Analysis of Polar Magnetic Storms

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Abstract

An approximate method for analyzing broad-scale current systems in the ionosphere is outlined which allows calculation of sources and sinks of current from surface observations. Such sources and sinks of ionospheric current have been suggested in connection with models for geomagnetic disturbance phenomena. Usual analytic methods yield only divergence-free current systems. The treatment of these current systems by the methods outlined yield the desired divergences of the sheet-current in the ionosphere.

In recent years a number of theories have been advanced to account for the morphology of polar magnetic storms. Particular attention has been devoted reproducing the average current systems that are inferred from magnetic perturbations at the surface to be flowing in the ionosphere. (*Martyn*, 1951; *Fejer*, 1961, 1963; *Kern*, 1961, 1962; *Obayashi*, 1961; *Axford and Hines*, 1961; and others).

These current systems are usually constructed by graphical methods, since the available data are poorly distributed at any given universal time. Analytic procedures are available, however, which can be easily and economically adapted to modern computing methods. In particular, harmonic analysis of magnetic perturbations can be used to obtain a scalar magnetic potential that corresponds to a given set of data. If the perturbations are nearly static in local time, the associated scalar magnetic potential can be related to a steady current distribution flowing in a thin spherical shell using standard methods from potential theory.

The earth's ionosphere is thought to be the seat of such currents as are related to the perturbations associated with polar magnetic storms. The electrical conductivity in the ionosphere is anisotropic, and is a relatively complicated function of both latitude and longitude. This complexity is usually ignored in studies of the basic mechanisms causing such disturbances, and we will perpetuate this tradition here. We caution the reader, however that the distribution of ionospheric conductivity can even in some simple cases control the phenomena that we are attempting to analyze.

A number of theories of geomagnetic disturbance phenomena have invoked currents along magnetic field lines resulting from dynamical effects in the magnetosphere. It becomes of interest to see whether sources and sinks of ionospheric currents can be discovered from surface observations. It is clear that the sum of the divergence of the ionospheric sheet current and a vertical current density at the top of the ionosphere must equal zero, if no charge is to be accumulating in the ionosphere. In a steady-state, the divergence of the ionospheric current sheet gives the currents supplied to the ionosphere by any processes in the magnetosphere.

One problem that arises in connection with the deduction of ionospheric currents from a magnetic perturbation at the earth's surface is that the intensity of air inferred current sheet is divergence-free. The inferred intensity $\mathbf{J}$ of the sheet current is given by

$$\mathbf{J} = \nabla \times (J\mathbf{r}) \quad (1)$$

where J is a scalar function of position on the spherical surface occupied by the current and $\mathbf{r}$ is a unit vector in the direction of the outward normal to the surface (*Chapman and Bartels*, 1940). It is normally required that J is strictly two dimensional, so that $\nabla J \mathbf{r} = 0$. We note that (1) can also be written

$$\mathbf{J} = \nabla J \times \mathbf{r} \quad (2)$$

If a portion of the current flowing in the ionosphere were derivable from a scalar potential, such that

$$\mathbf{J} = -\nabla \Phi \quad (3)$$

This portion of the current would not contribute to the magnetic potential external to the current sheet. This results from the vanishing of the line integral of (3) around any closed path in the surface. Hence the use of (1) as a representation of the ionospheric current, neglects the possible presence of sources and sinks of current in the ionosphere, for which $\nabla \cdot \mathbf{J}$ will not vanish.

As the starting point for our treatment, let us take both the electric field and the sheet current density to consist of both curl-free and divergence-free parts. Let

$$\mathbf{E} = -\nabla V + \nabla \times \Psi \mathbf{r} = -\nabla V + \nabla \Psi \times \mathbf{r} \quad (4)$$

$$\mathbf{J} = -\nabla \Phi + \nabla \times J\mathbf{r} = -\nabla \Phi + \nabla J \times \mathbf{r} \quad (5)$$

where V is the electrostatic potential, Ψ is a scalar field related to the time-dependent part of the electric field, Φ is a scalar potential for the curl-free part of the sheet current, and J is the current-function for the divergence-free part of the sheet current. J can be determined for any given perturbation of the magnetic field at the earth's surface using well known methods of potential theory (*Chapman and Bartels*, 1940, page 631). We have in addition the constitutive relation for an ionosphere with uniform height-integrated values of the direct conductivity k_1 and Hall conductivity k_2

$$\mathbf{J} = k_1 \mathbf{E} \pm k_2 \mathbf{E} \times \mathbf{r} \quad (6)$$

where the + sign is taken for the northern hemisphere, and the – for the southern hemisphere. We note at the outset that only horizontal gradients will be considered for the scalars V , Ψ , Φ , and J .

Now we have

$$\nabla \cdot \mathbf{E} = -\nabla^2 V \quad (7)$$

$$\nabla \cdot \mathbf{J} = -\nabla^2 \Phi = k_1 \nabla \cdot \mathbf{E} + k_2 \mathbf{r} \cdot \nabla \times \mathbf{E} \quad (8)$$

$$\text{Also} \quad \nabla \times \mathbf{E} = -(\nabla^2 \Psi) \mathbf{r} + \frac{\nabla \Psi}{r} \quad (9)$$

$$\nabla \times \mathbf{J} = -(\nabla^2 J)\mathbf{r} + \frac{\nabla J}{r} \quad (10)$$

Thus, from (8) using (7) and (9), we have

$$+\nabla^2 \Phi = k_1 \nabla^2 V + k_2 \nabla^2 \Psi \quad (11)$$

From the constitutive relation (6) taking only the + sign,

$$\begin{aligned} \nabla \times \mathbf{J} &= k_1 \nabla \times \mathbf{E} + k_2 \nabla \times (\mathbf{E} \times \mathbf{r}) = k_1 \nabla \times \mathbf{E} - k_2 (\nabla \cdot \mathbf{E})\mathbf{r} + k_2 \frac{\mathbf{E}}{r} \\ \nabla \times \mathbf{J} &= -(k_1 \nabla^2 \Psi - k_2 \nabla^2 V)\mathbf{r} + k_1 \frac{\nabla \Psi}{r} - k_2 \frac{\nabla V}{r} + k_2 \frac{\nabla \Psi \times \mathbf{r}}{r} \end{aligned} \quad (12)$$

Comparing this with (10), we see

$$\nabla^2 J = k_1 \nabla^2 \Psi - k_2 \nabla^2 V \quad (13)$$

and

$$\nabla J = k_1 \nabla \Psi - k_2 \nabla V + k_2 \nabla \Psi \times \mathbf{r} \quad (14)$$

We also have, from (5) and (6)

$$\nabla J = \mathbf{r} \times (\mathbf{J} + \nabla \Phi)$$

and

$$\nabla J = \mathbf{r} \times (k_1 \mathbf{E} + k_2 \mathbf{E} \times \mathbf{r}) + \mathbf{r} \times \nabla \Phi$$

or

$$\nabla J = -k_1 \mathbf{r} \times \nabla V + k_1 \nabla \Psi - k_2 \nabla V + k_2 \nabla \Psi \times \mathbf{r} + \mathbf{r} \times \nabla \Phi \quad (15)$$

But, comparing this with (14), we see that to preserve the identity

$$-k_1 \mathbf{r} \times \nabla V + \mathbf{r} \times \nabla \Phi = 0$$

or in general

$$k_1 \nabla V = \nabla \Phi \quad (16)$$

This result requires from (11) that

$$\nabla^2 \Psi = 0 \quad (17)$$

So that Ψ is a solution of Laplace's equation. We also note that from (13) and (17)

$$\nabla^2 J = -k_2 \nabla^2 V \quad (18)$$

We have then finally that

$$\nabla \cdot \mathbf{J} = k_1 \nabla \cdot \mathbf{E} = \frac{k_1}{k_2} \nabla^2 J \quad (19)$$

This result is independent of the existence of a divergence-free electric field. We have established that for the imposed two-dimensional geometry, the representation of an electric vector potential by $\Psi \mathbf{r}$ is subject to the condition that $\nabla^2 \Psi = 0$. Equations (8) and (19) also establish that the appropriate representation for the sheet current intensity is,

$$\mathbf{J} = \frac{k_1}{k_2} \nabla J + \nabla \times J \mathbf{r} \quad (20)$$

where we have set Φ in (5) equal to $-\frac{k_1}{k_2}J$.

If the electric field is curl-free (independent of time) we note that (18) gives in combination with (26) the sources of the electrostatic field. It has often been remarked that such charge distributions must be maintained by currents (*Kern and Vestine, 1962*). From (19) we see that the direct conductivity k_1 controls the magnitude of the static charge distribution associated with a given divergence of the sheet current.

Note that by introducing an additional scalar potential into the representation of the sheet current intensity, we have resolved the difficulty that $\nabla \cdot (\nabla \times J\mathbf{r}) = 0$, while $\nabla \cdot \mathbf{J} = k_1 \nabla \cdot \mathbf{E}$. The physical interpretation of the scalar potential Φ is that it represents a portion of the current flowing in the ionosphere that cannot contribute to magnetic perturbations at the earth's surface. As a consequence, the total current intensity in the ionosphere may be quite different from that inferred from a current function J constructed from the observed magnetic perturbations at the earth's surface.

The method for determining of $\nabla^2 J$ from an original scalar magnetic potential for a system of magnetic perturbations at the earth's surface is not obvious. It is, however, quite straightforward.

A spherical sheet current of radius a produces a scalar magnetic potential W at a radius $r < a$ given by

$$W(r) = -\frac{4\pi}{10} \sum_{n=0}^{\infty} \frac{n+1}{2n+1} J_n(a, \theta, \phi) \left(\frac{r}{a}\right)^n \quad (21)$$

where the $J_n(a, \theta, \phi)$ are terms in a spherical harmonic expansion of the scalar current function related to the divergence-free part of the surface current density by (1), with units of amperes. Conversely, for a scalar magnetic potential expanded in a surface harmonic representation over the surface of a sphere of radius r , we have $W(r) = \sum W_n(r, \theta, \phi)$. Then, since $J(a) = \sum J_n(a, \theta, \phi)$ we can write

$$J_n(a) = -\frac{10}{4\pi} \frac{2n+1}{n+1} \left(\frac{a}{r}\right)^n W_n(r) \quad (22)$$

for $r < a$ (*Chapman and Bartels, 1940*). This is the method by which $J(a)$ is inferred from the scalar potential for a given magnetic perturbation at the earth's surface.

Now we know that the Laplacian of W must vanish

$$\nabla^2 W(r) = \frac{2}{r} \frac{\partial W}{\partial r} + \frac{\partial^2 W}{\partial r^2} = 0 \quad (23)$$

Accordingly, taking the Laplacian of W , as given by eq. (21) we obtain

$$\nabla^2 W = -\frac{4\pi}{10} \sum_{n=0}^{\infty} \frac{n+1}{2n+1} \left[J_n \nabla^2 \left(\frac{r}{a}\right)^n + 2 \nabla J_n \cdot \nabla \left(\frac{r}{a}\right)^n + \left(\frac{r}{a}\right)^n \nabla^2 J_n \right]$$

Noticing that $\nabla J_n \cdot \mathbf{r} = 0$, and that $\nabla^2 W = 0$, this gives

$$J_n \nabla^2 \left(\frac{r}{a}\right)^n + \left(\frac{r}{a}\right)^n \nabla^2 J_n = 0$$

Hence

$$\nabla^2 J_n = -\frac{n(n+1)}{r^2} J_n \quad (24)$$

and

$$\nabla^2 J(a) = \sum_{n=0}^{\infty} \nabla^2 J_n = -\sum_{n=0}^{\infty} \frac{n(n+1)}{r^2} J_n \quad (25)$$

Using eq. (22), we can write

$$\nabla^2 J(a) = \frac{10}{4\pi} \sum_{n=0}^{\infty} \frac{n(2n+1)}{a^2} \left(\frac{a}{r}\right)^{n+2} W_n(r) \quad (26)$$

So that $\nabla^2 J(a)$ can be obtained directly from a spherical harmonic representation of the scalar potential for a given magnetic perturbation at the earth's surface.

Conversely, we can obtain the scalar magnetic potential for any assumed distribution of sources and sinks of current in the ionosphere.

From (26) $W_n(r) = \frac{4\pi}{10} \frac{a^2}{n(2n+1)} \left(\frac{r}{a}\right)^{n+2} \nabla^2 J_n$

Consequently, from (19)

$$W = \frac{4\pi}{10} \frac{k_2}{k_1} \sum_{n=0}^{\infty} \frac{a^2}{n(2n+1)} \left(\frac{r}{a}\right)^{n+2} (\nabla \cdot \mathbf{J})_n \quad (27)$$

where $(\nabla \cdot \mathbf{J})_n$ is the harmonic of n th order in a spherical harmonic expansion of an assumed distribution of sources and sinks of ionospheric current. Note that the series representation for $\nabla^2 J(a)$ will lead in many cases to a distribution on the sphere that resembles the distribution of J . Eq. (26), however, indicates that the effects of higher harmonics will be amplified in $\nabla^2 J(a)$ compared to $J(a)$. On inversion, the effects on $W(r)$ of higher harmonics in $\nabla^2 J$ will, on the other hand, be reduced.

Applications to satellite observations follow directly. It is clear, however, that analysis of surface observations would help in making effective use of data from satellites. To illustrate this problem, let us assume a series of satellite observations are available which enable us to construct the scalar potential of a steady ionospheric current system. We also assume that the satellite orbits at an altitude such that the currents supplying sources and sinks in the ionosphere $\mathbf{j}$ can be considered as a first approximation to be vertical at the satellite altitude.

Any scalar potential inferred from the satellite observations will be simply a best fit to the observed magnetic perturbations. Since $\nabla \times \mathbf{B} = 4\pi \mathbf{j}$ at satellite altitudes above regions in which the ionospheric current systems diverge, it can be expected that the satellite measurements will not give precisely the scalar potential appropriate to the sheet current. Departures should not be too great however, since appropriate fitting of a harmonic potential insures that $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$. Relatively small errors in position, the time dependent character of the perturbation field, and the time dependent nature of the accumulation of data points over a closed surface would probably frustrate any attempt to determine current densities at the satellite altitude by calculating line integrals (or solving $\nabla \times \mathbf{B} = 4\pi \mathbf{j}$). The analytic

procedures developed here can be applied to the scalar potentials derived from both satellite and surface observations to calculate divergence of the ionospheric sheet current. Vertical current densities follow immediately, and can be applied to calculation of current densities along magnetic field lines.

Conclusions

We have shown that steady currents flowing in a model ionosphere which has uniform height-integrated conductivities can be reconstructed from surface observations of the magnetic field perturbations. The usual potential analysis of the magnetic field perturbation yields a scalar current function which is directly related to the divergence-free portion of the sheet current. We showed that the Laplacian of this scalar current function is directly related to the divergence of the sheet current, and that the resultant current in the sheet is given by (20)

$$\mathbf{J} = \left(\frac{k_1}{k_2} \right) \nabla J + \nabla \times J \mathbf{r}$$

where k_1 and k_2 are respectively the height-integrated direct and Hall conductivities, and J is the scalar current function. The divergence of this sheet current is then

$$\nabla \cdot \mathbf{J} = (k_1/k_2) \nabla^2 J$$

This divergence can be immediately associated with vertical current densities due to current along magnetic field lines. The construction of models for polar ionospheric current systems thus enables us to calculate sources and sinks of currents in the ionosphere and to relate such sources and sinks to dynamical processes in the magnetosphere. A simple inversion of the analytical results permits us to calculate the magnetic perturbations on the surface that would result from a given pattern of sources and sinks of ionospheric current that can be expanded in a spherical harmonic representation. The results may also be of some interest in connection with analysis of satellite measurements of the geomagnetic field.

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