diffraction to the $r$ direction. The size of the ratio $L^*/D^*$ characterizes the pattern of sound beams (29), (30). The scaling relation (21) ensures that the sound is localized in the vicinity of the $x^*$ axis and the wavefronts are quasi-planar (31).

Equation (17) shows that the propagation speed is small compared with the sound speed and Eq. (19) does that the frequency is low compared with the eigenfrequency of the bubble. Accordingly, Eqs. (18) and (21) yield Eq. (2) in §2.1, i.e., the wavelength is large compared with the bubble radius and small compared with the diameter of the sound source.

3.2. Multiple scales and perturbation expansions

We nondimensionalize the independent variables $t, x,$ and $r$, as

$$t = \frac{t^*}{T^*}, \quad x = \frac{x^*}{L^*}, \quad r = \frac{r^*}{L^*}. \quad (22)$$

The near field of sound beam is defined as a region where the wavefronts are approximately planar (31), and the distance from the sound source is comparable with the wavelength. Therefore, in the near field, we ignore the spatial variations of dependent variables in the $r$ direction and employ the independent variables defined by only $t$ and $x$:

$$t_0 = t, \quad x_0 = x. \quad (23)$$

On the other hand, the far field is characterized by spherical wavefronts (31), and hence the wave motion in the far field should be described by not only the axial coordinate $x$ but also the radial coordinate $r$. Furthermore, the nonlinear and diffraction effects of sound beam appear in the far field (see §3.6). The independent variables appropriate to the far field are, therefore,

$$t_1 = \epsilon t, \quad x_1 = \epsilon x, \quad r_{1/2} = \sqrt{\epsilon} \Gamma r. \quad (24)$$
The definition of $x_1$ and $r_{1/2}$ implies that spatial variations along the axis of the beam are slower than those across the beam. In Eq. (24), $r_{1/2}$ accounts for the scaling relation (21) through the modification of $r$ in Eq. (22),

$$r = \frac{D^* r^*}{L^*} = \frac{r_{1/2}}{\sqrt{\epsilon}}, \quad \left( r_{1/2} \equiv \frac{r^*}{D^*} \right).$$

(25)

All dependent variables should now be regarded as functions of these extended independent-variables (23) and (24), and differential operators can be expanded as follows:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1}, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial x_0} + \epsilon \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial r} = \sqrt{\epsilon} \frac{\partial}{\partial r_{1/2}}.$$

(26)

Let us expand the dependent variables in power series by using $\epsilon (\ll 1)$:

$$R'/R_0 - 1 = \epsilon R_1 + \epsilon^2 R_2 + \cdots,$$

(27)

$$u_{G1}/U^* = \epsilon u_{G1} + \epsilon^2 u_{G2} + \cdots,$$

(28)

$$u_{L1}/U^* = \epsilon u_{L1} + \epsilon^2 u_{L2} + \cdots,$$

(29)

$$v_{G1}/U^* = \epsilon^{3/2} v_{G1} + \epsilon^{5/2} v_{G2} + \cdots,$$

(30)

$$v_{L1}/U^* = \epsilon^{3/2} v_{L1} + \epsilon^{5/2} v_{L2} + \cdots,$$

(31)

where $\epsilon$ denotes the small wave amplitude as mentioned at the beginning in §3. In the higher-order calculations, the terms of the derivatives of $v_{G1}$ and $v_{L1}$ with respect to $r$ appear, which result in the diffraction terms of a generalized KZK equation (see, §3.6 and the first terms of right-hand sides of $J_1$ and $J_2$ in Appendix). The expansions of the radial components of fluid velocities start with $O(\epsilon^{3/2})$ in Eqs. (30) and (31) because of the fact explained in the second paragraph in this subsection.

The expansions of density and pressure in the liquid phase are

$$\rho_{L1}/\rho_{L0} - 1 = \epsilon^2 \rho_{L1} + \epsilon^3 \rho_{L2} + \cdots,$$\n
(32)

$$\frac{p_{L1} - p_{L0}}{\rho_{L0} U^2} = \epsilon \rho_{L1} + \epsilon^2 \rho_{L2} + \cdots,$$\n
(33)

with

$$p_{L1} = \frac{\rho_{L1}}{V^2}, \quad p_{L2} = \frac{\rho_{L2}}{V^2}, \quad p_{L3} = \frac{\rho_{L3}}{V^2} + \frac{(n-1)p_{L1}^2}{2V^2}, \cdots$$\n
(34)

Equations (33) and (34) can be derived by substituting Eq. (32) into Eq. (11) with the aid of Eq. (17). The remaining variables, $p_{G1}, p_{G2}$, and $p^*$, are also nondimensionalized and expanded in $\epsilon$, and the expansion of $\alpha$ is presented in Eq. (37) below.

The nondimensional pressures in the unperturbed state, $p_{G0}$ and $p_{L0}$, are given by

$$p_{G0} \equiv \frac{p_{G0}}{\rho_{L0}^* U^2} = O(1), \quad p_{L0} \equiv \frac{p_{L0}^*}{\rho_{L0}^* U^2} = O(1).$$\n
(35)

The ratio of initial densities of the gas and liquid phases is assumed to be small as

$$\frac{\rho_{G0}^*}{\rho_{L0}^*} = O(\epsilon^2),$$\n
(36)

and hence this ratio does not affect the final result of the present analysis.

3.3. Nonuniform distribution of bubbles

We consider a nonuniformity of bubble distribution of the magnitude of $O(\epsilon)$ and with a spatial-variation scale of $O(1/\epsilon)$. Therefore, the void fraction can be expanded as

$$\alpha/\alpha_0 - 1 = \epsilon [\alpha_1(t_0, t_1, x_0, x_1, r_{1/2}) + \delta(x_1)] + \epsilon^2 \alpha_2 + \epsilon^3 \alpha_3 + \cdots,$$\n
(37)

where $\delta$ denotes a nonuniformity of bubble distribution at the initial state. Since the sound beam spreads only up to the range of $O(1/\sqrt{\epsilon})$ in the radial direction, $\delta$ depends on only the
axial coordinate \( x_1 \). Although the constant \( \alpha_0 \) should be small compared with unity due to the assumptions used in the derivation of basic equations shown in §2.2, it is treated as a quantity of \( O(1) \) because we seek the asymptotic solution with respect to the small parameter \( \epsilon \).

The initial nonuniformity of the void fraction \( \delta \) corresponds to that of the number density of bubbles \( n_0^* \). Therefore we have Eq. (3),

\[
\frac{n_0^*(x_1)}{n_0^*0} - 1 = \epsilon \delta(x_1) \ll 1, \tag{38}
\]

through the definition of void fraction

\[
\alpha \equiv \frac{4}{3} \pi R^3 n^*, \tag{39}
\]

where \( n^* \) is the number density of bubbles.

### 3.4. \( O(\epsilon) \): Near field

Substituting Eqs. (17)–(21), (26)–(33), and (37) into Eqs. (4)–(15), and then equating the coefficients of like powers of \( \epsilon \), \( \epsilon^{3/2} \), and \( \epsilon^2 \) in the resultant equations, we have the following set of linear equations from the coefficient of \( \epsilon^2 \):

(i) conservation equation of mass in gas phase,

\[
\frac{\partial \alpha_1}{\partial t_0} - 3 \frac{\partial R_1}{\partial t_0} + \frac{\partial u_{G1}}{\partial x_0} = 0, \tag{40}
\]

(ii) conservation equation of mass in liquid phase,

\[
\alpha_0 \frac{\partial \alpha_1}{\partial t_0} - (1 - \alpha_0) \frac{\partial u_{L1}}{\partial t_0} = 0, \tag{41}
\]

(iii) conservation equation of momentum in the \( x \) direction in gas phase,

\[
\beta_1 \frac{\partial u_{G1}}{\partial t_0} - \beta_1 \frac{\partial u_{L1}}{\partial t_0} - 3 \gamma \rho_{G0} \frac{\partial R_1}{\partial x_0} = 0, \tag{42}
\]

(iv) conservation equation of momentum in the \( x \) direction in liquid phase,

\[
(1 - \alpha_0 + \beta_1 \alpha_0) \frac{\partial u_{L1}}{\partial t_0} - \beta_1 \alpha_0 \frac{\partial u_{G1}}{\partial t_0} + (1 - \alpha_0) \frac{\partial p_{L1}}{\partial x_0} = 0, \tag{43}
\]

(v) Keller’s equation,

\[
R_1 + \frac{\Omega^2}{\Delta^2} p_{L1} = 0. \tag{44}
\]

As is clear from Eqs. (30) and (31), no variations of momentum in the radial direction appear in the leading order of approximation.

From Eqs. (40)–(44), the linear wave equation for \( R_1 \) can be derived:

\[
\mathcal{L}[R_1] \equiv \frac{\partial^2 R_1}{\partial t_0^2} - v_p^2 \frac{\partial^2 R_1}{\partial x_0^2} = 0, \tag{45}
\]

where \( \mathcal{L} \) is the differential operator with the phase velocity \( v_p \) defined by

\[
v_p = \sqrt{\frac{3 \alpha_0(1 - \alpha_0 + \beta_1) \gamma \rho_{G0} + \beta_1(1 - \alpha_0) \Delta^2 / \Omega^2}{3 \beta_1 \alpha_0(1 - \alpha_0)}}. \tag{46}
\]

Choosing \( v_p = 1 \) gives the explicit representation of \( U^* \) in Eq. (17) as

\[
U^* = \sqrt{\frac{3 \alpha_0(1 - \alpha_0 + \beta_1) \gamma \rho_{G0}^* / \rho_{L0}^* + \beta_1(1 - \alpha_0) R_{G0}^2 \omega_{B0}^2}{3 \beta_1 \alpha_0(1 - \alpha_0)}}. \tag{47}
\]

and the explicit form of \( L^* = U^* T^* \) is simultaneously obtained. We focus on only the right-running wave in the leading order of approximation and hence a retarded time (phase function) \( \phi_0 \) is introduced as

\[
\phi_0 \equiv t_0 - x_0. \tag{48}
\]
Putting $R_1 \equiv f(\phi_0; t_1, x_1, r_{1/2})$ reduces Eq. (45) to
\[
\frac{\partial f}{\partial t_0} + \frac{\partial f}{\partial x_0} = 0. \tag{49}
\]

From Eqs. (40)–(44) and $R_1 = f(\phi_0)$, we have the expressions of the first-order variables
\[
\alpha_1 = s_1 f, \quad \nu_{G1} = s_2 f, \quad u_{L1} = s_3 f, \quad p_{L1} = s_4 f,
\]
\[
s_4 = \frac{\Delta^2}{\Omega^2}, \quad s_1 = \frac{(1 - \alpha_0)[3\beta_1\alpha_0 - (1 - \alpha_0)s_4]}{\alpha_0(1 - \alpha_0 + \beta_1)}, \quad s_2 = s_1 - 3, \quad s_3 = \frac{\alpha_0 s_1}{1 - \alpha_0}, \tag{50}
\]

Here, constants of integration are dropped by using the conditions at $\phi_0 = t_0 - x_0 \to \infty$, where the bubbly liquid is at rest.

In the near field characterized by $t_0$ and $x_0$, all the first-order perturbations are governed by Eq. (49). Therefore, in the temporal and spatial scales of $O(1)$, the wave motion is linear, nondissipative, and nondispersive. Furthermore, the effects of diffraction and nonuniformity do not appear.

The results presented in this subsection are the same as those in §3.1 of the previous paper.

3.5. $O(e^{3/2})$: Velocity perturbations in radial direction

Equations of $O(e^{3/2})$ are composed of the conservation equations of momentum in the $r$ direction for gas and liquid phases,
\[
\beta_1 \frac{\partial \nu_{G1}}{\partial t_0} - \beta_1 \frac{\partial \nu_{L1}}{\partial t_0} - 3\gamma \nu_{G0} \Gamma \frac{\partial R_1}{\partial r_{1/2}} = 0, \tag{51}
\]
\[
(1 - \alpha_0 + \beta_1 \alpha_0) \frac{\partial \nu_{G1}}{\partial t_0} - \beta_1 \alpha_0 \frac{\partial \nu_{G1}}{\partial t_0} + (1 - \alpha_0) \Gamma \frac{\partial p_{L1}}{\partial r_{1/2}} = 0, \tag{52}
\]

respectively. Equations (51) and (52) contain the terms of derivatives of $\nu_{G1}$ and $\nu_{L1}$, and of gradients with respect to $r_{1/2}$. That is, the first-order perturbations of fluid velocities in the $r$ direction and the variations to the $r$ direction appear in the equations of $O(e^{3/2})$.

Clearly, $\nu_{G1}$ and $\nu_{L1}$ also depend on $\phi_0$, as in the case of $\alpha_1$, $\nu_{G1}$, $u_{L1}$, $p_{L1}$, and $R_1$. From Eqs. (51) and (52) with the aid of Eq. (50), we obtain
\[
\frac{\partial \nu_{G1}}{\partial \phi_0} = C_1 \frac{\partial f}{\partial r_{1/2}}, \quad \frac{\partial \nu_{L1}}{\partial \phi_0} = C_2 \frac{\partial f}{\partial r_{1/2}}, \tag{53}
\]

with
\[
C_1 = \frac{3(1 - \alpha_0 + \beta_1 \alpha_0)}{1 - \alpha_0 + \beta_1} - \frac{(1 - \alpha_0)^2 \Delta^2}{\alpha_0(1 - \alpha_0 + \beta_1)}, \tag{54}
\]
\[
C_2 = \frac{3\beta_1 \alpha_0 + (1 - \alpha_0) \Delta^2}{1 - \alpha_0 + \beta_1}. \tag{55}
\]

Equation (53) gives $\nu_{G1}$ and $\nu_{L1}$ as functions of $r_{1/2}$ and $\phi_0 = t_0 - x_0$. Accordingly, Eq. (53) describes the wave motion with the time scale of $T^*$, the length scale of $L^*$ in the axial direction, and the length scale of $D^*$ in the radial direction.

3.6. $O(e^2)$: Far field and generalized KZK equation

We shall proceed to the next-order calculations. The system of inhomogeneous equations in $O(e^2)$ is obtained as the corresponding set of Eqs. (40)–(44):
\[
\frac{\partial \alpha_2}{\partial t_0} - 3 \frac{\partial R_2}{\partial t_0} + \frac{\partial \nu_{G2}}{\partial x_0} = J_1, \tag{56}
\]
\[
\alpha_0 \frac{\partial \alpha_2}{\partial t_0} - (1 - \alpha_0) \frac{\partial \nu_{L2}}{\partial x_0} = J_2, \tag{57}
\]
\[
\beta_1 \frac{\partial \nu_{G2}}{\partial t_0} - \beta_1 \frac{\partial \nu_{L2}}{\partial x_0} - 3\gamma \nu_{G0} \frac{\partial R_2}{\partial x_0} = J_3, \tag{58}
\]
\[
(1 - \alpha_0 + \beta_1 \alpha_0) \frac{\partial u_2}{\partial t_0} - \beta_1 \alpha_0 \frac{\partial u_{g2}}{\partial \theta_0} + (1 - \alpha_0) \frac{\partial p_{l2}}{\partial x_0} = J_4, \quad (59)
\]

\[
R_2 + \frac{\Omega^2}{\Lambda^2} p_{l2} = J_5, \quad (60)
\]

where the inhomogeneous terms \(J_i (1 \leq i \leq 5)\) are composed of the first-order perturbations derived in the calculations of \(O(\varepsilon)\) and \(O(\varepsilon^{3/2})\); they are explicitly shown in Appendix.

Equations (56)–(60) are combined into the single equation for \(R_2\):

\[
\mathcal{L}[R_2] = J(\phi_0, t_1, x_1, r_{1/2}) = -\frac{1}{3} \frac{\partial J_1}{\partial \phi_0} + \frac{1}{3\alpha_0} \frac{\partial J_2}{\partial \phi_0} - \frac{1}{3\beta_1 (1 - \alpha_0)} \frac{\partial J_3}{\partial \phi_0} - \frac{1}{3\alpha_0 (1 - \alpha_0)} \frac{\partial J_4}{\partial \phi_0} - \frac{\Delta^2}{3\alpha_0 \Omega^2} \frac{\partial J_5}{\partial \phi_0}. \quad (61)
\]

We can rewrite \(J\) through straightforward calculations,

\[
J = -2 \frac{\partial }{\partial \phi_0} \left\{ \frac{\partial f}{\partial t_1} + \frac{\partial f}{\partial x_1} + [\Lambda_0 + \Lambda_4 \delta(\varepsilon \lambda_1)] \frac{\partial f}{\partial \phi_0} + \Lambda_1 \frac{\partial f}{\partial \phi_0} + \Lambda_2 \frac{\partial^2 f}{\partial \phi^2_0} + \Lambda_3 \frac{\partial^3 f}{\partial \phi^3_0} \right\}
\]

\[
+ \frac{C_1}{3} + \frac{1}{3\alpha_0} \frac{\partial f}{\partial \phi_0} \left\{ r_{1/2} \frac{\partial}{\partial r_{1/2}} \left( \frac{1}{2} \frac{\partial f}{\partial r_{1/2}} \right) \right\}. \quad (62)
\]

We impose the solvability condition for Eq. (61),

\[
J(\phi_0, t_1, x_1, r_{1/2}) = 0. \quad (63)
\]

This is a far field equation that describes the wave motion in the region of \(x = O(1/\varepsilon)\).

Combining the near field equation (49) and the far field equation (63) with Eq. (62), with the help of Eq. (26), yields

\[
\frac{\partial }{\partial t} \left( \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} + \varepsilon \left\{ [\Lambda_0 + \Lambda_4 \delta(\varepsilon \lambda_1)] \frac{\partial f}{\partial t} + \Lambda_1 \frac{\partial f}{\partial t} + \Lambda_2 \frac{\partial^2 f}{\partial t^2} + \Lambda_3 \frac{\partial^3 f}{\partial t^3} \right\} \right)
\]

\[
= -\frac{1}{2\xi} \frac{\partial }{\partial \xi} \left( \frac{\partial f}{\partial \xi} \right). \quad (64)
\]

Equation (64) is finally transformed into

\[
\frac{\partial }{\partial \eta} \left( \frac{\partial f}{\partial X} + \Lambda_1 \frac{\partial f}{\partial \eta} + \Lambda_2 \frac{\partial^2 f}{\partial \eta^2} + \Lambda_3 \frac{\partial^3 f}{\partial \eta^3} \right) = \frac{\Gamma^2}{2\xi} \frac{\partial }{\partial \zeta} \left( \frac{\partial f}{\partial \zeta} \right), \quad (65)
\]

through the variable transformation

\[
X = \varepsilon x, \quad \eta = t - [1 + \varepsilon [\Lambda_0 + \Lambda_4 \delta(\varepsilon \lambda_1)] \varepsilon, \quad \xi = \sqrt{\varepsilon} r. \quad (66)
\]

Here, the coefficients are given by

\[
\Lambda_0 = \frac{(1 - \alpha_0) \Delta^2 \Omega^2}{6\alpha_0 \Omega^2} = -\Pi_0, \quad (67)
\]

\[
\Lambda_1 = -\Pi_1, \quad (68)
\]

\[
\Lambda_2 = -\frac{1}{6\alpha_0} \left( 4\mu + \frac{3\Lambda^2 \Omega^2}{\Omega^2} \right) = \Pi_2, \quad (69)
\]

\[
\Lambda_3 = \frac{\Delta^2}{6\alpha_0} = -\Pi_3, \quad (70)
\]

\[
\Lambda_4 = \frac{[\beta_1 + (1 - \alpha_0)^2] \Lambda^2 \Omega^2 - 3\beta_1 \alpha_0^2}{6\alpha_0 (1 - \alpha_0)(1 - \alpha_0 + \beta_1)}, \quad (71)
\]

where \(\Pi_i (i = 0, 1, 2, 3)\) are the coefficients in the original KdVB equation\(^{12}\); the difference of signs \([i.e., \Lambda_i = -\Pi_i (i = 0, 1, 3)]\) is caused by the differences of definitions of phase functions in Eqs. (48) and (66). Since the explicit representation of the coefficient of nonlinear term \(\Lambda_1(= -\Pi_1)\) is so complex and has already been shown in the previous paper\(^{12}\), we do not show it in this paper.
Equation (65) may be called a generalized KZK equation (i.e., KZK equation\(^{(17),(18)}\) with dispersion and nonuniform effects) or a generalized KP equation (i.e., KP equation\(^{(19)}\) with dissipation and nonuniform effects). The right-hand side of Eq. (65) signified by the Laplacian with respect to the normalized radial coordinate \(\zeta\) describes the weak diffraction effect. The weak nonuniformity of number density of bubbles \(\delta\) appears in the retarded time \(\eta\) in Eq. (66), and hence \(\delta\) provides a small correction of the phase velocity in the same way as the advection coefficient \(\Lambda_0\). The second, third, and fourth terms in the left-hand side of Eq. (65) stand for the weak nonlinearity, weak dissipation, and weak dispersion, respectively, as in the case of the original KdVB equation. The generalized KZK equation (65) governs the nonlinear propagation of diffracted sound beam with the dissipation and dispersion in nonuniform bubbly liquids in the field characterized by \(X\), \(\eta\), and \(\zeta\), where the temporal, spatial (axial), and spatial (radial) scales are of \(O(1/\epsilon)\), \(O(1/\epsilon)\), and \(O\left(1/\sqrt{\epsilon}\right)\), respectively.

For plane waves, the diffraction term does not appear (i.e., the dependence of \(\zeta\) in \(f\) vanishes), then the original KdVB equation for plane wave propagation with the nonuniformity is obtained. On the other hand, neglecting the nonuniformity reduces Eq. (65) to the KZK equation with the dispersion effect (or KP equation with the dissipation effect). Accordingly, the double limit of \(\Gamma \to 0\) and \(\delta \to 0\) retrieves the original KdVB equation in uniform bubbly liquids.

4. Conclusions

We have derived the generalized KZK equation with dispersion and nonuniform effects (or generalized KP equation with dissipation and nonuniform effects) from the basic equations of two-fluid model of bubbly flows. This nonlinear wave equation governs the long range propagation of ultrasound beam in nonuniform bubbly liquids for the case of the long wavelength with the low frequency. The derivation procedure is based on the two types of extensions to the original method\(^{(12)}\), i.e., adding the scaling relation of the typical diameter of beam to the original scaling relations, and assuming that the initial nonuniformity of number density of bubbles and its spatial variation are as small as the wave amplitude. The former introduces the weak diffraction of beam and the latter does the weak nonuniformity of bubble distribution.

The generalized KZK equation derived here accounts for the nonuniform effect of bubble distribution and is based on the two-fluid model basic equations, and has not been derived so far in the studies of bubbly flows. In a forthcoming paper, a numerical computation of the generalized KZK equation and the corresponding extension for the NLS equation\(^{(12)}\) will be carried out.

Acknowledgements

This work was carried out by the aid of Research on Advanced Medical Technology, Ministry of Health, Labor and Welfare (H19-nano-010). The first author was financially supported from the Japan Society for the Promotion of Science, Research Fellowship for Young Scientists. The authors would like to express their deepest gratitude toward these grants.
Appendix. Inhomogeneous terms in §3.6

We show the inhomogeneous terms $J_i$ ($1 \leq i \leq 5$) in Eqs. (56)–(60). The diffraction parameter $\Gamma$ is included in $J_1$ and $J_2$, and the nonuniformity $\delta$ in $J_1, J_2, J_3, J_4$:

\begin{align*}
J_1 &= -\frac{\Gamma}{r_{1/2}} \frac{\partial r_{1/2} u_G}{\partial t} + \delta \left( \frac{3}{2} \frac{\partial R_1}{\partial t} - \frac{\partial u_G}{\partial x_0} \right) + K_1, \\
J_2 &= (1 - \alpha_0) \frac{\Gamma}{r_{1/2}} \frac{\partial r_{1/2} u_L}{\partial t} - \delta \alpha_0 \frac{\partial u_L}{\partial x_0} + K_2, \\
J_3 &= \delta \left[ 3 \gamma p_G \frac{\partial R_1}{\partial x_0} - \beta \frac{\partial u_G}{\partial x_0} (u_G - u_L) \right] + K_3, \\
J_4 &= \delta \left[ \alpha_0 \frac{\partial u_L}{\partial x_0} + \beta \alpha_0 \frac{\partial (u_G - u_L)}{\partial x_0} + \alpha_0 \frac{\partial p_L}{\partial x_0} \right] + K_4, \\
J_5 &= K_5,
\end{align*}

where $K_i$ ($1 \leq i \leq 5$) are the corresponding inhomogeneous terms in the derivation of the original KdVB equation (12).

References


