The Structure of Three-dimensional Hydromagnetic Waves
in a Uniform Cold Plasma

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(Read Nov. 6, 1963; Received March 30, 1964)

Abstract

Characteristics of the three-dimensional hydromagnetic waves caused by the local source in a uniform cold plasma are studied. There are two modes of hydromagnetic waves. The one is an isotropic mode which spreads in all directions from the source and transmits the disturbance of magnetic field component parallel to field lines. The other is a mixed transverse mode consisting of two parts, the pure and the converted transverse modes, and communicates the signals of normal components of fields. It is emphasized that the former part is a one-dimensional wave propagating parallel to lines of force through the source region, while the latter is the converted one-dimensional wave from the three-dimensional isotropic mode with the geometrical attenuation. The space-time relations of disturbances belonging to the converted mode are obtained. The structures of hydromagnetic waves generated by the particular external forces, the partial compression and the azimuthal drag, are also presented. In a case of reflection of the transverse wave on the anisotropic conducting sheet, the quantitative relation between the incident electric field and the induced currents in the sheet is shown. Furthermore, space and time variations of hydromagnetic oscillations excited by the azimuthal stress and confined within the region between two parallel infinitely conducting solid are illustrated. With relation to the characters of three-dimensional hydromagnetic waves, qualitative discussion on some geomagnetic disturbances are given.

1. Introduction

From the analysis of the characteristic equations, it is wellknown that there are two modes of hydromagnetic waves (HM-waves) in a cold plasma; the isotropic (the fast) and the transverse (the Alfvén) modes (Denisse and Delcroix 1963). With relation to geomagnetic pulsations, HM-oscillations in the earth's magnetosphere have been investigated for the axisymmetric case (Dungey 1954, Akasofu 1956, Kato and Watanabe 1956 and Obayashi and Jacobs 1958). Corresponding to existence of two modes of HM-waves in a cold plasma, the poloidal and torsional oscillations occur independently in this two-dimensional case. On the other hand, based on the ray characteristics, investigations on propagation of HM-disturbances in the model magnetosphere have been made by several workers (Dessler et al. 1960, Bazer and Hurley 1963, Namikawa et al. 1963 a, b, Kitamura 1963 and Stegelmann (89)
and Kenschitzki 1964), and trajectories of ray path, the travel time and the rise time of disturbances were obtained.

In the three-dimensional propagation of HM-waves, however, separation of disturbances into the isotropic and the transverse modes cannot be possible for the components of original electric, magnetic and velocity fields (Grad 1959, Lighthill 1960 and MacDonald 1961). It should be therefore noted that a wave front of the isotropic or the transverse mode obtained by the method of ray-theory does not coincides with that of the original field components. After Grad’s style, “a fast wave front will generally leave behind it a transverse residue which then propagates one-dimensionally.” Although in a uniform plasma the state of compression and of rarefaction and the parallel component of vorticity are transmitted independently as the isotropic and the transverse modes, respectively, de-coupling in this sense becomes to be impossible if the space variation of the local Alfvén velocity is appreciable. When the magnitude of Alfvén velocity varies with altitude, there appears a resonance level depending on the frequency of disturbances and the pressure effect becomes to be important within the region of resonance (Tamao 1964a).

Shifting the subject to geomagnetic disturbances, the complicated local time and latitudinal dependence of the observed short period disturbances may be strongly related to the three-dimensional structure of HM-waves in the magnetosphere. It is the purpose of the present paper to give the preliminary bases on the hydromagnetic interpretation of the observed geomagnetic signals. In the following sections the problems concerning the three-dimensional structure of HM-waves caused by the local source, their reflection on the anisotropic conducting sheet and their normal modes of oscillations will be discussed.

2. Fundamental Equations.

Let us take a cylindrical coordinates \((r, \varphi, z)\) with \(z\)-axis along the uniform stationary magnetic field, \(B\). Using the condition of completely frozen of the lines of force into the material and neglecting the effect of thermal pressure, the vector equation for the velocity field, \(v\), is

\[
4\pi \rho \frac{\partial^2 v}{\partial t^2} = [\text{rot rot}(v \times B)] \times B + \frac{\partial F}{\partial t},
\]

where \(F\) is the applied external force and \(\rho\) the density of plasma. Alternatively, we have the following equation for the electric field, \(E\), which is perpendicular to field lines from the condition of \(E = B \times v\),

\[
\frac{\partial E}{\partial t} = B \times j + B \times \mathbf{F},
\]

where \(j\) is the normal component of electric current.

Let us now represent the electric field, \(E\) and the perturbation magnetic field, \(h\), in terms of a vector potential \(A\), then

\[
E = -\frac{\partial A}{\partial t}
\]
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and

\[ h = \text{rot} A \quad (2.4) \]

where \( B \cdot A = 0 \), since \( E \cdot B = 0 \).

Using the Maxwell equation, eq. (2.2) becomes

\[ \text{rot} \perp \text{rot} A + V^{-2}(\partial^2 A / \partial t^2) = 4\pi B^{-2} F \times B, \quad (2.5) \]

where \( \perp \) stands for the normal direction to field lines and \( V = B / \sqrt{4\pi \rho} \). Taking the divergence of eq. (2.5), we have

\[ \left( \frac{\partial^2}{\partial z^2} - \frac{1}{V^2} \frac{\partial^2}{\partial t^2} \right) \text{div} A \perp = -4\pi B^{-2} \text{div}(F \times B). \quad (2.6) \]

On the other hand, operating \( \text{rot}_z \) on the both sides of eq. (2.5), then

\[ \left( p^2 - \frac{1}{V^2} \frac{\partial^2}{\partial t^2} \right) \text{rot}_z A \perp = -4\pi B^{-2} \text{rot}_z(F \times B), \quad (2.7) \]

since

\[ \text{rot}_z \text{rot} \perp \text{rot} A = -p^2 \text{rot}_z A \perp, \]

where \( p^2 \) is the three-dimensional Laplacian.

From eqs. (2.6) and (2.7), it is understood that \( \text{div} A \perp \) (or \( \text{rot}_z v \perp \)) will be communicated one-dimensionally along the field lines through the source, while \( \text{rot}_z A \perp \) (or \( \text{div} v \perp \)) spreads in all directions from the source. We call here the former as the transverse mode and the latter the isotropic mode, respectively. If the space-time relations of \( \text{div} A \perp \) and of \( \text{rot}_z A \perp \) for the applied source can be obtained, the original \( A \perp \) (therefore \( E, h \) and \( v \)) will be determined as the solution of the two-dimensional Poisson's equations with the source of \( \text{div} A \perp \) and of \( \text{rot}_z A \perp \) (see Appendix B).

3. The Isotropic Waves, the Pure and the Converted Transverse Waves.

We shall use the notation \( \overline{A} \) to representing the Laplace transform of a function \( A(t) \), then

\[ \overline{A}(p) = \int_0^{\infty} A(t)e^{-pt}dt. \]

From eq. (2.7), the Laplace transform of the parallel component of magnetic field, \( \overline{h}_z \), is satisfied by

\[ [p^2 - (p/V)^2] \overline{h}_z = 4\pi B^{-1} \text{div}\overline{F} \perp. \quad (3.1) \]

On the other hand, using the relation of \( \overline{h}_z = \text{rot}_z \overline{A} \) we have the following equation for the original vector potential \( \overline{A} \perp \)

\[ \left[ \frac{\partial^2}{\partial z^2} - (p/V)^2 \right] \overline{A} \perp = -4\pi B^{-2}(\overline{F} \times \overline{B}) + \text{rot}_z \overline{h}_z \quad (3.2) \]

from eq. (2.5).

From the just above equation, it can be seen that \( A \perp \) holds two characters concurrently, that is, the one-dimensional wave along the field lines and the spherical wave as a
result of the source function $\text{rot}_{\perp} h$. Signals resulting from the first term of the righthand side of eq. (3.2) is the one-dimensional wave propagating only along the lines of force which pass through the source region and we call it as the pure transverse wave (PT-mode). One-dimensional disturbances caused by the second term, $\text{rot}_{\perp} \vec{h}$, are the converted one from the three-dimensional isotropic wave and hereafter we refer them as the converted transverse waves (CT-mode).

For simplicity, let us suppose that the external force $\vec{F}$ is acting only on the plane of $z=0$ and its time behaviour is the step function. Then, the solution at a point $P(r, \varphi, z)$ of eq. (3.1) is

$$\vec{h}(r, \varphi, z, \rho) = \frac{1}{pB} \int e^{-\left(\frac{\rho}{r}\right)^{2}} \text{div} \vec{F}_{\perp} dS,$$  

(3.3)

where $R = \sqrt{\rho^{2} + z^{2}}$, $\rho = \sqrt{r^{2} + \rho^{2} - 2\rho r \cos(\varphi - \theta)}$, $dS = \rho d\rho d\theta$ and $r$ and $\theta$ are the radial and the azimuthal coordinates of the source point, respectively. Performing the inverse transform,

$$h_{z}(r, \varphi, z, t) = -\frac{1}{B} \int \frac{H\left(t - \frac{R}{V}\right)}{R} \text{div} \vec{F}_{\perp}(\rho, \theta) dS,$$  

(3.4)

where $H(t)$ is the Heaviside step function.

Next, we consider the time variations of disturbances belonging to the CT-mode. Substituting the solution of $\vec{h}$ into eq. (3.2) and using eq. (2.3), $r$-component of the electric field becomes

$$E_{r}(r, \varphi, z, t) = -\frac{V}{2B} \frac{1}{r} \frac{\partial}{\partial \varphi} \left[ \text{div} \vec{F}_{\perp} dS \left[ \int_{-\infty}^{z} H\left(t - \frac{z - z_{0} + R_{0}}{V}\right) \frac{dz_{0}}{R_{0}} \right] \right] + \int_{z}^{\infty} H\left(t - \frac{z_{0} + R_{0}}{V}\right) \frac{dz_{0}}{R_{0}} \right]$$  

(3.5)

where $R_{0} = \sqrt{\rho^{2} + z_{0}^{2}}$. From the property of the Heaviside function, we have

$$E_{r}(r, \varphi, z, t) = -\frac{V}{2B} \frac{1}{r} \frac{\partial}{\partial \varphi} \left[ \text{div} \vec{F}_{\perp} dS \left[ \int_{z_{1}}^{z} \frac{dz_{0}}{\sqrt{\rho^{2} + z_{0}^{2}}} + \int_{z}^{z_{2}} \frac{dz_{0}}{\sqrt{\rho^{2} + z_{0}^{2}}} \right] \right]$$  

(3.6)

where

$$z_{1} = -\frac{1}{2} \cdot \frac{(Vl - z)^{2} - \rho^{2}}{Vl - z} 

\leq 0$$  

(3.7)

and

$$z_{2} = \frac{1}{2} \cdot \frac{(Vl + z)^{2} - \rho^{2}}{Vl + z} \geq 0.$$

(3.8)

In order to clear the space-time relation of $E_{r}$ represented by eq. (3.5), let us assume in Fig. 1 that $\text{div} \vec{F}_{\perp}$ is only acting on a point $O$. Signals of the isotropic mode started from the point $O$ will spread in all directions as is shown in this figure. At the instant of time of $t = t_{0} = R/V$, namely when the isotropic wave front reaches to the point $P$, the first signal will be observed at this point. This first signal is, however, suffered the space attenuation
The spherical wave fronts of the isotropic mode from the point source $O$ and the paths of the converted transverse mode arriving the point $P$ for several instants of time. The direction of the uniform stationary magnetic field coincides with the $z$-axis.

Fig. 1. The spherical wave fronts of the isotropic mode from the point source $O$ and the paths of the converted transverse mode arriving the point $P$ for several instants of time. The direction of the uniform stationary magnetic field coincides with the $z$-axis.

of $R^{-1}$ during its travel within a distance of $OP$, since it is communicated by the isotropic mode. After the time of $t=t_0$, there are continuous arrival of signals of the CT-mode at the point of $P$. Their path consists of two parts, the isotropic part and the transverse part. The former is the ray-path of the isotropic mode from $O$ to a crossed point on the line of force through the point $P$, except the line of $OP$. The latter is the path along this field line, from the crossed point to the observational point $P$ and on this path there is no space attenuation. Let us consider the instant of time of $t=V^{-1}(R_1+z_1-z)$. At this instant the signals, which propagate along the paths $OP_1P$ and $OP_2P$, respectively, just arrive simultaneously at the point $P$. The intensity of signal at this time is a summation of all contributions from the signals of CT-mode converted from the isotropic disturbances, of which ray-paths were involved within a triangle of $OP_1P_2$. The signal which contributes to point $P$ with the largest intensity is such that it propagates along the direction perpendicular to the field lines by
a distance \( \bar{\omega} \) in the isotropic mode, then after crossing the field line through the point \( P \), it reaches to the observed point one-dimensionally without any geometrical attenuation.

Making the integration of \( z_0 \) and the differentiation of \( \varphi \) in eq. (3.6), we have

\[
E_r = -\frac{V}{2B} \int d\theta \int d\rho \rho \, \Pi_1(\bar{\omega}, z, t) \, \text{div} F_{\perp} \sin(\varphi - \theta).
\]  
(3.9)

In the same way,

\[
E_\varphi = -\frac{V}{2B} \int d\theta \int d\rho \rho \, \Pi_1(\bar{\omega}, z, t) \, \text{div} F_{\perp} \sin(\varphi - \theta),
\]  
(3.10)

where

\[
\Pi_1(\bar{\omega}, z, t) = \frac{1}{\bar{\omega}^2 + z_1^2} + \frac{1}{\bar{\omega}^2 + z_2^2} + \frac{1}{\bar{\omega}^2} \left[ \frac{z_2}{\sqrt{\bar{\omega}^2 + z_2^2}} - \frac{z_1}{\sqrt{\bar{\omega}^2 + z_1^2}} \right].
\]  
(3.11)

Using eq. (2.4), we have also

\[
h_r = -\frac{1}{2B} \int d\theta \int d\rho \rho \, \Pi_z(\bar{\omega}, z, t) \, \text{div} F_{\perp} \sin(\varphi - \theta),
\]  
(3.12)

and

\[
h_\varphi = -\frac{1}{2B} \int d\theta \int d\rho \rho \, \Pi_z(\bar{\omega}, z, t) \, \text{div} F_{\perp} \sin(\varphi - \theta),
\]  
(3.13)

where

\[
\Pi_z(\bar{\omega}, z, t) = \frac{1}{\bar{\omega}^2 + z_2^2} + \frac{1}{\bar{\omega}^2 + z_1^2} + \frac{1}{\bar{\omega}^2} \left[ \frac{2z}{\sqrt{\bar{\omega}^2 + z_2^2}} - \frac{z_1}{\sqrt{\bar{\omega}^2 + z_1^2}} - \frac{z_3}{\sqrt{\bar{\omega}^2 + z_3^2}} \right].
\]  
(3.14)

On the \( z=0 \) plane, \( h_{\perp}=0 \) for all time since \( \Pi_z=0 \).

Time variations of the functions of \( F=\bar{\omega}^2 \Pi_1 \) and \( G=\bar{\omega}^2 \Pi_z \) with the parameter of \( z/\bar{\omega} \)

![Fig. 2](image-url). The time variations of signals of the CT-mode caused by the point source with the time behaviour of the step function. The function \( F \) represents the variations of the electric and velocity fields. The function \( G \) corresponds to the normal components of the magnetic perturbations.
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are illustrated in Fig. 2. After the sufficient long time, there are following general tendency such as \( F\rightarrow 2; \) \( I_1\rightarrow 2\omega^{-2} \) and \( G\rightarrow 2(z/\omega)/\sqrt{1+(z/\omega)^2}; \) \( II_2\rightarrow 2z/\omega^2 \sqrt{\omega^2+z^2} \). Therefore the intensity of \( E_\perp \) only depends on the horizontal distance \( \omega \), while \( h_\perp \) varies considerably with the value of \( z/\omega \). If \( z\gg \omega \), then \( |E_\perp|=|h_\perp|=2\omega^{-2} \). It should be noted from fig. 2 that the discontinuous increase of electric field begins simultaneously with the arrival of the isotropic mode \( (t_0=R/V) \), but the intensity of the normal components of magnetic field increases continuously from the zero level. As a distinct character of signals of CT-mode, it is pointed out that the rise time of signals is finite even in the case of disturbances emanated from a point source. This character of CT-mode will contribute to increase the rise time of geomagnetic ssc, particularly on the ionospheric currents induced by \( E_\perp \).

Summarizing the three-dimensional HM-waves caused by the local source in a uniform cold plasma:

i) isotropic wave (IP-mode)

This is a spherical wave and communicates the disturbance of the parallel component of magnetic field, \( h_\parallel \) and the state of compression and rarefaction. There is the geometrical attenuation of \( R^{-1} \).

ii) transverse wave

a) pure transverse wave (PT-mode)

Signals of the normal components of fields, \( E_\perp, h_\perp \) and \( v_\perp \) are transmitted one-dimensionally along the field lines which pass through the source region. There is no space attenuation. If the time variation of the applied force is a step function, behind the wave front of this mode there are the stationary disturbances of the normal components of fields. This mode accompanies also the electric current flowing parallel to the field lines.

b) converted transverse mode (CT-mode)

This is the one-dimensional wave converted from the isotropic mode and also transmits the signals of the normal components of fields. Since the predecessor of this mode is the spherical wave, the signals of this mode are suffered the partial attenuation. If we consider the disturbances on a plane perpendicular to the field lines, the signals are confined within the finite domain of which area increases with the progress of time and the intensity of signal decreases outwards in proportion to \( \omega^{-2} \) from the center of the domain, after the sufficiently long time.

c) mixed transverse wave (MT-mode)

In the mode of propagation along the field lines, the intensity of signals of the normal components of fields is superposition of PT- and CT-modes. The domain of existence of disturbances of the normal components at any time on the normal plane to the field lines is just the one for the CT-mode. After long time the distribution of perturbation fields coincides with that of the so-called transverse mode which was discussed in section 2.

4. The Structure of Three-dimensional Hydromagnetic Waves Caused by the Particular Source.

As the examples of HM-disturbances caused by the asymmetric source, we consider two cases of the external force, the radial inward force and the azimuthal drag. Let us as-
sume in the followings that the time variation of the applied force is a step function.

(a) the radial inward force

Taking the external force such as

\[ \mathbf{F} = -\mathbf{I}F(r, \varphi) \delta(z) H(t) \]  

(4.1)

where

\[ F(r, \varphi) = \begin{cases} r^{-1}(r-a), & \text{for } |\varphi| \leq \varphi_0 \\ 0, & \text{for } |\varphi| > \varphi_0 \end{cases} \]  

(4.2)

the parallel component of the perturbation magnetic field due to the isotropic mode is

\[ h_z = \frac{1}{B} \varphi_1 \int_{-\varphi_0}^{\varphi_1} \frac{d\theta}{r^2} \left[ H\left( t - \frac{R_a}{V} \right) + \frac{a - r\cos(\varphi - \theta)}{V} \delta\left( t - \frac{R_a}{V} \right) \right], \]  

(4.3)

where \( R_a = \sqrt{r^2 + a^2 - 2ar \cos(\varphi - \theta)} \). This perturbation has the same sign with that of the stationary magnetic field \( \mathbf{B} \). The electric and magnetic fields of PT-mode are

\[ E_r = -\frac{2\pi V}{B} F(r, \varphi) H\left( t - \frac{z}{V} \right), \]  

and

\[ h_r = \frac{2\pi}{B} F(r, \varphi) H\left( t - \frac{z}{V} \right) \]  

(4.4, 4.5)

for \( z > 0 \),

and the sign of \( h_z \) becomes reverse for \( z < 0 \).

On the other hand, the components of electric field of CT-mode are

\[ \begin{align*}
E_r &= \frac{1}{2V} \varphi_1 \int_{-\varphi_0}^{\varphi_1} \frac{r}{B} \frac{\partial A}{\partial \varphi} d\theta, \\
E_\varphi &= -\frac{1}{2V} \varphi_1 \int_{-\varphi_0}^{\varphi_1} \frac{r}{B} \frac{\partial A}{\partial r} d\theta,
\end{align*} \]  

(4.6)

where

\[ A = \int_{-\varphi_0}^{\varphi_1} \frac{\partial}{\partial \theta} [\rho F(\rho, \theta)] d\rho \int_r^{z_2} \frac{dz_0}{\sqrt{\omega_a^2 + z_0^2}}. \]  

(4.7)

The equation of the electric field lines of CT-mode is \( A(r, \varphi, z, t) = \text{const} \). Carrying the integrations of \( z_0 \) and \( \rho \),

\[ A(r, \varphi, z, t) = \int_{-\varphi_0}^{\varphi_1} a - r\cos(\varphi - \theta) \left[ \frac{z_2}{\sqrt{\omega_a^2 + z_2^2}} - \frac{z_1}{\sqrt{\omega_a^2 + z_1^2}} \right] d\theta, \]  

(4.8)

where \( \omega_a = \sqrt{a^2 + r^2 - 2ar \cos(\varphi - \theta)} \) and \( \omega \) in the expression of \( z_1 \) and \( z_2 \) are replaced by \( \omega_a \).

Since \( z_1 = -z_2 \) and \( |z_1| > \omega_a \) after the sufficiently long time, we have approximately

\[ A(r, \varphi) = \int_{-\varphi_0}^{\varphi_1} \frac{a - r\cos(\varphi - \theta)}{a^2 + r^2 - 2ar \cos(\varphi - \theta)} d\theta. \]  

(4.9)

From the relation of \( \mathbf{E} = \mathbf{B} \times \mathbf{v} \) and the eq. (4.6), the scalar function \( A(r, \varphi) \) is a two-dimensional velocity potential. If we can obtain a complex function \( Z(r, \varphi) \) such that its imaginary part coincides with the function \( A(r, \varphi) \), the real part of \( Z \) gives the stream function. As such a complex function we have
where \( x = r/a \). From this equation, the velocity potential is

\[
Z(r, \varphi) = C + iA = a^{-1} \ln \left\{ \frac{1-2x \cos \varphi \cos \psi + x^2 \cos 2\varphi + i2x \sin \varphi (\cos \varphi - x \cos \varphi \varphi_0)}{(1-x \cos(\varphi + \varphi_0))^2 + x^2 \sin^2(\varphi + \varphi_0)} \right\}, \quad (4.10)
\]

and the stream function is

\[
A = a^{-1} \tan^{-1} \left[ \frac{2x \sin \varphi_0 (\cos \varphi - x \cos \varphi_0)}{1 - 2x \cos \varphi \cos \varphi_0 + x^2 \cos 2\varphi_0} \right]. \quad (4.11)
\]

Representing the electric field in terms of the stream function \( C \), we have \( \mathbf{E} = -(V/2B) \nabla C \) which coincides with the expression (B.7) derived from the potential \( \Phi \) in Appendix B.

The electric field lines (dotted line) and the stream lines (full line) for the case of \( \varphi_0 = \pi/4 \) which were calculated from eqs. (4.11) and (4.12) are shown in Fig. 3. Viewing from the upper side of \( z > 0 \), the stream lines consist of two vortexes of which rotational sense are clockwise for \( \varphi > 0 \) and anti-clockwise for \( \varphi < 0 \), respectively. The pattern of the electric field lines is equivalent with that derived from the concentration of positive charges at \( \varphi = \pi/4, \ r = a \) and of negative charges at \( \varphi = -\pi/4, \ r = a \).

Fig. 3. The stationary stream lines (full line) and the electric field lines (dotted line) of the CT-mode on the plane perpendicular to the field lines, caused by the partial compression. The thick full line represents the source region.
(b) the azimuthal force

As the external force in eq. (A.7), we assume

\[ P_\perp = I_\perp Q(r, \phi) \delta(z) H(t) \]  

(4.13)

where \( Q(r, \phi) = \sin \phi [H(r-a)-H(r-b)] \) and \( b > a \).

The solution of eq. (A.7) becomes

\[ U(r, \phi, z, t) = \frac{1}{4\pi} \int_0^{2\pi} \cos \theta d\theta \int_a^b \frac{\delta \left( \frac{t-R}{V} \right)}{R} d\rho, \]

(4.14)

where \( R = \sqrt{r^2 + \rho^2 - 2r\rho \cos (\phi-\theta)} \). Since \( \partial h_z/\partial t = -U \), we have

\[ h_z = -\frac{1}{4\pi} \int_0^{2\pi} \cos \theta d\theta \int_a^b \frac{H \left( \frac{t-R}{V} \right)}{R} d\rho \]

(4.15)

and therefore \( h_z > 0 \) for \( \pi/2 < \phi < (3\pi/2) \) and \( h_z < 0 \) for \( 0 < \phi < \pi/2 \) and \( (3\pi/2) < \phi < 2\pi \).

The solution of eq. (A.8) which results from the first term of the right-hand side (PT-mode) is

\[ \left\{ \begin{array}{l}
C_r = 0 \\
C_\phi = \frac{V}{2} \sin \phi [H(r-a)-H(r-b)] H \left( t - \frac{z}{V} \right).
\end{array} \right. \]

(4.16)

On the other hand, the solutions of CT-mode resulting from the second term of eq. (A.8) are

\[ C_r = -\frac{V}{8\pi} \int_0^{2\pi} \cos \theta d\theta \int_a^b \frac{r-\rho \cos (\phi-\theta)}{\rho^2} F(\omega/z, t) d\rho \]

(4.17)

and

\[ C_\phi = -\frac{V}{8\pi} \int_0^{2\pi} \cos \theta d\theta \int_a^b \frac{\rho \sin (\phi-\theta)}{\rho^2} F(\omega/z, t) d\rho \]

(4.18)

where \( F(\omega/z, t) = \omega^2 II_1(\omega, z, t) \) as was defined in section 3.

Since \( F(\omega/z, t) \) tends to 2 after long time, we have the following stationary expressions as a sum of PT- and CT-modes.

\[ \begin{array}{ll}
(C_r)_{\text{pt}} & = \frac{(V/4) \ln(b/a)}{\cos \phi} \quad r < a \\
(C_r)_{\text{ct}} & = \frac{(V/4) \sqrt{r^2-a^2}}{2r^2} \cos \phi \quad a \leq r \leq b \\
(C_r)_{\text{ct}} & = -\frac{(V/4) b^2-a^2}{2r^2} \cos \phi \quad r > b
\end{array} \]

(4.19)

and

\[ \begin{array}{ll}
(C_\phi)_{\text{pt}} & = \frac{(V/4) \ln(b/a)}{\sin \phi} \quad r < a \\
(C_\phi)_{\text{ct}} & = \frac{(V/4) \sqrt{r^2-a^2}}{2r^2} \sin \phi \quad a \leq r \leq b \\
(C_\phi)_{\text{ct}} & = -\frac{(V/4) b^2-a^2}{2r^2} \sin \phi \quad r > b.
\end{array} \]

(4.20)

The upper part of Fig. 4 illustrates schematically the direction of the applied drag and of the generated electric field. The pattern of the electric field is equivalent to the polarization field resulting from the charge accumulation such that on the inner boundary of the source region there are positive charges for \( \pi/2 < \phi < 2\pi \) and negative charges for \( 0 < \phi < \pi \).
and the situation is just reverse on the outer boundary. In the lower part of Fig. 4, the inner part of the generated vortex motion is shown schematically. Fig. 5 shows the relative radial variations of the components of fields, where the word of "transverse" means the PT-mode, "isotropic" the CT-mode and "transverse plus isotropic" the MT-mode, respectively.

Next, we shall discuss shortly the hydromagnetic oscillations within a region which extends infinitely in the lateral direction but limited in direction along the field lines by the two parallel horizontal plane \((z = \pm l)\). For simplicity, let us assume that there are the infinite conducting solid for \(|z| \geq l\). Let us further suppose that the external force such as given by eq. (4.13) is suddenly applied. Oscillations excited from this situation are separated into three parts. Two one-dimensional oscillations along the field lines which correspond to the PT- and CT-modes, respectively, and the ordinary three-dimensional normal modes of the
isotropic wave. Analytic derivation was given in Appendix C. A summary of the result is given in Table 1 wherein the functional dependence of the time variations and the space variations on the horizontal plane are shown for the perturbations of magnetic field. This result will be referred in section 6, with relation to the variation of rotational sense of the geomagnetic disturbance vectors.

5. Reflection of the Transverse Hydromagnetic Waves on the Anisotropic Conducting Sheet.

Hydromagnetic disturbances generated in the outermost magnetosphere will propagate earthwards and reach the ionospheric level. During the transmission through the ionosphere they will be suffered considerable modification. Since the wave length of disturbances will be sufficiently larger than the thickness of the ionosphere, we can approximate the ionosphere as an infinitely thin anisotropic conductor. Signals penetrated through the ionosphere
will be observed at the earth’s surface as the electromagnetic waves. However, their wave length is longer than the earth’s radius. Then, it may be possible to interprete that the observed geomagnetic disturbances are the magnetic fields due to the induced ionospheric currents at the arrival of HM-disturbances.

In this section we shall consider the above-mentioned circumstance for the case of HM-waves caused by a local source. Let us suppose now the following simple model. The half space of \( z<0 \) is the HM-region and the other half space of \( z>0 \) the electromagnetic (EM-region) region. The boundary plane of \( z=0 \) is an anisotropic electric conductor and the direction of the stationary magnetic field coincides with \( z \)-axis. As will be shown in Appendix B, the intensity of signals of the isotropic mode is negligible in comparison with that of the transverse mode at the level which is sufficiently distant from the source along the field lines. If we suppose that the incident disturbances, with infinite extension in the horizontal direction, reach the plane of \( z=0 \), we can use the transverse mode derived from the scalar function \( \Phi \) which is obtained in Appendix B, as the incident MT-mode. Let us assume further that the source of this MT-mode is the same as the inward force in section 4 and lies on the plane of \( z=-l \).

We have now the following solutions for the respective disturbances in the Laplace transformed form.

I) HM-region

i) incident transverse wave

\[
\mathbf{E} = -\frac{V}{2aB} \cdot \frac{e^{-k(z+l)}}{p} \mathbf{\hat{r}} \mathbf{\nabla} \chi 
\]  

(5.1)

and

\[
\mathbf{h} = \frac{1}{2aB} \cdot \frac{e^{-k(z+l)}}{p} \mathbf{\hat{r}} \mathbf{\hat{z}} \mathbf{\nabla} \chi 
\]  

(5.2)

where

\[
\chi(r,\phi) = \ln \left[ \frac{a^2 + r^2 - 2a \cos(\phi - \phi_0)}{a^2 + r^2 - 2a \cos(\phi + \phi_0)} \right]  
\]  

(5.3)

and \( k = \frac{p}{V} \).

ii) reflected transverse wave

\[
\mathbf{E} = -e^{k\phi} \mathbf{\hat{r}} \mathbf{\nabla} \Phi', \quad \mathbf{h} = -(k/p)e^{k\phi} \mathbf{\hat{r}} \mathbf{\hat{z}} \mathbf{\nabla} \Phi' 
\]  

(5.4)

where the scalar function \( \Phi'(r,\phi, \phi) \) is determined from the boundary conditions and \( \mathbf{\hat{z}} \) the unit vector in the \( z \)-direction.

iii) reflected isotropic wave

\[
\mathbf{E} = -e^{k'\phi} \mathbf{\hat{r}} \mathbf{\nabla} \Psi, \quad \mathbf{h} = p^{-1} \mathbf{\hat{r}} \mathbf{\hat{z}} \mathbf{\nabla} \Psi 
\]  

(5.5)

The scalar function \( \Psi(r,\phi, \phi, \phi') \) is a solution of

\[
\mathbf{\hat{r}} \mathbf{\nabla} \mathbf{\hat{r}} \Psi + \lambda^2 \Psi = 0 
\]  

(5.6)

where \( k' = \sqrt{\lambda^2 + k^2} \) and \( \lambda \) is the positive number.

II) EM-region
The differential equation satisfied by the electric field, \( \mathbf{E} \), is
\[
\nabla \times \nabla \times \mathbf{E} - \kappa^2 \mathbf{E} = 0, \tag{5.7}
\]
where \( \kappa = (p/c) \) and \( c \) is the velocity of light.

Defining the vector potential \( \mathbf{A} \) such as \( \mathbf{E} = -\partial \mathbf{A} / \partial t \), \( \mathbf{h} = \nabla \times \mathbf{A} \) and \( \text{div} \mathbf{A} = 0 \), the equation of \( \mathbf{A} \) is
\[
\nabla \times \nabla \times \mathbf{A} - \kappa^2 \mathbf{A} = 0. \tag{5.8}
\]

As the two independent solenoidal solutions, we have
\[
\mathbf{A} = \nabla \times (1, S) + \nabla \times (1, T), \tag{5.9}
\]
where the first term corresponds to the poloidal-type magnetic field and the second the toroidal-type. Scalar functions \( S \) and \( T \) satisfy
\[
(r^2 - \kappa^2)(S, T) = 0. \tag{5.10}
\]

As the solutions of this equations, we assume
\[
\begin{align*}
S(r, \varphi, z) &= p^{-1} e^{-\sqrt{\omega^2 + \kappa^2} z} S'(r, \varphi, \rho) \\
T(r, \varphi, z) &= p^{-1} e^{-\sqrt{\omega^2 + \kappa^2} z} T'(r, \varphi, \rho)
\end{align*} \tag{5.11}
\]
then the differential equations for \( S' \) and \( T' \) are
\[
(r^2 + \kappa^2)(S', T') = 0. \tag{5.12}
\]

For simplicity, neglecting the factor of \( \exp[-\sqrt{\omega^2 + \kappa^2} z] \), we have the following formal solutions for the \( r \), \( \varphi \), and \( z \)-components of fields.

i) toroidal-type
\[
\mathbf{E} = \begin{pmatrix}
\sqrt{\lambda^2 + \kappa^2} \partial T'/\partial r \\
\sqrt{\lambda^2 + \kappa^2} r^{-1} \partial T'/\partial \varphi \\
\lambda^2 T'
\end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix}
-(rp)^{-1} \kappa^2 \partial T'/\partial \varphi \\
(rp)^{-1} \kappa^2 \partial T'/\partial r \\
0
\end{pmatrix} \tag{5.13}
\]

ii) poloidal-type
\[
\mathbf{E} = \begin{pmatrix}
-r^{-1} \partial S'/\partial \varphi \\
\partial S'/\partial r \\
0
\end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix}
-(rp)^{-1} \sqrt{\lambda^2 + \kappa^2} \partial S'/\partial r \\
-(rp)^{-1} \sqrt{\lambda^2 + \kappa^2} \partial S'/\partial \varphi \\
(rp)^{-1} \kappa^2 S'
\end{pmatrix} \tag{5.14}
\]

(III) conducting sheet current

Let us assume that the stationary magnetic field directs to the positive z-axis. The electric current flowing in the anisotropic conductor is
\[
\mathbf{J} = \sigma_1 \mathbf{E} - \sigma_2 \mathbf{E} \times \mathbf{1}, \tag{5.15}
\]
where \( \sigma_1 \) and \( \sigma_2 \) are the height-integrated Pedersen and Hall conductivities, respectively, and it is defined here that \( \sigma_2 \) is positive. Representing the components of \( \mathbf{J} \) with the electric field in the EM-region,
\[
\mathbf{J}_r = \sigma_1 \left[ \sqrt{\lambda^2 + \kappa^2} \frac{\partial T'}{\partial r} - \frac{1}{r} \frac{\partial S'}{\partial \varphi} \right] - \sigma_2 \left[ \sqrt{\lambda^2 + \kappa^2} \frac{1}{r} \frac{\partial T'}{\partial \varphi} + \frac{\partial S'}{\partial r} \right] \tag{5.16}
\]
The boundary conditions on the plane of $z=0$ are

$$
1_z \times (\mathbf{E}_+ - \mathbf{E}_-) = 0
$$

and

$$
1_z \times (\mathbf{H}_+ - \mathbf{H}_-) = 4\pi \mathbf{J},
$$

where the suffixes $+$ and $-$ stand for $z>0$ and $z<0$, respectively.

Since the general solutions of $S'$ and $T'$ are represented by the Fourier-Bessel integral, we must also expand the incident scalar function $\chi(r, \phi)$ into the Fourier-Bessel integral. The Fourier-Bessel integral of $\chi$ is

$$
\chi(r, \phi) = 8 \sum_{n=1}^{\infty} \sin n\phi_0 \sin n\phi \int_0^{\infty} f_n(\alpha r) J_n(\alpha) \lambda^{-1} d\lambda.
$$

In the same way, we have

$$
T'(r, \phi) = 8 \sum_{n=1}^{\infty} \sin n\phi_0 \sin n\phi \int_0^{\infty} f_n'(\alpha r) T_n'(\alpha, \beta) \lambda^{-1} d\lambda
$$

and so on. Then, we have the following relations

$$
\varphi_n = S_n',
$$

$$
(5.20)
$$

$$
\sqrt{\lambda^2 + \kappa^2} T_n' - \Phi_n' = -(V/2aB) f_n(\alpha a),
$$

$$
(5.21)
$$

\[4\pi \sigma_1 p \sqrt{\lambda^2 + \kappa^2} + \alpha_1 k\Phi_n' = -(1/2aB) f_n(\alpha a)
$$

and

$$
4\pi \sigma_2 p \sqrt{\lambda^2 + \kappa^2} T_n' + [4\pi \sigma_1 p + \alpha_2 \sqrt{\lambda^2 + \kappa^2}] S_n' + k\varphi_n' = 0,
$$

$$
(5.22)
$$

$$
(5.23)
$$

from the boundary conditions. Solving this simultaneous equation, we have

$$
\varphi_n = S_n' = -\frac{\alpha_2 \sqrt{\lambda^2 + \kappa^2}}{\alpha_1 + \sqrt{\lambda^2 + \kappa^2} + \sqrt{\lambda^2 + k^2}} T_n',
$$

$$
(5.24)
$$

$$
(5.25)
$$

\[T_n' = -\frac{J_n(\alpha a)}{\alpha B} \frac{1}{2akB + \sqrt{\lambda^2 + \kappa^2} T_n'},
$$

$$
(5.26)
$$

where

$$
g(\lambda, \mu) = (\alpha_1^2 + \alpha_2^2) \sqrt{\lambda^2 + \kappa^2} + \alpha_1 [\lambda^2 + \sqrt{\lambda^2 + \kappa^2} (\lambda^2 + k^2)] k \sqrt{\lambda^2 + \kappa^2}
$$

$$
+ [k^2 + \kappa^2 \sqrt{\lambda^2 + \kappa^2}] [\sqrt{\lambda^2 + \kappa^2} + \sqrt{\lambda^2 + k^2}]
$$

$$
(5.27)
$$

and $\alpha_1 = 4\pi \sigma_1 p$, $\alpha_2 = 4\pi \sigma_2 p$.

From eq. (5.24) we see that if $\sigma_2 = 0$, then $S_n' = \varphi_n = 0$. It is therefore concluded that when the incident wave on the isotropic conducting sheet is the transverse mode, the reflected wave is also the transverse and the transmitted disturbance into the EM-region is the toroidal-type, of which magnetic effect is negligible.

As was mentioned in the first paragraph of this section, the displacement current is
negligible for the ordinary geomagnetic disturbances and it can be considered that the velocity of light is effectively infinite in the EM-region with the sufficient approximation. With this approximation we have

$$T_n' = -\frac{J_n(\lambda a)}{abk} \cdot \frac{k + \beta_1^{-1}(\lambda + \sqrt{\lambda^2 + k^2})}{\beta_1k + [\lambda + k + \sqrt{\lambda^2 + k^2}] + \beta_1^{-1}(\lambda + \sqrt{\lambda^2 + k^2})}$$

(5.28)

where $\beta_1 = 4\pi\sigma_1 V$, $\beta_3 = 4\pi\sigma_3 V$ and $\sigma_3 = \sigma_1 + (\sigma_3^2/\sigma_1)$.

Performing the inverse transform, then

$$T_n'(\lambda, t) = -\frac{J_n(\lambda a)}{abk} \cdot \frac{V}{2\pi i} \int_{-\infty}^{+\infty} \beta_1 k \cdot [\lambda + k + \sqrt{\lambda^2 + k^2}] + \beta_1^{-1}(\lambda + \sqrt{\lambda^2 + k^2}) \cdot \frac{e^{\imath \omega t}}{k} \, dk.$$  

(5.29)

In the followings, we consider the stationary solutions. The stationary part of $T_n'$ is a contribution from the pole of $k=0$ in the integral of eq. (5.29) and we have

$$T_n'(\lambda) = -\frac{J_n(\lambda a)}{abk} \cdot \frac{V}{1 + \beta_1} \cdot$$

(5.30)

Since there is no stationary part of $S_n'$ from eq. (5.24), the steady Hall current becomes

$$J_{Hr} = -\sigma_1 \frac{1}{r} \frac{\partial J}{\partial \varphi} \quad \text{and} \quad J_{Hv} = \sigma_1 \frac{\partial J}{\partial r}$$

(5.31)

where

$$J = 8 \sum_{n=1}^{\infty} \sin n\varphi_0 \sin n\varphi \int_0^\lambda T_n'(\lambda) J_n(\lambda r) \lambda^{-1} \, d\lambda = -\frac{V \chi(r, \varphi)}{aB(1 + 4\pi\sigma_1 V)}$$

(5.32)

The stationary part of the inverse transform of $\lambda p^{-1}S'$ does not vanish. From eqs. (5.14), (5.24) and (5.26) it is seen that the stationary poloidal-type magnetic field will be observed in the EM-region of which main source is the induced Hall current given by eq. (5.31), if $\sigma_3$ is larger than $\sigma_1$.

On the other hand, since the electric field of the incident MT-mode is $E_0 = -\frac{V}{2aB} \frac{\partial \chi}{\partial \varphi}$ from eq. (5.1), expressing the Hall current in terms of $E_0$, we have

$$J_H = \frac{2\sigma_1}{1 + 4\pi\sigma_1 V} E_0 \times \hat{z}$$

(5.33)

from eqs. (5.31) and (5.32). It can be seen from this equation that if $4\pi\sigma_1 V \ll 1$, the effective intensity of electric field contributing to the Hall current is nearly twice times of the incident field, while if $4\pi\sigma_1 V \gg 1$ the contributing electric field becomes to be reduced considerably. The stationary part of the scalar function $\varphi'$ which determines the reflected transverse mode is

$$\varphi' = \frac{1 - 4\pi\sigma_1 V}{2(1 + 4\pi\sigma_1 V)} \cdot \frac{V}{aB} \chi.$$  

(5.34)

Then, the stationary electric field of the reflected transverse mode, $E_{ref}$, at the level being just above the conducting sheet becomes nearly

$$E_{ref} \approx E_0 \quad \text{if} \quad 4\pi\sigma_1 V \ll 1$$

and
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\[ E_{\text{ref}} = -E_0, \quad \text{if} \quad 4\pi \sigma_i V \gg 1. \]

These two limiting values correspond to the cases of augmentation and of depression of the resultant electric field as was mentioned just above. It is noted here that there is no stationary electric field of the reflected isotropic mode. From eq. (5.33) the equation of the stationary Hall current becomes \( \chi(r, \varphi) = \text{const.} \) and the current lines coincide with the stream lines in the HM-region.

6. Comment on Geomagnetic Disturbances.

In this section we shall discuss qualitatively some geomagnetic disturbance phenomena with relation to the results obtained in the above sections.

It may be possibly important in the three-dimensional propagation of HM-waves in the magnetosphere that there always occurs the conversion from the isotropic waves to the transverse disturbances, and the latitudinal dependence of the amplitude of the observed geomagnetic disturbances may be strongly related with the space distribution of signals of the mixed transverse mode. The space-time relations of the wave front of HM-waves hitherto obtained by several workers, with the method of ray characteristics, do not those for the original field of \( E \) or \( h \), but for the \( \text{div} v \) (isotropic wave) or for the \( \text{div} E \) (transverse wave). From this point of view it is questionable to apply the result of the separated wave front obtained by ray theory to the interpretation of the observed geomagnetic disturbances, as was made by Kitamura (1963).

The results obtained in the above sections are those for the uniform plasma and of course, their direct application to the non-uniform magnetosphere is limited. However, as the second approximation from the wave theory it may be possible to consider that the isotropic wave emanated from the local source transmits independently through the non-uniform magnetosphere, and the converted transverse waves from this independent isotropic mode propagate one-dimensionally along the curved field lines and reach the ionospheric level. In Fig. 6 the illustration of this situation is shown for the point source \( Q \) on the equatorial plane. If we do not consider the effects of partial reflection and refraction of the isotropic wave due to the non-uniformity, the first signal observed at the point \( P \) is the one which propagates along the dotted line \( QP_1P \) and its mode is the isotropic one during its entire path. Its amplitude will be small, because of large geometrical attenuation during its passage. Signals of the CT-mode arrive continuously at the point \( P \), in a time sequence such as \( QP_4P, QP_3P \) and so on. The signal with the largest intensity at point \( P \) is that transmitted from \( Q \) to \( P_0 \) on the equatorial plane in the form of an isotropic mode, and then propagates one-dimensionally along the curved path of \( P_0P \) without any space attenuation. Signals of the CT-mode originated from the isotropic wave, whose ray-path crosses the field line through a point \( P \) in the southern hemisphere, produce the disturbance at \( P \) with decreasing intensity. Then, the time variation of the disturbances observed at \( P \) will have a form given in the upper-left side of Fig. 6. The above-mentioned interpretation may give a plausible explanation for the observed character of geomagnetic disturbances that the amplitude of disturbances generally increases with the geomagnetic latitude, since signals
Fig. 6. The schematic illustration on the propagation of the three-dimensional hydromagnetic waves from the point source Q, in the space with the curved field lines. The curved full lines are the field lines. Circles around the point Q represent the wave front of the isotropic mode. Dotted lines stand for the paths of the isotropic mode. The conversion from the isotropic mode to the one-dimensional wave along the field lines occurs at the points of $P_4$, $P_3$, $P_2$ and so on, successively. In the upper-left, the time variation of signals at the point $P$, is shown schematically, when the time behaviour of the source is represented by the step function.

of the CT-mode reaching the higher latitude suffer less geometrical attenuation before their conversion from the isotropic mode. Of course, in addition to these direct geomagnetic perturbations due to the CT-mode, we should consider also the effect of ionospheric currents which induced by the electric field of CT-mode.

If the primary origin of the geomagnetic disturbances are hydromagnetic perturbations generated by the interaction between the solar stream and the geomagnetic field, the geographical and the local time distribution of observed disturbances may be strongly associated with the difference of acting stress and the structure of generated waves. Let us now suppose that the two external forces, mentioned in section 4, are corresponding to the partial compression on the daylight side and to the azimuthal drag due to the solar stream, respectively. The stream motions caused by the partial compression are similar to those given in Fig. 3. Their rotational sense is counter-clockwise in the forenoon and clockwise in the afternoon side, if we interpret the left part of the figure as the forenoon and the right part the afternoon. The induced ionospheric Hall currents due to the electric field accompanying with these vortex motion give a possible explanation of the equivalent current system of the preliminary reverse impulse of ssc (Tamao 1964b). The current lines and the magnetic field lines of the mixed transverse wave on the plane perpendicular to the field lines which caused by the azimuthal drag are shown schematically in the upper part of Fig. 7, viewing
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Fig. 7. The upper figure: The schematic pattern of the stationary current lines and corresponding magnetic field lines of the mixed transverse mode on the plane normal to lines of force, caused by the azimuthal drag, looking to the equatorial plane.

The lower figure: The schematic picture of the ionospheric Hall currents induced by the incidence of the mixed transverse waves generated by the azimuthal drag, looking from above the north pole.

to the equator in the northern hemisphere. The ionospheric Hall currents induced by the incidence of the associated electric field are also shown in the lower part of the same figure, looking from the above. The pattern of these currents is similar to the equivalent current system of the DS-part of ssc, as was discussed by several workers for the polar part of some geomagnetic disturbances (Axford and Hines 1961, Piddington 1962a, b and Nagata and Kokubun 1962).

Analysing the magnetograms at the geomagnetic ssc (including the accompanying pulsations), Wilson and Sugiura (1961) showed that the sense of rotation of the horizontal vectors associated with ssc is anti-clockwise in the morning and clockwise in the evening, respectively, for the high latitude observatory in the northern hemisphere. As a possible mechanism for this difference of the sense of polarization, they suggested that due to the impact of the solar wind the equatorial portion of the geomagnetic field lines in the outer-
most magnetosphere near the morning and the evening sides are blown to the night side and the resulting motions of the field lines are such as those shown by the dotted lines in the upper part of Fig. 7, looking from above the north pole. When these perturbations of the field lines are transmitted to the earth, the rotation of the magnetic vector are just the same as the observed ones. However, the rotation of a field line on the plane perpendicular to the field lines is not identical with that of the magnetic perturbation vector at a point fixed on the plane. As was shown in the Table 1 (and Appendix C), all the time variations

Table 1  Hydromagnetic oscillations of the magnetic field perturbations within the region, confined with the two lateral infinitely extending planes, and caused by the sudden application of the azimuthal external force on the plane of \( z = 0 \).

<table>
<thead>
<tr>
<th>space variations on the ( z = \text{cost.} ) plane</th>
<th>PT-mode</th>
<th>CT-mode</th>
<th>Isotropic mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>source function ( Q(r, \phi) )</td>
<td>( \omega^2 )</td>
<td>( \omega^2 )</td>
<td>a) ( \omega^{-2} )</td>
</tr>
<tr>
<td>a) const.</td>
<td>a) const.</td>
<td>b) ( \omega^{-1} K_v (\beta_n \omega / V) )</td>
<td></td>
</tr>
<tr>
<td>b) ( \cos \left( \frac{(2m+1)\pi}{2l} Vt \right) )</td>
<td>b) ( \cos \left( \frac{(2m+1)\pi}{2l} Vt \right) )</td>
<td>c) const.</td>
<td></td>
</tr>
<tr>
<td>time variations</td>
<td></td>
<td></td>
<td>a) ( \cos \left[ \frac{(2m+1)\pi}{2l} Vt \right] )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>b) ( \cos \left[ \frac{(2m+1)\pi}{2l} Vt \right] )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>c) ( t^{-1} \sin \left[ \frac{(2m+1)\pi}{2l} Vt \right] )</td>
</tr>
</tbody>
</table>

of the normal components of magnetic field in the HM-oscillations caused by the azimuthal stress, of which directions are westward in the morning and eastward in the evening sides, respectively, are proportional to \( \cos \omega_0 t \), except the transient isotropic normal modes with the decreasing amplitude of \( t^{-1} \). There is no phase difference between \( h_r \) and \( h_\phi \) for these stationary oscillations. In order to explain the observed variation of the polarization, it may be required to take into account the effect of the radial pressure of the solar wind as well as the azimuthal drag simultaneously, or to introduce the effect of the anisotropy of the ionospheric conductivity. Alternatively, as a tentative explanation for the like behaviour of the polarization of pc5 pulsations, Kato and the author (1964) considered the forced HM-oscillations caused by the fluctuation of the pressure of the solar wind, in which the push and the drag occur alternately on the daylight side.

In the interpretation of characteristics of the observed geomagnetic short period disturbances (such as ssc, si and pulsations) it must be important to clarify the mechanism of the problems such as the generation of HM-waves due to the interaction between the solar wind and the magnetosphere, the propagation character of HM-waves through the magnetosphere and the transmission into the ionosphere. Particularly, the geographical distribution of the geomagnetic disturbances may be related strongly with the three-dimensional structure of HM-waves in the magnetosphere. In the discussion of the ionospheric currents corresponding to the geomagnetic perturbations, it will be very important to determine the space distribution of the electric field of incident waves.
Acknowledgement

The author wishes to express his thanks to Prof. Y. Kato for his encouragement in the course of this study and to Dr. T. Namikawa, Osaka City University, for his valuable discussion.

References

Kato, Y. and Tamao, T. (1964) to be published.
Tamao, T. (1964a) to be published.

Appendix A.

Assuming the external force $\mathbf{F}$ is perpendicular to the field lines and defining $\mathbf{C} = B \mathbf{v}$, eq. (2.1) becomes

$$V^{-2} \partial^2 \mathbf{C}_\perp \partial t^2 - \left[ \text{rot} \right. \left. \text{rot} \left( \mathbf{C} \times \mathbf{1}_z \right) \right] \times \mathbf{1}_z = \frac{\partial \mathbf{P}_\perp}{\partial t}$$

(A.1)

where $\mathbf{P}_\perp = 4 \pi B^{-1} \mathbf{F}_\perp$.

Operating $\text{rot}$ in both sides of eq. (A.1), the $z$-component of the resulting equation is

$$\frac{\partial^2 W}{\partial z^2} = \frac{1}{V^2} \frac{\partial^2 W}{\partial t^2} - \left[ \frac{1}{r} \frac{\partial}{\partial \varphi} \left( \frac{1}{V^2} \right) \frac{\partial^2 C_r}{\partial t^2} - \frac{\partial}{\partial r} \left( \frac{1}{V^2} \right) \frac{\partial^2 C_\varphi}{\partial t^2} \right] = - \text{rot}_z \frac{\partial \mathbf{P}_\perp}{\partial t},$$

(A.2)

where

$$W = \text{rot}_z \mathbf{C}_\perp.$$  

(A.3)

Taking the divergence of eq. (A.1),

$$P^2 U - \frac{1}{V^2} \frac{\partial^2 U}{\partial t^2} - \left[ \frac{1}{r} \frac{\partial}{\partial \varphi} \left( \frac{1}{V^2} \right) \frac{\partial^2 C_r}{\partial t^2} + \frac{1}{r} \frac{\partial}{\partial \varphi} \left( \frac{1}{V^2} \right) \frac{\partial^2 C_\varphi}{\partial t^2} \right] = - \text{div} \frac{\partial \mathbf{P}_\perp}{\partial t}$$

(A.4)
where

\[ U = \text{div} C_\perp. \]  \hspace{1cm} (A.5)

If the space variation of the Alfvén wave velocity, \( V \), is sufficiently small, eqs. (A.2) and (A.4) become

\[ \frac{\partial^2 W}{\partial z^2} - \frac{1}{V^2} \frac{\partial^2 W}{\partial t^2} = -\text{rot}_r \frac{\partial P_\perp}{\partial t}, \]  \hspace{1cm} (A.6)

and

\[ r^2 U - \frac{1}{V^2} \frac{\partial^2 U}{\partial t^2} = -\text{div} \frac{\partial P_\perp}{\partial t}, \]  \hspace{1cm} (A.7)

respectively. Using eq. (A.4) and rewriting the \( r \)- and \( \phi \)-components of eq. (A.1), the equation of original field, \( C \), is

\[ \left( \frac{\partial^2}{\partial z^2} - \frac{1}{V^2} \frac{\partial^2}{\partial t^2} \right) C_\perp = \left[ \frac{\partial P_\perp}{\partial t} + \text{grad}_\perp U \right] \]  \hspace{1cm} (A.8)

which corresponds to eq. (3.2).

**Appendix B.**

Now, assuming the formal solution of

\[ A_\perp = \text{grad}_\perp \Phi + \text{rot}_z (1, \Psi), \]  \hspace{1cm} (B.1)

then

\[ \nabla_\perp \Phi = \text{div} A_\perp \quad \text{and} \quad \nabla_\perp \Psi = -\text{rot}_z A_\perp \]  \hspace{1cm} (B.2)

where \( \nabla_\perp \) is the two-dimensional Laplacian on the plane perpendicular to the lines of force. If we have the solutions of the transverse mode (\( \text{div} A_\perp \)) for eq. (2.6) and of the isotropic mode (\( \text{rot}_z A_\perp \)) for eq. (2.7), the scalar functions, \( \Phi \) and \( \Psi \) can be determined from eq. (B.2), as the solutions of two-dimensional Poisson equations. However, it should be noted here that in order to be consistent with the result obtained in section 3, the domain of the non-vanishing of \( \Phi \) and \( \Psi \) on the \( z = \text{const} \) plane must be limited area at the finite time after the external force has been applied. This requirement is the natural consequence of the space-time relation of waves with the finite velocity of propagation.

Equations determining the scalar functions of \( \Phi \) and \( \Psi \) are

\[ \left[ \frac{\partial^2}{\partial z^2} - (p/V)^2 \right] \nabla_\perp \Phi = -\frac{4\pi}{pB} \left[ \frac{1}{r} \frac{\partial}{\partial r} (rF_\rho) - \frac{1}{r} \frac{\partial F_z}{\partial \phi} \right] \]  \hspace{1cm} (B.3)

and

\[ \left[ \nabla^2 - (p/V)^2 \right] \nabla_\perp \Psi = -\frac{4\pi}{pB} \left[ \frac{1}{r} \frac{\partial}{\partial r} (rF_\rho) + \frac{1}{r} \frac{\partial F_z}{\partial \phi} \right]. \]  \hspace{1cm} (B.4)

When the applied external force is the radial inward one such as used in section 4, then after the sufficiently long time the solution of eq. (B.3) is

\[ \Phi = \frac{V}{2aB} e^{-(p/V)^2 \frac{z}{\rho^2}} \chi(r, \phi), \quad z > 0 \]  \hspace{1cm} (B.5)

where
The Structure of Three-dimensional Hydromagnetic Waves

The electric field of this transverse mode is

$$\mathbf{E}_\perp = -\text{grad}_\perp \frac{\partial \Phi}{\partial t} = -\frac{V}{2aB} H\left(t - \frac{z}{V}\right) \mathbf{F}_\perp. \quad (B.7)$$

On the other hand, from eq. (B.4) the solution of the isotropic mode is

$$\Psi = -\frac{1}{4\pi B \rho} \int \ln[r^2 + r'^2 - 2rr' \cos(\varphi - \varphi')] r' \, dr' \, d\varphi'$$

$$\times \int \frac{\partial}{\partial \rho} [\rho F(\rho, \theta)] \frac{e^{-(\rho'/V) R'}}{R'} \, d\rho \, d\theta, \quad (B.8)$$

where

$$R' = \sqrt{\rho^2 + r'^2 - 2\rho r' \cos(\varphi - \theta) + z^2}.$$ 

Since

$$\int_0^\infty \frac{\partial}{\partial \rho} [\rho F] \frac{e^{-(\rho'/V) R'}}{R'} \, d\rho = \frac{a - r \cos(\varphi - \theta)}{R_a} \left[ \frac{1}{R_a} + \frac{\rho}{V} \right] e^{-(\rho/V) R_a},$$

where $R_a = \sqrt{r'^2 + a^2 - 2ar' \cos(\varphi - \theta) + z^2}$, the contribution from the isotropic mode can be neglected in comparison with that of the transverse mode at the level of the sufficiently large $z$.

Appendix C.

Considering the relation of $\tilde{h}_z = -\rho^{-1} U$, $U$ must vanish at $z = \pm l$ from the boundary condition of $\tilde{h}_z = 0$ at $z = \pm l$. The solutions of the equation of

$$\rho^2 \tilde{U} - (\rho/V)^2 \tilde{U} = -\frac{1}{r} \frac{\partial Q}{\partial \varphi} \delta(z), \quad (C.1)$$

which vanish at $z = \pm l$, are

$$\tilde{U} = -\frac{1}{2\pi} \int_0^{2\pi} \int_0^l \frac{\partial Q}{\partial \theta} \, d\rho \sinh[l \sqrt{\lambda^2 + k^2}] \sinh[(z-l) \sqrt{\lambda^2 + k^2}] \frac{J_0(\lambda \tilde{\rho}) \lambda \, d\lambda}{\sqrt{\lambda^2 + k^2} \sinh[2l \sqrt{\lambda^2 + k^2}]}$$

for $z > 0 \quad (C.2)$

and

$$\tilde{U} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^l \frac{\partial Q}{\partial \theta} \, d\rho \sinh[l \sqrt{\lambda^2 + k^2}] \sinh[(z+l) \sqrt{\lambda^2 + k^2}] \frac{J_0(\lambda \tilde{\rho}) \lambda \, d\lambda}{\sqrt{\lambda^2 + k^2} \sinh[2l \sqrt{\lambda^2 + k^2}]}$$

for $z < 0 \quad (C.3)$

where $J_0$ is the Bessel function of the first kind and $k = \rho/V$. From the condition of the completely reflection the electric field $\mathbf{E}_\perp$, vanishes at the boundary surfaces. $\mathbf{C}_\perp$ also vanishes at $z = \pm l$, since $\mathbf{C}_\perp = \mathbf{E} \times \mathbf{1}_r$. The equations for the $r$- and $\varphi$-components of $\mathbf{C}_\perp$ are

$$\frac{\partial^2 C_r}{\partial z^2} - k^2 C_r = -\frac{\partial U}{\partial r} \quad (C.4)$$

and

$$\frac{\partial^2 C_\varphi}{\partial z^2} - k^2 C_\varphi = -Q(r, \varphi) \delta(z) \frac{1}{r} \frac{\partial U}{\partial \varphi}, \quad (C.5)$$
respectively. The solutions of these equations satisfying the boundary conditions are

\[
C_r = \frac{1}{2\pi} \frac{\partial}{\partial r} \int_0^{2\pi} d\theta \int_0^{\infty} \frac{\partial Q}{\partial \theta} d\rho \Pi(\lambda, \omega, p) \tag{C.6}
\]

and

\[
C_r = \frac{V \sinh(l/V) \sinh[(l-z)p/V]}{p \sinh(2lp/V)} Q(r, \varphi) + \frac{1}{2\pi} \frac{1}{r} \frac{\partial}{\partial \varphi} \int_0^{2\pi} d\theta \int_0^{\infty} \frac{\partial Q}{\partial \theta} d\rho \Pi(\lambda, \omega, p), \tag{C.7}
\]

respectively, for \( z > 0 \), where

\[
II(\lambda, \omega, p) = \frac{V \sinh(l/V) \sinh[(l-z)p/V]}{p \sinh(2lp/V)} \int_0^{\infty} J_0(\lambda \bar{\omega}) \lambda^{-1} d\lambda - \int_0^{\infty} \frac{\sinh[l \sqrt{\lambda^2 + (p/V)^2}] \sinh[(l-z) \sqrt{\lambda^2 + (p/V)^2}]}{\sqrt{\lambda^2 + (p/V)^2} \sinh[2l \sqrt{\lambda^2 + (p/V)^2}]} J_0(\lambda \bar{\omega}) \lambda^{-1} d\lambda. \tag{C.8}
\]

Differentiating with respect to \( r \) and \( \varphi \) and using the properties of the Bessel function, we have

\[
C_r = -\frac{1}{4\pi} \int_0^{2\pi} d\theta \int_0^{\infty} \frac{\partial Q}{\partial \theta} d\rho \cdot \frac{r - \rho \cos(\varphi - \theta)}{\bar{\omega}} - I(\bar{p}, \lambda, z) \tag{C.9}
\]

and

\[
C_r = f_1(\bar{p}, z) Q(r, \varphi) - \frac{1}{4\pi} \int_0^{2\pi} d\theta \int_0^{\infty} \frac{\partial Q}{\partial \theta} d\rho \cdot \frac{\rho \sin(\varphi - \theta)}{\bar{\omega}} I(\bar{p}, \lambda, z), \tag{C.10}
\]

where

\[
I(\bar{p}, \lambda, z) = f_1(\bar{p}, z) \int_{-\infty}^{\infty} H_1^{(1)}(\lambda \bar{\omega}) d\lambda - \int_{-\infty}^{\infty} f_2(\lambda, \bar{p}, z) H_1^{(1)}(\lambda \bar{\omega}) d\lambda \tag{C.11}
\]

\[
f_1(\bar{p}, z) = \frac{V \sinh(lp/V) \sinh([(l-z)p/V])}{p \sinh(2lp/V)} \tag{C.12}
\]

and

\[
f_2(\lambda, \bar{p}, z) = \frac{\sinh[l \sqrt{\lambda^2 + (p/V)^2}] \sinh([(l-z) \sqrt{\lambda^2 + (p/V)^2}]}{\sqrt{\lambda^2 + (p/V)^2} \sinh[2l \sqrt{\lambda^2 + (p/V)^2}]} \tag{C.13}
\]

and \( H_1^{(1)} \) is the Hankel function of the first kind of first degree. Replacing the real integrals of \( \lambda \) with the contour integrals on the complex \( \lambda \)-plane, we have

\[
\int_{-\infty}^{\infty} H_1^{(1)}(\lambda \bar{\omega}) d\lambda = \bar{\omega}^{-1}
\]

and

\[
\int_{-\infty}^{\infty} f_2(\lambda, \bar{p}, z) H_1^{(1)}(\lambda \bar{\omega}) d\lambda = -\frac{1}{\Pi l} \sum_{n=0}^{\infty} (-1)^{n+1} \sin \left[ \frac{(2n+1)\pi(l-z)}{2l} \right] K_1(\lambda_n \bar{\omega}) \lambda_n^{-1}
\]

where \( \lambda_n = \sqrt{[(2n+1)\pi/2l]^2 + (p/V)^2} \) and \( K_1 \) is the modified Bessel function of the second kind of first degree. Performing the inverse transform, then

\[
C_r = -\frac{1}{4\pi} \int_0^{2\pi} d\theta \int_0^{\infty} \frac{\partial Q}{\partial \theta} d\rho \cdot \frac{r - \rho \cos(\varphi - \theta)}{\bar{\omega}} \left\{ (V/\bar{\omega}) I_l \right\} 
\]

\[
-(1/\pi l) \sum_{n=0}^{\infty} (-1)^{n+1} \sin \left[ \frac{(2n+1)\pi(l-z)}{2l} \right] I_l \}
\]

and
where

\begin{equation}
I_1 = \frac{1}{2\pi i} \int_{\gamma-\infty}^{+\infty} \frac{\sinh(lp/V)\sinh(l-z)p/V}{p\sinh(2lp/V)} e^{pt} dp
\end{equation}

and

\begin{equation}
I_2 = \frac{1}{2\pi i} \int_{\gamma-\infty}^{+\infty} \frac{K_0(\tilde{\beta}_n\tilde{\omega})}{\tilde{p}\lambda_n} e^{pt} dp
\end{equation}

Using the relation of $\tilde{\mathbf{h}}_\perp = p^{-1}\partial \mathbf{C}/\partial z$, the solution of the magnetic perturbations are

\begin{equation}
h_r = \frac{1}{4\pi} \int_0^{2\pi} d\theta \int_0^{\infty} d\rho \rho \sin(\varphi-\theta) \left[ I_1 \right]
\end{equation}

\text{and}

\begin{equation}
h_\varphi = -I_2 Q(r, \varphi) + \frac{1}{4\pi} \int_0^{2\pi} d\theta \int_0^{\infty} d\rho \rho \sin(\varphi-\theta) \left[ I_3 \right]
\end{equation}

where

\begin{equation}
I_3 = \frac{1}{2\pi i} \int_{\gamma-\infty}^{+\infty} \frac{\sinh(lp/V)\cosh(l-z)p/V}{p\sinh(2lp/V)} e^{pt} dp
\end{equation}

Evaluating the integrals on the $p$-plane,

\begin{equation}
I_1 = 2 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)\pi} \sin\left[ \frac{(2m+1)\pi(l-z)}{2l} \right] \sin\left[ \frac{(2m+1)\pi}{2l} Vt \right]
\end{equation}

\text{and}

\begin{equation}
I_3 = 1 + 2 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)\pi} \cos\left[ \frac{(2m+1)\pi(l-z)}{2l} \right] \cos\left[ \frac{(2m+1)\pi}{2l} Vt \right]
\end{equation}

for the time of $t>z/V$.

On the other hand, the integral of $I_3$ can be replaced by the contour integral consisting the three parts, the one around the pole of $p=0$, the integrals around the branch points of $p=\pm i\beta_n$ and the line integrals along the cuts of the branch points, as is shown in Fig. 8. Making these integrals, we have

\begin{equation}
I_3 = \frac{VK_0(\beta_n\tilde{\omega}/\tilde{\beta}_n)}{\beta_n} - \frac{V\tilde{\omega}}{\pi \beta_n \tilde{\omega}} \cos\beta_n t - \frac{\tilde{\omega}}{2\beta_n} \frac{\sin\beta_n t}{t}
\end{equation}

for the time of $t>\tilde{\omega}/V$, where the first term is the contribution from the pole of $p=0$, the second the branch points of $p=\pm i\beta_n$ and the third the branch cuts. The integrals along the branch cuts were obtained with the approximation that the main contribution to the line integral comes from the parts near the branch points due to the condition of $t-(\tilde{\omega}/V)>0$.

In eq. (C.19) The first term means the one-dimensional oscillations of the PT-mode, the
second the one-dimensional oscillations due to the CT-mode and the third infinite series the normal mode oscillations of the three-dimensional isotropic wave.