On the Influence of the Hall Current to the Electrical Conductivity of the Ionosphere, II

By Motokazu HIRONO
Geophysical Institute, Kyoto University

Abstract

It is shown that there is a narrow region along line of zero dip near the E layer, of about 15° of latitude wide in which conductivity is very great, as a result of polarization by the Hall current. The effect of this belt is discussed by the dynamo theory, giving the result that this belt will cause enhanced diurnal variation of magnetic field near the geomagnetic equator.

1. Introduction. In the previous paper [1], hereafter referred to as I, we made a preliminary calculation on the influence of the Hall Current (in place of wick we write H.C. in the following) to the electrical conductivity of the ionosphere. The present paper describes a further examination of the problem.

T.G. Cowling [2] showed that, when there are few negative ions in the ionosphere, equ. (5) in I holds for ions and electrons separately. Therfore, mean velocity of charged particles 1 is expressed in the following equation

\[ \overline{C}_1 = \frac{e_1}{m_1} \cdot \frac{\nu_1}{\nu_1^2 + \omega_1^2} E_x - \frac{1}{H} \frac{\omega_1^2}{\nu_1^2 + \omega_1^2} \mathbf{h} \times E + \frac{e_1}{m_1} \frac{E'}{\nu_1} \]  

(1)

where \( E = E^0 + e_0 \times H \), \( h = H/H \)

When \( E \) is perpendicular to \( H \)

\[ \overline{C}_1 = \frac{e_1}{m_1} \cdot \frac{\nu_1}{\nu_1^2 + \omega_1^2} E - \frac{1}{H} \frac{\omega_1^2}{\nu_1^2 + \omega_1^2} \mathbf{h} \times E \]  

(2)

Current intensity \( J \) is expressed by the equation

\[ J = \sum n_j e_j \mathbf{C}_j \quad (j = e, +, -) \]

\[ = \left( \sum n_j \frac{e_j^2}{m_j} \cdot \frac{\nu_j}{\nu_j^2 + \omega_j^2} \right) E - \frac{1}{H} \left( \sum n_j e_j \frac{\omega_j^2}{\nu_j^2 + \omega_j^2} \right) \mathbf{h} \times E \]  

(3)

now let us put

\[ \sigma' = \sum n_j \left( \frac{e_j^2}{m_j} \right) \left( \frac{\nu_j}{\nu_j^2 + \omega_j^2} \right) \]

(4)

and

\[ \sigma'' = -\left( \frac{1}{H} \right) \sum n_j e_j \left( \frac{\omega_j^2}{\nu_j^2 + \omega_j^2} \right) \]  

then considering \( n_e + n_- = n_+ \) and \( \omega_2^-/(\nu_2^- + \omega_2^-) = \omega_2^+/(\nu_2^+ + \omega_2^+) \)

we have

\[ \sigma'' = (n_e/H) \left( \omega_2^-/(\nu_2^- + \omega_2^-) - \omega_2^+/(\nu_2^+ + \omega_2^+) \right) \]  

(5)

using (1) and (5), equation (3) is written as

\[ J = \sigma' E + \sigma'' \mathbf{h} \times E \]  

(6)
Cowling [2] called $\sigma'$ and $\sigma''$, direct and transverse conductivity respectively. In the following, as in I, we treat the case $\lambda<1$ near the E region. The ratio $\sigma''/\sigma'$ with gas density $n$ (particles/c.c.) is shown in Table 1.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\sigma''/\sigma'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{10}$</td>
<td>$3 \times 10^{-3}$</td>
</tr>
<tr>
<td>$10^{11}$</td>
<td>0.177</td>
</tr>
<tr>
<td>$10^{12}$</td>
<td>3.0</td>
</tr>
<tr>
<td>$10^{13}$</td>
<td>17.6</td>
</tr>
<tr>
<td>$10^{14}$</td>
<td>4.1</td>
</tr>
</tbody>
</table>

When $\sigma'' \ll \sigma'$, as in the upper part of the F1 layer and in the F2 layer ($n<10^{11}$), current flows almost in the direction of $\mathbf{E}$. But when $\sigma'' > \sigma'$, as near the E layer ($n=6 \times 10^{12}$), then an impressed electromotive force $\mathbf{E}_0$ produces not only direct current, but also a considerable H.C., and we must consider the effect of the polarization field $\mathbf{E}_1$ by the latter current. Then $\mathbf{E}=\mathbf{E}_0+\mathbf{E}_1$, provided $\mathbf{E}$ is perpendicular to $\mathbf{H}$, therefore equation (5) becomes

$$\mathbf{J}=\sigma'(\mathbf{E}_0+\mathbf{E}_1)+\sigma''\mathbf{h} \times (\mathbf{E}_0+\mathbf{E}_1)$$  \hspace{1cm} (7)

2. Conductivity of the E region.

Let right-handed rectangular axis o-xyz be taken, so placed that ox, oy and oz are directed to the south, east and upward respectively and unit vectors in the direction of ox, oy and oz be $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ respectively. According to the “Dynamo theory” concerning the diurnal variation of the geomagnetic field, both horizontal air velocity $\mathbf{v}_0$ and magnetic field $\mathbf{H}$, produce primarily dynamo field $\mathbf{v}_0 \times \mathbf{H}=v_H \mathbf{i} - u_H \mathbf{j} - v_H \mathbf{k}$.

A. Schuster [3] in 1908 showed the possibility that $-v_H \mathbf{k}$ might produce a three dimensional current system. But we think that, as the thickness of the ionosphere, even when $\mathbf{E}$ and $\mathbf{F}$ layers are combined, is sufficiently thin compared with the horizontal uniformity of the diurnal current system, $-v_H \mathbf{k}$ is counteracted by the resulting electric field $\mathbf{E}_k$ and $-v_H \mathbf{E}_k=0$. When we discuss the diurnal variation, therefore, we can eliminate both $-v_H \mathbf{k}$ and $\mathbf{E}_k$ from $\mathbf{E}$ in equation (1) and agree with Cowling’s opinion [4]. In I we discussed the effect of the vertical component of the H.C., in the following we discuss effect of the horizontal component of that in addition. As in I we assume that the axes of rotation and magnetism of the earth coincide. We treat first the case, $\mathbf{E}_o=\mathbf{E}_o \cdot \mathbf{j}$ close to the E region.

(i) Conductivity near the geomagnetic equator.

Fig. 1 shows a meridional cross section of the ionospheric E layer near the geomagnetic equator 0. In this region magnetic lines of force describe curves as AB and CD. When $\mathbf{E}_o$ is uniform in sufficiently wide region and $E_o > 0$, H.C. tends to flow in the direction of arrows in Fig. 1, but the current will not flow in the direction counteracted by the resulting electric field, because $\sigma'$ will decrease rapidly downward in the lower part of the E layer for several ten kms in distance. In this case $\mathbf{E}_t=\mathbf{E}_t \cdot \mathbf{n}$ where $\mathbf{n}=\mathbf{h} \times \mathbf{j}$; substituting this in (7) we get

$$\mathbf{J}=\sigma'(\mathbf{E}_0+\mathbf{E}_t)+\sigma''(\mathbf{n} \mathbf{E}_0+\mathbf{h} \times \mathbf{n} \cdot \mathbf{E}_t)$$  \hspace{1cm} (8)

moreover, in this case $\mathbf{n} \cdot \mathbf{J}=0$ ensues.  \hspace{1cm} (9)
Substituting equation (8) in the upper equation, we have
\[ E_t = -\left(\sigma''/\sigma'\right) E_0 \]
from this relation and (8) it follows that
\[ \mathbf{J} = \sigma'' \mathbf{E}_n \]
where \( \sigma'' = \sigma' \left(1 + \left(\sigma''/\sigma'\right)^2\right) \).
\( \sigma'' \) is considered to be resultant direct conductivity, and identical with equ. (11) in I for \( \phi = 0 \).

In Figs. 2 and 3, \( \sigma''/n_e \) and \( \sigma''/\sigma' \) are shown with \( n \) and for \( \lambda < 1 \).

From these figures it is evident that \( \sigma''/n_e \) is maximum in the lower part of the \( E \) region.

(ii) Conductivity in middle latitudes.

Let conductivity parallel to the magnetic lines of force be \( \sigma_{||} \), then \( \sigma_{||}/\sigma' \) with \( n \) is shown in Fig. 5. Fig. 4 shows that of the same kind as Fig. 1, in middle latitudes. We consider an element ABCD characterized by magnetic lines of force AB and CD. Electron density between \( E \) and \( F \) layers will be not less than one twentieth of maximum density of the \( E \) layer. When an uniform electromotive force \( \mathbf{E}_0 = E_0 \mathbf{j} \) (\( E_0 > 0 \)), which is also uniform for longitude, is impressed between AD in the \( E \) layer, current flows not only in the direction of \( \mathbf{j} \) but also in the direction of ADCB. Let
polarization electric field resulting from H.C. be \( E_1 \) and as current between AD, we have, using equation (1)
\[
J = (a'\epsilon_{1x}/\sin^2 \phi + a''\epsilon_i \sin \phi) i + (a'\epsilon_0 + a''\epsilon_i) j
\]
where \( E_i = E_{1x}/\sin \phi + E_0 \). Using the above equation and \( k \cdot J = 0 \), we eliminate \( E_1 \) and get
\[
J = (a'\epsilon_{1x}/\sin^2 \phi + a''\epsilon_i \sin \phi) i + (a'\epsilon_0 + a''\epsilon_i) j
\]
where \( \phi \) is the dip of geomagnetic field.
As current density I integrated with height in every layer, we have for AD approximately
\[
I = \int_a^b a'\epsilon_{1x}/\sin^2 \phi m + \int_a^b a''\epsilon_0 \sin \phi m \]
where \( \phi_m \) is the mean value of \( \phi \) in AD. Considering that, if AD, BC are not much less than AB, CD, the resistances between AB and CD are far less than those between AD and BC, and that, in the F region, as \( a'' \ll a' \), even when similar \( E_0 \) is impressed there, its influence to \( i \cdot J \) is negligible, for BC we have
\[
I = -\int_a^b a''\epsilon_0 \sin \phi m
\]
From above two equations \( E_{1x} = a \cdot \sin^3 \phi_m \cdot E_0 \)
where \( a = -\int_a^b a''\epsilon_0 \sin \phi m \)
Substituting (13) in (11) and denoting mean value in the E layer by suffix E,
\[
J_E = (a'\epsilon_{1x}/\sin^2 \phi + a''\epsilon_i \sin \phi) E_0 j + (a'\epsilon_0 - a''\epsilon_i) E_0 j
\]
As component of \( J \) in \( i \) flows in an opposite direction for E and F region, their effect to the magnetic field is negligible.
When \( h < 1 \), \( n_e = 1.5 \times 10^6 \), \( \int_a^b a''\epsilon_0 dh \approx 4.2 \times 10^{-14} \), \( n_e = 6 \times 10^{-9} \),
\[
\int_a^b a''\epsilon_0 dh = 1.76 \times 10^{-13} \cdot n_e = 2.64 \times 10^{-8} \text{ (e.m.u.)}
\]
According to Cowling [2], in the absence of induction drag, for F1 layer \( \int a''\epsilon_0 dh \approx 4 \times 10^{-8} \), for F2 layer \( 10^{-7} \), so for F1 and F2 layers combined \( 1.4 \times 10^{-7} \). When BC contains F1 layer only \( a = -0.58, a'' = 2.28 a' \) for \( \phi_m = \pi/4 \) and when BC contains both F1 and F2 layers \( a = -0.188, a'' = 1.4 a' \) for \( \phi_m = \pi/4 \). Consequently \( a'' \) at middle latitudes will be several times of \( a' \) and we may take \( \int a''\epsilon_0 dh \approx 10^{-8} \) as an order of magnitude. When \( \epsilon_{1x} = \epsilon_0 \cdot i \) we can treat the problem in a similar way, and get the same order of magnitude.

(iii) Latitude at which \( a'' \) type is transformed to \( a' \) type.

Geomagnetic lines of force are approximated by \( r = a \cdot \sin^2 \theta / \sin^2 \theta_i \) (where \( \theta = \theta_i \) at \( r = a \)). Let line of force which started from h km height above the equator, meet with 100km level at latitude \( L_0 \), and the relation between \( L \) and \( h \) is shown in Fig. 6.

h=200km (F1), 300km (F2) correspond L=7°.2, 10° respectively. At latitudes L<7°.2,
lines of force, which crosses $E$ layer, will not enter into the $F$ region as shown in Fig. 1 therefore $\sigma_y$ will not greatly differ from $\sigma_y^{\mu\nu}$. But at latitudes $L > 7.02$, as BC of the above mentioned element ADBC enter into the $F$ region, $\sigma_y$ will decrease rapidly from $L = 7.02$ towards higher latitudes tending to $2\sigma^i$.

Up to the present we considered the effect of H.C. for an element AD-CB. According to the principle of superposition, the above mentioned result will hold for the diurnal current also. When an electromotive force $E_0 = (E_{0x}, E_{0y})$ is imposed, we can take $I_x = \sigma_x E_{0x}$, $I_y = \sigma_y E_{0y}$ with respect to the electric current producing the diurnal magnetic variation by reason mentioned in (ii). At the equator, for the mean value in the $E$ layer, in which local scale height is 10km, $\sigma_y = \sigma_y^{\mu\nu}$ and using equation (9) $\sigma_y^{\mu\nu} = 40\sigma^i$. As $L$ increases, $\sigma_y$ will slowly decrease and beyond $L = 7.02$ $\sigma_y$ will rapidly decrease tending to $2\sigma^i$. Under the assumption that vertical component of H.C. does not flow, and using equ. (11) in I, we get for $(\sigma_y/\sigma^i)_{yK}$ curve A in Fig. 7. Such an assumption will be fit, when the effective layer is sufficiently thin compared with current system. Curve A is considered to be a fairly good approximation for the above consideration, but curve B corresponding to $\sigma_y = 2\sigma^i$ will be better for the latitudes $L > 7.02$.

Let mean value in $0^\circ \leq L \leq 7.05$ of $\sigma_y$ be $\bar{\sigma}_y$ and that in middle latitudes be $\bar{\sigma}_m$ then for horizontally uniform electron density we get $\bar{\sigma}_y \approx 6\bar{\sigma}_m$.

Curve C is written for the case $\lambda = 10$ using equ. (11) in I, for reference.

3. A consideration based on the Dynamo theory.

According to the previous section, there is a narrow belt along the geomagnetic equator, of about 15° of latitude wide, in which $\sigma_y$ is very great. Moreover in this belt $\sigma_x$ is very great according to equ. (14) in I, and they are ohmic. By the Dynamo theory we discuss the influence of this belt to the current. Let air velocity in the spherical shell containing $E$ region be $u = \partial \phi / \partial x = \partial \phi / a \partial \theta$, $v = \partial \phi / \partial y = \partial \phi / a \sin \theta \partial \varphi$, Fig. 7.
current function be $R$, electrostatic potential be $S$, then we have the next relation [5],

$$
\begin{align*}
\nu H_r - \frac{\partial S}{\partial \theta} &= \frac{1}{K} \frac{\partial R}{a \sin \theta \partial \phi} \\
-\nu H_{\theta} - a \frac{\partial S}{\partial \phi} &= -\frac{1}{K} \frac{\partial R}{a \sin \theta \partial \phi}
\end{align*}
$$

where $a$ is the radius of the shell, $K$ is conductivity integrated with height. We assume the next distribution of $K$, I $K=K_1$ (const.) for $0<\theta<\theta_1$, II $K=\mu K_1$ for $\theta_1<\theta<\pi-\theta_1$, III $K=K_1$ for $\pi-\theta_1<\theta<\pi/2$

We take, as the velocity potential of the air, for mathematical convenience

$$
\phi = k \cdot P_1^1 \cdot \sin \phi
$$

and investigate the outline of the influence of region II.

Eliminating $S$ from (15) we have

$$
\frac{\partial^2 R}{\sin \theta \partial \phi^2} + \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial R}{\partial \phi} \right) = aK \left\{ \frac{\partial (\nu H_r)}{\partial \phi} + \frac{\partial (\nu H_{\theta} \sin \theta)}{\partial \theta} \right\}
$$

(17)

We write $R=\sum \sum R^n_m$, $R^n_m=r^n_m P^n_m \sin m \phi$.

A particular integral of equation (17) in regions I, III is $R^1_1=(1/6) CK_1 k P_2^1 \sin \phi$, in region II is $R^1_1=(1/6) Cu K_1 k P_2^1 \sin \phi$ (18)

where $H_2=C \cdot \cos \theta$. Next we consider the solution of the equation

$$
\frac{\partial^2 F}{\sin \theta \partial \phi^2} + \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial F}{\partial \phi} \right) = 0.
$$

(19)

Let $R=f(\theta) \cdot F(\phi)$, $M^2$ be constant of separation and (19) gives

$$
\frac{\partial^2 F}{FM^2} = -\frac{1}{f} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial F}{\partial \phi} \right) = M^2.
$$

(20)

In order that $F$ be single valued function of $\phi$, $M$ must be integer and $F=\sin M \phi \cos M \phi$

Let two independent solution of (21) for $M$ be $\phi_{1M}(\theta)$, $\phi_{2M}(\theta)$, then complete integral of (19) is

$$
\sum \sum \{ \gamma_{1M} \phi_{1M}(\theta) + \gamma_{2M} \phi_{2M}(\theta) \} (\gamma_{1M} \sin M \phi + \gamma_{1M} \cos M \phi)
$$

where $\gamma_{1M}$ are constants. It is well known that among periodic variations of geomagnetic field, the diurnal one is the greatest, therefore for simplicity we assume that $M=1$ and take $F=\sin \phi$. In this case, bounded solutions of equation (19) in every region are (6) I $\{ C_1 \sin \theta/(1+\cos \theta) \} \cdot \sin \phi$

II $\{ C_2 \cot \theta + C_3 \cosec \theta \} \cdot \sin \phi$, III $\{ C_4 \sin \theta/(1-\cos \theta) \} \cdot \sin \phi$

Therefore a solution of (17) which is bounded in every region, is obtained by summing (18) and above functions in every region respectively. For this solution by the symmetry of current against equator $C_5=0$, $C_4=C_1$. Considering the continuity of normal component of current and tangential component of electric field along the boundary between I, II and II, III we can determine $C_1$ and $C_2$, then

$$
\begin{align*}
\text{I} & \quad R=\frac{1}{6} CK_1 k \left\{ P_2^1 (\theta) + \frac{(\mu-1)}{\sin \theta_1 (1+\mu \cos \theta_1)} P_2^1 (\theta) \right\} \sin \phi \\
\text{II} & \quad R=\frac{\mu}{6} CK_1 k \left\{ P_2^1 (\theta) - \frac{(\mu-1) \sin \theta_1}{1+\mu \cos \theta_1} P_2^1 (\theta) \right\} \cot \theta \sin \phi
\end{align*}
$$

(22)
From above equations we can determine eastward current $V$, $V = -\frac{\partial R}{\partial \theta} = V_\theta(\theta) \cdot \sin \phi$. $V_\theta(\theta)$ is represented in Fig. 8 in unit of $(1/6) CK_1 k$ for various values of $\theta_1$ and $\mu$.

![Fig. 8](image)

The more $\mu$ increases, the more total amount of current increases, and the more current focus shifts towards equator and the more the current in lower latitudes concentrates towards equator. But, as it was shown by D.F. Martyn that some diurnal currents flow in $F_1$ and $F_2$ layers [7], it is impossible to determine $\mu$ directly from this result. Nevertheless the value $2 \leq \mu \leq 6$ might be appropriate.

4. Discussion of the result.

It was shown that there is a narrow region along the geomagnetic equator, of about $15^\circ$ latitude wide, in which conductivity is very great, as a result of polarization by the H.C., and this belt concentrates some part of diurnal electric current, giving enhanced diurnal variation of geomagnetic field. In reality, however, the axes of rotation and magnetism do not coincide and there exist fair local anomalies of magnetic field. In the real ionosphere the high conductive belt will lie near the $E$ layer, along the line of zero dip at that level. According to precise statistical research by Prof. M. Hasegawa and Dr. M. Ota [8], a belt of enhanced geomagnetic diurnal variation lies approximately along the geomagnetic equator, but a little discrepancy exists between them. Investigation about this discrepancy will be reported in a later paper.

Acknowledgement

The writer wishes to express his hearty thanks to Prof. M. Hasegawa for his
encouragement in this work, and to Dr. S. Hayami for his advice. It is a deep pleasure to record here a debt of gratitude to Prof. T.G. Cowling for his many valuable discussions with respect to part I of this work and to Dr. D.F. Martyn for his valuable suggestions.

Many thanks are also due to Mr. H. Maeda for his assistance in this work.

References