Electromagnetic Core-Mantle Coupling*

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(Received March 13, 1972)

The time-scale of the variation in the length of the day is comparable to, or shorter than, the electromagnetic decay time, τ_η , of the mantle. This suggests that an electromagnetic theory for the origin of these fluctuations should not only depend on the electromechanical coupling time, τ_I , introduced by ROCHESTER (1960), but also on τ_η . It is shown that, when τ_I is small compared with τ_η , the fluctuation time for perturbations in the angular velocity of the mantle is of order $\tau_c = \tau_I^{2/3} \tau_\eta^{1/3}$, and not τ_η . The possibility that τ_I is not large compared with τ_η for the case of the Earth is considered. Also, by means of a simple shellular model of the core, the rôle of Alfvén waves in modifying the coupling of core to mantle is discussed. The relationship of this theory with that of BRAGINSKII (1970) is briefly noted. A new account is given of mantle induction which, although only valid at small magnetic Reynolds numbers (based on the conductivity of the mantle), is sufficiently general to permit the coupling of the mantle to an arbitrary slow motion on the core surface to be treated.

1. Introduction

It is generally conceded that the main geomagnetic field stems from electric currents created by a self-excited dynamo process in the earth's liquid core, a fluid commonly held to be a good electrical conductor. Since the mantle is not perfectly insulating, these currents must partially leak into the mantle, each element of which accordingly experiences a Lorentz force. The mantle as a whole will therefore be subject to an electromagnetic couple, Γ_L , where 'L' stands for leakage. Even were the core a perfect conductor, which effectively shorted out potential leakage currents before they could reach the mantle, the flow, $\mathbf{u}$, in the core (measured relative to the mantle) would redistribute the pre-existing lines of force

* The material presented here formed the basis of an invited paper at the International Conference on the Core-Mantle Interface held at Melbourne, Florida, from March 7-March 9, 1972.

emanating from the core surface. Currents would be induced in the mantle which would subject it to a couple, Γ_F , where 'F' stands for frozen field, recognizing that flux tubes are convected with the perfectly conducting core in its motion. This provides a second electrical means of coupling core to mantle.

Before the magnitudes of Γ_L and Γ_F can be assessed, the electrical properties of the mantle should be modelled. The conductivity must not be so large that the secular variation is too drastically attenuated. For high frequencies, the electrical time constant of a radially symmetric mantle is (ROBERTS, 1954)

$$\tau_\eta^{sv} = \mu \left[\int_a^{r_e} [\sigma(r)]^{1/2} dr \right]^2, \quad (1.1)$$

where a is the radius of the core, r_e the radius of the earth, σ is the conductivity of the mantle, and μ is its permeability (mks units). The suffix η in Eq. (1.1) stands for the magnetic diffusivity, $\eta = 1/\mu\sigma$. If we suppose that all conducting matter in the mantle is confined to the lowermost shell $a < r < b = a + L$, we could use Eq. (1.1) to define a mean conductivity σ^{sv} for the mantle, by writing $\tau_\eta^{sv} = \mu \sigma^{sv} L^2$. This is not, of course, the only way in which a mean conductivity can be defined. For example, we might introduce

$$\bar{\sigma} = \frac{1}{L} \int_a^b \sigma dr, \quad \tau_\eta = \mu \bar{\sigma} L^2. \quad (1.2, 3)$$

The distinction between such obviously similar quantities might seem unjustified by the paucity of available data. It does appear, however, that the success or failure of the electromagnetic theory of core-mantle coupling is a delicate matter, and that some notice must be taken of these refinements. If σ is uniformly distributed in $a < r < b$, then $\tau_\eta = \tau_\eta^{sv}$. Otherwise τ_η exceeds τ_η^{sv} (Schwartz inequality), and in the geophysically interesting case, in which σ increases rapidly with depth, it does so by perhaps an order of magnitude. For example, suppose

$$\sigma(r) = \sigma(a)(b-r)^{2\alpha} L^{2\alpha};$$

then $\tau_\eta/\tau_\eta^{sv} = (\alpha+1)^2/(2\alpha+1)$, or about 3 for $\alpha=5$, and 6 for $\alpha=10$. Thus, although τ_η^{sv} may be only 4 or 5 years (McDONALD, 1957), τ_η may lie between 10 and 30 years. This, as we shall see, affects Γ_F and therefore also a significant time constant τ_I to be defined presently.

Yet another definition is sometimes apposite, particularly when diffusion from the core surface, S , is studied in its early stages, following a disturbance at the top of the core. This is simply the diffusion time,

$$\tau_\eta^a = \mu a^2 \sigma(a), \quad (1.4)$$

based on the conductivity at the bottom of the mantle. In practice this is the largest of the three, being 40–400 years for $\sigma(a)=10^2-10^3$ mho/m. A related (but shorter) time-constant, which is particularly relevant to core-mantle coupling, is defined by Eq. (3.10) below.

The strength of the core-mantle coupling depends on these times, and also on the appropriate mean of the vertical field, B_r , on S . Extrapolation from the earth's surface suggests that the rms field intensity does not exceed 5 gauss, and we will therefore take $B_r=3.5$ gauss in our estimates which follow.

The magnitude of Γ_L is difficult to assess with certainty since it depends on the details of the dynamo mechanism. It is clear, however, that the same potential differences which can create a toroidal field of magnitude, B_T , in the highly conducting core, can only generate a toroidal field of order B_T/ξ in the poorly conducting mantle, where ξ is the ratio of the conductivity of the core, σ_c , to an appropriately defined conductivity for the mantle. (Thus, for example, if $\xi=1000$, fields of 350 gauss in the core will be associated with fields of only 0.35 gauss in the mantle.) The Maxwell stress across S will therefore be of order $B_r B_T/\mu\xi$, and it follows that

$$\Gamma_L \simeq 4\pi a^3 B_r B_T / \mu \xi, \quad (1.5)$$

or about $4.10^{18} \text{ kg} \cdot \text{m}^2/\text{s}^2$.

To estimate Γ_F , we note that the electric field created along S by the motion, $\mathbf{u}$, on the core surface is of order uB_r , and this sets up currents of density σuB_r in the mantle, and an associated Lorentz force per unit volume of order σuB_r^2 . Summing, we see that the resultant couple on the mantle is

$$\Gamma_F \simeq 4\pi a^3 B_r^2 u \tau_\eta / \mu L, \quad (1.6)$$

where (Section 2) the appropriate definition of τ_η is provided by Eq. (1.3). If, however, we simply take τ_η as τ_η^{sv} (5 years, say), we find that Γ_F is about $8.3 \cdot 10^{21} (u/\text{ms}^{-1}) \text{ kg} \cdot \text{m}^2/\text{s}^2$.

If the angular velocity of the mantle is unchanging in the mean, the total torque must vanish:

$$\Gamma_T \equiv \Gamma_L + \Gamma_F = 0. \quad (1.7)$$

Using the estimates made above we recognize that flow speeds of order $5 \cdot 10^{-4} \text{ m/s}$ are required. This should be compared with the velocity $3.5 \cdot 10^{-4} \text{ m/s}$ implied by the westward drift on the core equator. The balance Eq. (1.7) is not unexpected on current dynamo theory which suggests (*e.g.*, ROBERTS, 1971) that $B_T \simeq R B_r$, where R is the magnetic Reynolds number based on the zonal shearing motion in the core. If we take this shear to be

of order u/a , then $R \simeq \mu \sigma_c u a$, and Eq. (1.5) gives $\Gamma_L \simeq 4\pi \sigma a^4 B_r^2 u$, where σ is an appropriately defined mantle conductivity. This should be compared with $\Gamma_F \simeq 4\pi \sigma a^3 L B_r^2 u$, as predicted by Eq. (1.6).

A particularly simple and significant example of the flux redistribution mechanism is provided by the bodily rotation of the entire core surface, as a rigid shell, with angular velocity ω relative to the mantle. In a steady state, Γ_F is parallel to ω , and we have

$$\Gamma_F = K\omega, \quad (1.8)$$

where, by (1.6),

$$K \simeq 4\pi a^4 B_r^2 \tau_\eta / \mu L, \quad (1.9)$$

and is positive. Using the estimate of Γ_F above, we see that $K \simeq 2.9 \cdot 10^{28} \text{ kg} \cdot \text{m}^2/\text{s}$.

This illustration is useful in demonstrating how the balance Eq. (1.7) is preserved. If Eq. (1.7) does not hold at some instant, the angular velocity, ω_m , of the mantle will change and, since the total angular momentum of the core-mantle system is preserved, some (and eventually all) the core will tend to move in the opposite sense. The velocity, u , with which the flux lines are carried along S is altered and Γ_F will change as fast as electromagnetic diffusion in the mantle will allow. In the simplest picture, the core rotates rigidly with angular velocity ω_c and, by Lenz's law, the relative angular velocity

$$\omega \equiv \omega_c - \omega_m, \quad (1.10)$$

adjusts itself to that value for which Eq. (1.7) holds.

This picture, which is essentially that painted by BULLARD *et al.* (1950) provides a satisfying explanation of the westward drift. When applied to other observations, however, difficulties arise. The length of the day alters irregularly by a few milliseconds per decade, and occasionally even more rapidly; for example, in 1897 a change of 3 milliseconds appears to have been effected in about a year.

Taking the moment of inertia, I_m , of the mantle to be $7.2 \cdot 10^{37} \text{ kg} \cdot \text{m}^2$ and the angular acceleration, $\dot{\omega}_m$, to be 10^{-20} s^{-2} (corresponding to a change in length of day of about 3 milliseconds over a decade), it is seen that fluctuations of order $10^{18} \text{ kg} \cdot \text{m}^2/\text{s}^2$ are required in the torque balance Eq. (1.7). In other words, changes are required which are of the same order as Γ_L and Γ_F separately. The situation is exacerbated when it is recalled that estimates Eq. (1.5) and Eq. (1.6) depend on the assumption that leakage and flux redistribution take place over the entire mantle boundary. It seems, however, that the excitation is sometimes limited to local areas of S . For

example, MOORE (1954) has argued that the 1897 event referred to above is correlated with the appearance of a particularly strong secular variation focus below South Africa. YUKUTAKE (private communication, 1972) associates this secular variation with the westward drift of an extremely large non-dipole anomaly below Africa, which presumably changed its motion during 1897. If we suppose that the scale, l , of the accelerating field is of order 10^3 km, the associated couple is diminished by a factor of order $\pi l^2/4\pi a^2 = (l/2a)^2 \simeq 1/50$. This particular difficulty would be removed if B , and/or σ were assumed to be larger than we have supposed.

In addition to this obstacle over magnitudes, the short period over which ω_m fluctuates presents difficulties. We may first notice the existence of a new time-scale. This is most easily displayed by temporarily supposing that changes in ω_m are so gradual that the core has time to adjust continuously to the changing state of the mantle, and therefore always rotates as a solid body with angular velocity, ω_c . In the same spirit we may neglect diffusion in the mantle, supposing that, as ω changes, Γ_F always takes the steady state value Eq. (1.8). The equations of motion of mantle and core are then

$$I_m d\omega_m/dt = K(\omega_c - \omega_m) + \Gamma_L = -I_c d\omega_c/dt, \quad (1.11)$$

where $I_c \simeq 8.5 \cdot 10^{36} \text{ kg} \cdot \text{m}^2$ is the moment of inertia of the core. It is convenient to introduce the reduced moment of inertia

$$I = I_m I_c / (I_m + I_c), \quad (1.12)$$

($I \simeq 7.7 \cdot 10^{36} \text{ kg} \cdot \text{m}^2$), where upon Eq. (1.11) yields

$$I d\omega/dt = -K\omega - \Gamma_L, \quad (1.13)$$

which reveals the existence of the new electromechanical time-scale

$$\tau_I = \frac{I}{K} = \frac{I}{4\pi\sigma a^4 L B_r^2}. \quad (1.14)$$

Taking the value of K given below Eq. (1.9), we see that $\tau_I \simeq 8.5$ years; Rochester's value for K gives $\tau_I \simeq 25$ years. This is embarrassingly long compared with the time-scale of the fluctuations in the length of the day. We must recall, however, that we have used τ_γ^{sv} to obtain K . Remembering that τ_γ may be larger than τ_γ^{sv} by a factor of 5, or even 10, we find that the discrepancy is substantially reduced (RODEN, 1963).

The argument which led to Eq. (1.13) is deficient in at least two significant ways: the electromagnetic response of the mantle has been neglected, and the magnetohydrodynamic response of the core has been ignored. We first restore the former, continuing, however, to suppose that

the core is rigid. We specialize to the case in which a steady state is perturbed at $t=0$ by a couple $\delta\Gamma(t)$ parallel to the initial angular velocity, and that a perturbation $\delta\omega$ in ω results. It is clear that, when diffusion is included, Eq. (1.13) is no longer viable, but must be replaced by an equation of the form

$$I \frac{d}{dt} \delta\omega(t) = - \int_0^t G(t') \delta\omega(t-t') dt' + \delta\Gamma(t), \quad (1.15)$$

where G is an appropriately defined Green's function incorporating diffusive "memory effects"; it must satisfy

$$K = \int_0^\infty G(t') dt'. \quad (1.16)$$

The precise form of G is not of special interest here; it is considered in Section 3. It depends on the form of the poloidal field on S and the distribution of conductivity in the mantle. It is shown in Section 3 that

$$G(t) = \frac{K}{\tau_\eta} \left[\left(\frac{\tau_\eta}{\pi t} \right)^{1/2} - e^{t/\tau_\eta} \operatorname{erfc} \left(\frac{t}{\tau_\eta} \right)^{1/2} \right], \quad (1.17)$$

may be taken as an archetype; here

Fig. 1 The form of the Green's function (G) and its integral (H) which determine the electromagnetic "memory effects" of the response of the angular velocity of the mantle to variations in applied torque; see Eqs. (1.15), (1.17) and (1.19).

$$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2} du, \quad (1.18)$$

is the complementary error function. The choice Eq. (1.17) is typical in the sense that $G(t) = O(t^{-3/2})$ for $t \rightarrow \infty$ and is $O(t^{-1/2})$ for $t \rightarrow 0$. It is shown in Fig. 1 together with $H(t)/K$, where

$$H(t) \equiv \int_0^t G(t') dt' = K \left[1 - e^{t/\tau_\eta} \operatorname{erfc} \left(\frac{t}{\tau_\eta} \right)^{1/2} \right]. \quad (1.19)$$

Clearly, $H \rightarrow 0$ as $t \rightarrow 0$ and $H \rightarrow K$ as $t \rightarrow \infty$. It may be seen that H attains half its full value in a time of order $0.59\tau_\eta$. It is clear from Eq. (1.15) that no significant error will be made in accepting Eq. (1.13) when τ_I is large compared with τ_η .

Suppose that $\delta I(t)$ is a constant couple δI_0 suddenly 'switched on' at $t=0$. Clearly $\delta\omega$ will ultimately settle down to the steady state value $\delta\omega_0 = \delta I_0/K$, but the manner in which it does so depends very much on the ratio $\chi = \tau_I/\tau_\eta$, of characteristic times. It is evident that initially $\delta\omega$ will increase linearly with t but, sooner (small τ_η) or later (large τ_η), a restoring couple will arise to halt this growth, and this will ultimately bring $\delta\omega$ to $\delta\omega_0$. If $\tau_\eta \gg \tau_I$, the angular velocity has time to reach very large values before deceleration commences. To see this, we note that at small time, t , the currents induced by the differential motion between mantle and core will have penetrated a distance into the mantle of order $\varepsilon = (\eta t)^{1/2}$, rather than to the full depth L used to obtain Eq. (1.6). The couple is therefore smaller by a factor of order $(\eta t)^{1/2}/L = (t/\tau_\eta)^{1/2}$. In other words, by Eq. (1.8), $\delta I_F \simeq K(t/\tau_\eta)^{1/2} \delta\omega$. But, during the time that this is small compared with δI_0 , we have $\delta\omega \simeq t \delta I_0/I$, so that $\delta I_F \simeq \delta I_0(t^3/\tau_\eta \tau_I^2)^{1/2}$. This illustrates two points. First, for sufficiently short times, $\delta\omega$ grows on neither the τ_η nor the τ_I time-scale, but on a compromise time-scale, τ_c , where

$$\tau_c = \tau_\eta^{1/3} \tau_I^{2/3}. \quad (1.20)$$

In particular, this scale measures the time which must elapse before δI_F can become a sensible fraction of δI_0 . Second, the angular velocity at which δI_F becomes of the same order as δI_0 is approximately $\tau_c \delta I_0/I = \tau_c \delta\omega_0/\tau_I = \delta\omega_0(\tau_\eta/\tau_I)^{1/3}$ and, when $\tau_\eta \gg \tau_I$, this is large compared with the ultimate steady-state value, $\delta\omega_0$. The final transition to this state will take a time of order τ_η to accomplish.

For any χ , the presence of the τ_c time-scale may be detected for sufficiently small t . For example, using Eq. (1.17) for G , we readily find that

$$\delta\omega = \frac{\delta\omega_0}{\chi} \left(\frac{t}{\tau_\eta} \right) \left[1 - \frac{18}{15\sqrt{\pi}} \left(\frac{t}{\tau_c} \right)^{3/2} + O(t^2) \right], \quad t \rightarrow 0. \quad (1.21)$$

It is interesting also to observe from Eq. (1.15) that ohmic diffusion *always* decides the ultimate behaviour of $\delta\omega$, and that in consequence $\delta\omega$ approaches its steady state value $\delta\omega_0$ from *above*. For example, using Eq. (1.17) for G , we readily find that

$$\delta\omega \sim \delta\omega_0 \left[1 + \left(\frac{\tau_\eta}{\pi t} \right)^{1/2} + \pi\chi \left(\frac{\tau_\eta}{\pi t} \right)^{3/2} + O(t^{-5/2}) \right], \quad t \rightarrow \infty, \quad (1.22)$$

where we have neglected terms of exponentially small order.

For $\tau_\eta \ll \tau_I$ ($\chi \gg 1$), Eq. (1.15) gives for all t , except the very small and the very large,

$$\delta\omega = \delta\omega_0 (1 - e^{-t/\tau_I}), \quad (1.23)$$

in agreement with the fast diffusion theory of ROCHESTER (1960). The temporal behaviour is determined by the larger time constant τ_I , and the electromagnetic field is able to adjust 'instantaneously' to the changing mechanical state.

For $\tau_\eta \gg \tau_I$ ($\chi \ll 1$), Eq. (1.15) gives, for all except large times

$$\delta\omega = \frac{\delta\omega_0}{3\chi^{1/3}} \left[e^{t/\tau_c} \operatorname{erfc} \sqrt{\frac{t}{\tau_c}} + \omega e^{\omega^2 t/\tau_c} \operatorname{erfc} \omega \sqrt{\frac{t}{\tau_c}} + \omega^2 e^{\omega^2 t/\tau_c} \operatorname{erfc} \omega^2 \sqrt{\frac{t}{\tau_c}} \right], \quad (1.24)$$

where ω and ω^2 are the complex cube roots of unity. This early evolution is *not* determined by the larger time scale, τ_η , but by the compromise scale Eq. (1.20) which is weighted towards the *smaller* characteristic period, τ_I . For large t , Eq. (1.22) holds; in these final stages the mechanical state is able to adjust 'instantaneously' to the changing electromagnetic field.

One must wonder how reasonable it is to suppose, as we have done, that $\delta\Gamma_0$ is applied suddenly at $t=0$, for on the present theory $\delta\Gamma$ is created either by leakage or flux rearrangement at the core-mantle interface, and for both these processes the accelerating fields grow at a rate proportional to η . We should not forget, however, that the scale, l , of the disturbance on the core-mantle boundary is probably small and, although this is as we have seen a disadvantage when we consider the magnitude of the accelerating couple, it does imply that its growth rate, $\tau_E = l^2/\eta$, may be small compared with τ_η . In this case the evolution of $\delta\omega$ should be closely similar to that described above. We have, however, considered briefly in Section 3 another extreme case in which l is of the same order as L and $\tau_E = \tau_\eta$. Once again it is found that τ_c determines the run of $\delta\omega$ for small t , and once more it is shown that $\delta\omega$ approaches $\delta\omega_0$ from above for large t . It is no longer true (for evident physical reasons) that, when χ is small, $\delta\omega$ is initially large, compared with $\delta\omega_0$. It is significant, however, that $\delta\omega$ reaches values of order $\delta\omega_0$ in a time of order τ_c . This may be seen from

Fig. 2 The response of the angular velocity of the mantle to a step function torque and to a gradually applied torque. It is supposed that the electromagnetic diffusion time, τ_η , is equal to the electromechanical time constant, τ_I , given by (1.14). For comparison purposes, the response predicted by the fast diffusion theory of Rochester is shown dotted [see Eq. (1.23)].

the asymptotic expansion for $\delta\omega$ for small χ ; it is

$$\delta\omega = \frac{\delta\omega_0}{3} \left[3 - e^{t/\tau_c} \operatorname{erfc} \sqrt{\frac{t}{\tau_c}} - e^{\omega^2 t/\tau_c} \operatorname{erfc} \omega \sqrt{\frac{t}{\tau_c}} - e^{\omega t/\tau_c} \operatorname{erfc} \omega^2 \sqrt{\frac{t}{\tau_c}} \right]. \quad (1.25)$$

The behaviour of $\delta\omega/\delta\omega_0$ as a function of t/τ_η for $\tau_\eta = \tau_I$ is shown in Fig. 2 for both the step function applied torque and for the more gradual diffusive excitation. For comparison purposes the result Eq. (1.23) of the fast diffusion theory is also shown. Since $\tau_\eta = \tau_I$ is the "cross-over case", the evidence of the τ_c time-scale is not particularly strong; it is necessary to examine expressions such as Eqs. (1.24) and (1.25) to see its significance in the case of small τ_I/τ_η . Even so, it does not appear that Eq. (1.23) well represents the solution for the step function torque, even in the case $\tau_\eta = \tau_I$.

The main feature to emerge from this discussion is the presence of the τ_c time-scale. This governs the growth of $\delta\omega$ when $\tau_I \ll \tau_\eta$; in the case $\tau_I \gg \tau_\eta$, the evolution time is τ_I . Thus, either τ_I is comparable with ($\tau_I \gg \tau_\eta$) or smaller than ($\tau_I \ll \tau_\eta$) the time-scale of the fluctuations is the length of the day (a few years say), or these fluctuations do not owe their existence to electromagnetic coupling. This appears to require either a larger value of B_r than we have supposed or a distribution of conductivity heavily weighted towards the base of the mantle: probably $\sigma(a)$ must exceed 1000 mho/m (RODEN 1963).

A purely mathematical objection should be met. In the limit $\sigma \rightarrow \infty$,

the status of the mantle as a perfect conductor is no different from that of the core. It is therefore surprising that the response time depends on σ alone, and not equally on the core conductivity, σ_c . The difficulty may be traced to the double limiting operation $\sigma \rightarrow \infty$, $\sigma_c \rightarrow \infty$. We have supposed that $\sigma_c = \infty$ as the limit is taken. If instead $\sigma_c/\sigma \equiv \xi \neq 0$, the diffusion time must depend on σ_c as well as σ . In fact τ_η should be replaced by $a^2/(\eta^{1/2} + \eta_c^{1/2})^2$ where $\eta_c = 1/\mu\sigma_c$ is the diffusivity of the core. Moreover, estimate Eq. (1.14) for τ_I is erroneous; τ_I must be proportional to $\sqrt{(\eta\eta_c)}$. Thus τ_c is symmetric in η and η_c .

Some feeling for the time-scale on which the core responds to a change in the angular velocity of the mantle follows from writing τ_I another way: by Eq. (1.14), we have

$$\tau_I = \left(\frac{2LI}{15aI_c} \right) \frac{\tau_A^2}{\tau_\eta}. \quad (1.26)$$

It follows that

$$\tau_A = 5.5 \sqrt{(\tau_I \tau_\eta)}, \quad (1.27)$$

or equivalently

$$\tau_A = 5.5 \chi^{-1/6} \tau_c. \quad (1.28)$$

Here

$$\tau_A = a/V \quad (1.29)$$

is the Alfvén time-scale, defined as the time which must elapse before an Alfvén wave, travelling with the speed

$$V = B_r / \sqrt{(\mu\rho)} \quad (1.30)$$

appropriate to the poloidal field strength and mean core density ρ , can pass from core surface to core centre. Ignoring the inconsequential constant, we see from Eq. (1.27) that τ_A is intermediate between the two time-scales previously isolated. It is possible therefore that the solid core theory just described is not entirely satisfactory.

To clarify the effect of Alfvénic radiation into the core, it is convenient to suspend Eqs. (1.26) to (1.28) temporarily, supposing that τ_A is independent of τ_I and τ_η . If τ_A is small compared with τ_I and τ_η , the solid core theory described earlier becomes attractive. If τ_A is large compared with τ_I and τ_η , we may anticipate two opposing effects. If δI_0 is suddenly applied to the mantle at $t=0$ and simultaneously (to preserve total angular momentum) an equal couple is applied to the upper layers of the core, the lower core cannot respond before the long time necessary for Alfvén waves to

reach them has elapsed. The earlier theory will therefore, in essence, apply with I_c (and therefore I) replaced by the much smaller moment of inertia of the upper layers concerned; τ_I will be correspondingly smaller, and the equalizing couple will therefore grow more rapidly. A pseudo-steady state will therefore soon be attained in which (since $\delta\Gamma$, K and therefore $\delta\omega_0$ are much as before) the difference in angular velocity between core and mantle is comparable with its value for the solid core model considered earlier. Because of the wide disparity of the moments of inertia, however, little of this velocity difference represents a change in the angular velocity of the mantle. The mantle's share in $\delta\omega_0$ will only increase gradually as the disturbance seeps into the core on the (assumed) long time-scale, τ_A , and the true final equilibrium is reached.

Summarizing, when τ_A is large and excitation occurs in the outer layers of the core, we would expect a sudden change in the rate of westward drift, but not a sudden change in the angular velocity of the mantle. Some support for this picture is to be found in the work of VESTINE (1952) and VESTINE and KAHLE (1968). From an analysis of the westward drift of the eccentric dipole field, VESTINE and KAHLE (1968) infer that only the uppermost 200 km of the core participates in the angular momentum exchange with the mantle. A further discussion is given by BALL *et al.* (1968), who also adduce evidence for a phase lag of 7 ± 2 years between the fluctuations in ω_m and the motion of the eccentric dipole. This they attribute to electromagnetic time delay, through conduction in the mantle. As YUKUTAKE observes (private communication, 1972) however, the basic implication that the drift velocity of the eccentric dipole is representative of the whole geomagnetic field is arguable.

Although this qualitative description is devoted to the case of large τ_A , one might surmise that it has some measure of validity even when Eq. (1.26) to Eq. (1.28) hold.

Instead of seeking an excitation mechanism at the top of the core, the existence of the Alfvén wave mechanism opens up the possibility that the mantle acquires its angular momentum as a by-product of reflection at the core surface of an upward travelling Alfvén wave generated deep in the core by a convulsion of the dynamo mechanism. Diffusive effects in the mantle influence the phase relationship between flow and field during reflection (Section 4), and this affords an alternative mechanism for at least part of the phase lag reported by Ball, Kahle and Vestine. We may also note that, on this picture, the angular velocity difference between core and mantle will not necessarily increase rapidly initially; unlike the previous model, there is no net applied torque in the upper layers to represent local sources of disturbance.

Some of these notions are given a more quantitative expression in

Section 4 through a crude shellular model (ROBERTS, 1954) in which fluidity is represented by a core structured as a continuous set of rigid spherical shells each of which is free to move about the geocentre. If the ideas of HIGGINS and KENNEDY (1971), that the core possesses a high degree of static stability, stand the test of time, such a model might well provide a viable basis for further progress. If, however, the core is in a gravitationally neutral state, a better model exists (BRAGINSKIĬ, 1970; see also ROBERTS and SOWARD, 1972), which allows properly for the dominance of Coriolis forces. In this picture, a disturbance in the core would generate inertial waves of ultra-high frequency, MAC waves which vary slowly, and rapid torsional oscillations. Here MAC stands for Magnetic, Archimedian (bouyancy) and Coriolis forces; see BRAGINSKIĬ (1967), and ROBERTS and SOWARD (1972). The inertial waves are probably of little interest because of their extreme (semi-diurnal) frequency. The MAC waves are probably a significant ingredient of the dynamo process, but have periods of the order of hundreds of years which render them irrelevant here. The torsional oscillation may be pictured as an Alfvén wave carried across a core structured as a continuous set of rigid *cylindrical* shells each of which is free to move about a common axis parallel to the earth's angular velocity, Ω . Unlike the MAC waves, for which inertia plays little part, the Coriolis force has no influence on the torsional oscillation, and the inertial terms are required to balance the Lorentz force arising from the stretching of the lines of poloidal field by the zonal motions.

It is convenient to picture a disturbance in the core in its component parts: inertial, MAC and torsional waves. The torsional oscillation communicates with the rest of the core rapidly. For example, BRAGINSKIĬ (1970) integrated several explicit examples for which the r.m.s. radial field emanating from the core was only a little more than half a gauss; the fundamental periods ranged from 70 to 150 years, depending on model. Scaling in the obvious way, it seems that, for a mean vertical field of 3.5 gauss at the core surface, the periods are of the order of 10 to 20 years. [The smaller of these times allows for the rapid increase in asymmetric fields with depth into the core which is a consequence of the dynamo theory developed by BRAGINSKIĬ. The Alfvén speed is increased so far beyond Eq. (1.29) that the inner part of the core can be considered as rigid for the time-scales of interest here.] The fact that the period of the torsional oscillation is close to the diffusion time of the mantle probably implies that ohmic dissipation in the mantle is the most potent mechanism for damping these waves (Section 4). It might also have a bearing on the phase lag required by BALL *et al.* (1968).

It may also be noted that, unlike the shellular model considered earlier,

every cylinder is in direct contact with the mantle so that, no matter where the source of excitation lies in the core, its geostrophic (torsional) component transmits a torque to the mantle almost at once. This may partially account for the observed sudden onset of fluctuations in angular speed of the mantle. It would seem, however, that the findings of VESTINE (1952) and VESTINE and KAHLE (1968) mentioned above would require re-interpretation in terms of the cylindrical shells furthest from the axis of rotation, rather than in the uppermost spherical shells as described earlier. The mechanical effects of reduced inertia while the disturbance remains near S would not, however, seem to be very different. The fluctuations in westward drift should be most marked at small core latitudes.

It would appear from this qualitative discussion that, although further study of the magnetohydrodynamical problem is called for, the final results may not be radically different from the solid core model considered above.

It seems clear that, if electromagnetic coupling fails, it does so by at most a factor of 10 in torque magnitude, and a factor of 2 in time-scale. HIDE (1969) has argued, however, that the situation is sufficiently unsatisfactory to warrant a search for other viable mechanisms. In particular, he has called attention to the probable non-sphericity of the core boundary (see also GARLAND, 1957), and has pointed out that the interaction of core flow with the bumps on S must create stresses on S . In contrast with estimates by ELSASSER and TAKEUCHI (1955) who had earlier considered this mechanism and found it wanting, Hide found plausible reasons why these 'topographical' stresses should be large, of order $\rho\Omega\mu h$ in fact, where h is the mean height of the bumps. This should be compared with the magnetic surface stresses which, as we have seen, are of order $B_r B_t / \mu \xi \simeq B_t^2 / \mu \xi R$. On current ideas of core magnetohydrodynamics, B_t^2 / μ is of order $\rho\Omega u a$ (e.g., ROBERTS and SOWARD, 1972). Thus the magnetic stresses on S are of order $\rho\Omega u (a/\xi R)$. If $h/a > 1/\xi R \simeq 10^{-8}$ (say), Hide's topographical coupling should be the more important; Hide estimates h/a to be about $3 \cdot 10^{-4}$ ($h \simeq 1$ km). It is clear that, if electromagnetic coupling proves quantitatively satisfactory, it will still be necessary to understand why Hide's mechanism should not be more effective.

The remainder of this article consists in the main of calculations in support of statements already made. In Section 2, we present a method of evaluating the electromagnetic torque for slow changes in field. This is sufficiently flexible to admit spatially varying fluid flow over S , and not just solid body rotation. In Section 3, rapidly varying disturbances are envisaged, and the response of the mantle to applied torques is considered, on the assumption that the core is rigid: the presence of the new time-scale, τ_e , is exhibited. In Section 4, some remarks are made about the magnetohydrodynamic problems raised by core-mantle interactions.

2. Torques Produced by Slow Rearrangement of Flux Tubes

Select a frame of reference fixed in the mantle, which for simplicity has the simple structure defined in Section 1. Let the core be a perfectly conducting fluid moving with speed $\mathbf{u}$. The fluid motions on S rearrange the flux tubes emanating from the core and therefore cause the earth's field to fluctuate. As ROBERTS and SCOTT (1965) observed from a comparison of time-scales, this probably accounts for much of the observed secular variation at the earth's surface. They noted that the hypothesis required the vertical components of secular variation and main field at the base of the mantle to be related by

$$\frac{\partial}{\partial t}(rB_r) + \text{div}(r\mathbf{B}, \mathbf{u}) = 0, \text{ on } S. \quad (2.1)$$

In evaluating the second term, u_r may be set zero, but not $\partial u_r / \partial r$ (except when poloidal velocities, *i.e.*, ζ in Eq. (2.14) below, are excluded). Roberts and Scott suggested that Eq. (2.1) might be used as a constraint to improve the accuracy of secular variation analyses; others thought that Eq. (2.1) could lead to a complete knowledge of $\mathbf{u}$ on the core surface. Both ROBERTS and SCOTT (1965) and BACKUS (1969) emphasized that this could not be the case, Backus providing a particularly clear description of what could, and could not, be legitimately inferred about $\mathbf{u}$ on S . For our present discussion, Eq. (2.1) provides a boundary condition to which the solutions of the induction equation,

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B}, \quad (2.2)$$

in the mantle must be subjected. Once the induced field has been determined, the couple $\mathbf{\Gamma}_F$ can be evaluated.

The general theory is intricate, although it is in principle easy to obtain the Laplace transform of $\mathbf{B}$ and therefore, by convolution, $\mathbf{\Gamma}_F$ also. Significant simplification occurs if it is supposed, as we shall now do, that $\mathbf{u}$ on S is small compared with a/τ_η ($\simeq 0.02$ m/s); this is certainly true of the westward drift for which $u \simeq 3.5 \cdot 10^{-4}$ m/s at the core equator, but is possibly not as well satisfied for local turbulence near the core surface. Since the magnetic Reynolds number $u\tau_\eta/a$ is small, we may solve the induction problem by iteration. To leading order ($\tau_\eta = 0$), the mantle is an insulator, and the field within it is that obtained by downward extrapolation of the field measured on and near the earth's surface ($r = r_e$), *i.e.*, the field, $\mathbf{B}$, is

$$\mathbf{B}_0 = -\nabla \Phi_0, \quad r \geq a, \quad (2.3)$$

where

$$\Phi_0 = r_e \sum_{n=1}^{\infty} \sum_{m=0}^n [g_{nm}(t) \cos m\phi + h_{nm}(t) \sin m\phi] (r_e/r)^{n+1} \lambda_{nm} P_n^m(\theta). \quad (2.4)$$

Here P_n^m is Ferrar's associated Legendre function, and λ_{nm} is the normalization factor necessary to convert it to the geophysically more common Schmidt form; g_{nm} and h_{nm} are the usual harmonic coefficients; (r, θ, ϕ) are spherical polar coordinates.

In the next order, we write

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1, \quad (2.5)$$

and suppose that $|\mathbf{B}_1|/|\mathbf{B}_0|$ is of order $u\tau_n/a$, that is to say, small. The induction equation then gives, on linearization,

$$\nabla^2 \mathbf{B}_1 = \eta^{-1} \partial \mathbf{B}_0 / \partial t, \quad (2.6)$$

which implies that

$$\nabla^2 \mathbf{j}_1 = 0, \quad (2.7)$$

where $\mathbf{j}_1 = \text{curl} \mathbf{B}_1 / \mu$ is the current induced in the mantle by $\mathbf{B}_0$. Continuity across S of the tangential component, $\mathbf{r} \times \mathbf{E}$, of electric field $\mathbf{E}$ requires, to the same order that

$$\mu \eta \mathbf{r} \times \mathbf{j}_1 = -(\mathbf{r} \cdot \mathbf{B}_0) \mathbf{u}, \quad \text{on } S, \quad (2.8)$$

while $j_{1r} = 0$ on $r = b$.

It is useful to divide $\mathbf{B}$ into its toroidal and poloidal parts, writing

$$\mathbf{B} = \text{curl} T \mathbf{r} + \text{curl}^2 S \mathbf{r}. \quad (2.9)$$

Then Eq. (2.7) gives at once

$$\nabla^2 T_1 = 0, \quad \nabla^4 S_1 = 0, \quad (2.10, 2.11)$$

while Eq. (2.8) shows that

$$\frac{\partial}{\partial r} (r T_1) = -\frac{r}{\eta} \phi, \quad \nabla^2 S_1 = \frac{1}{\eta} \zeta, \quad \text{on } S. \quad (2.12, 2.13)$$

Here, following BACKUS (1969), we have represented $\mathbf{u}$ on S by

$$\mathbf{B}_0 \cdot \mathbf{u} = r \nabla \zeta(\theta, \phi) + \mathbf{r} \times \nabla \phi(\theta, \phi). \quad (2.14)$$

On using our knowledge of Φ_0 at the earth's surface, and by assuming a form for $\mathbf{u}$ consistent with Eq. (2.1), we may in principle expand ϕ and ζ in harmonic series with known coefficients:

$$\phi = \sum_{n=1}^{\infty} \sum_{m=0}^n (p_{nm}^c \cos m\phi + p_{nm}^s \sin m\phi) P_n^m(\theta), \quad (2.15)$$

$$\zeta = \sum_{n=1}^{\infty} \sum_{m=0}^n (k_{nm}^c \cos m\phi + k_{nm}^s \sin m\phi) P_n^m(\theta). \quad (2.16)$$

Equations (2.10) to (2.13) now yield

$$T_1 = \frac{a}{\eta} \sum_{n=1}^{\infty} \sum_{m=0}^n (p_{nm}^c \cos m\phi + p_{nm}^s \sin m\phi) \left[\frac{(b/r)^{n+1} - (r/b)^n}{n(b/a)^{n+1} + (n+1)(a/b)^n} \right] \times P_n^m(\theta), \quad (2.17)$$

$$r^2 S_1 = \frac{1}{\eta} \sum_{n=1}^{\infty} \sum_{m=0}^n (k_{nm}^c \cos m\phi + k_{nm}^s \sin m\phi) \left(\frac{a}{r} \right)^{n+1} P_n^m(\theta). \quad (2.18)$$

Equation (2.18) may be solved completely subject to the demand (say) that the entire field observed at the earth's surface is contained in S_0 (then S_1 and $\partial S_1 / \partial r$ vanish on $r=b$). The addition of this potential field would not affect the couple, Γ_F , and we do not therefore trouble to take this last step.

The electromagnetic couple on the mantle is

$$\Gamma = \int \mathbf{r} \times (\mathbf{j} \times \mathbf{B}) dV. \quad (2.19)$$

This may be reduced to the surface integral

$$\Gamma = \frac{1}{\mu} \int (\mathbf{r} \times \mathbf{B})(\mathbf{B} \cdot d\mathbf{S}). \quad (2.20)$$

To obtain Eq. (2.20) the fact that $\mathbf{r}$ and $d\mathbf{S}$ are parallel has been used; $d\mathbf{S}$ is drawn outwards from the mantle.

In the present case Γ_{F_0} is clearly zero since $\mathbf{j}_0 = 0$. To compute Γ_{F_1} , we treat the toroidal and poloidal parts of $\mathbf{B}_1$ separately. The contribution from S_1 is computed from Eq. (2.19), which gives

$$\Gamma_{F_1} = \int \mathbf{r} \times (\mathbf{j}_1 \times \mathbf{B}_0) dV. \quad (2.21)$$

We restrict ourselves for simplicity to the z -component. Then, by Eqs. (2.18) and (2.21), we have

$$\Gamma_{F_1}^{(S)} = -\frac{1}{\mu} \int \mathbf{r} \frac{\partial \Phi_0}{\partial r} \frac{\partial}{\partial \phi} r^2 S_1 dV,$$

or

$$\Gamma_{F_{1z}}^{(S)} = -\frac{4\pi a^4}{\mu\eta} \sum_{n=1}^{\infty} \sum_{m=1}^n \frac{m(n+1)}{(2n-1)(2n+1)\lambda_{nm}} (k_{nm}^c h_{nm} - k_{nm}^s g_{nm}) \times \\ \times \left(\frac{r_e}{a}\right)^{n+2} \left[1 - \left(\frac{a}{b}\right)^{2n-1}\right]. \quad (2.22)$$

To evaluate the contribution from T_1 , we use Eq. (2.20) which gives

$$\Gamma_{F_{1z}}^{(T)} = -\frac{a^3}{\mu} \int \frac{\partial \Phi_0}{\partial r} \frac{\partial T_1}{\partial \theta} \sin^2 \theta d\theta d\phi, \quad (2.23)$$

or

$$\Gamma_{F_{1z}}^{(T)} = -\frac{4\pi a^4}{\mu\eta} \sum_{n=1}^{\infty} \sum_{m=0}^n \frac{1}{\lambda_{nm}} \left(\frac{r_e}{a}\right)^{n+2} \left[\frac{(n+1)(n+2)(n+m+1)}{(2n+3)(2n+1)q_{n+1}} (g_{nm} p_{n+1,m}^c + \right. \\ \left. + h_{nm} p_{n+1,m}^s) - \frac{(n+1)(n-1)(n-m)}{(2n-1)(2n+1)q_{n-1}} (g_{nm} p_{n-1,m}^c + h_{nm} p_{n-1,m}^s) \right], \quad (2.24)$$

where

$$q_n = [n(b/a)^{n+1} + (n+1)(a/b)^n] / [(b/a)^{n+1} - (a/b)^n]. \quad (2.25)$$

It is worth noting that, in the particular case of solid body rotation about Oz , for which $\zeta = 0$ and

$$\frac{\partial \phi}{\partial \theta} = -\omega r \frac{\partial \Phi_0}{\partial r} \sin \theta, \quad (2.26)$$

only T_1 contributes to Γ_F , and expression Eq. (2.23) gives

$$\Gamma_{F_{1z}} = \frac{a^2}{\omega \mu} \int \frac{\partial \phi}{\partial \theta} \frac{\partial T_1}{\partial \theta} \sin \theta d\theta d\phi,$$

or by Eqs. (2.15) and (2.17)

$$\Gamma_{F_{1z}} = \frac{2\pi a^3}{\omega \mu \eta} \sum_{n=1}^{\infty} \sum_{m=0}^n [(p_{nm}^c)^2 + (p_{nm}^s)^2] \frac{[2n(n+1) - m(2n+1)]}{(2n+1)\lambda_{nm}^2 q_n} = K\omega. \quad (2.27)$$

It is clear from Eqs. (2.25) and (2.27) that Γ_{F_z} has the same sign as ω , a result which is not unexpected from the physical standpoint, but which is not transparent in the formulation of ROCHESTER (1960).

The Eqs. (2.22) and (2.24) of the two components of Γ_{F_z} completes the theory. As already noted, the formulation is less general than that of ROCHESTER (1960) since high velocities on S are excluded; it is more general in the sense that an arbitrary surface velocity may be assumed, and not merely a solid body rotation.

3. Electromechanical Response of the Mantle to a Sudden Change

Suppose that at time $t=0$ the fluid velocity $\mathbf{u}$ on the surface of the core suddenly changes. For time short compared with τ_T^2 , the currents induced by this disturbance will not have penetrated far from S , and they may, correct to leading order, be considered to lie in a boundary layer, or electromagnetic skin, at the base of the mantle. The thickness of this layer will be proportional to $\eta_a^{1/2}$, where η_a is the magnetic diffusivity at the base of the mantle. (This may be small compared with the mean diffusivity through the mantle; see section 1.)

We introduce the Laplace transformation, denoted by a superimposed bar, and defined for a function $f(t)$ by

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t) dt.$$

In the boundary layer approximation just described, the Laplace transform of Eq. (2.2) gives

$$\mathbf{r} \times \bar{\mathbf{B}}^{(+)} = (\mathbf{r} \times \bar{\mathbf{B}}_a) \exp[-(s/\eta_a)^{1/2}(\mathbf{r}-\mathbf{a})], \quad (3.1)$$

where $\mathbf{B}_a(\theta, \phi, t)$ is the field on $r=a$; θ and ϕ dependencies will be suppressed below but are, of course, to be understood. Naturally B_r is constant through the boundary layer, since $\text{div } \mathbf{B}=0$.

We have written $\mathbf{B}^{(+)}$ on the left of Eq. (3.1) to emphasize that the curl is taken on the mantle side of S , *i.e.*, the limit $r \rightarrow a+$ is considered. While on the topic of surface discontinuities, we should note that, since the core is assumed perfectly conducting, eddy currents will be produced on its surface as $\delta\omega$ varies. Thus $\mathbf{r} \times \mathbf{B}$ will be discontinuous as, therefore, will be the surface stresses and the expression (2.20) for the couple, $\mathbf{\Gamma}$. This reflects the fact that surface currents contribute to the net couple on the core. The correct expression for $\mathbf{\Gamma}$ is obtained if $\mathbf{B}^{(+)}$ is used in Eq. (2.20) irrespective of whether the couple on the core or the mantle is under consideration (although, of course, in the former case the sign of $\mathbf{\Gamma}$ must be reversed, since $d\mathbf{S}$ is oppositely directed).

From Eq. (3.1), we obtain

$$\mathbf{r} \times (\partial \bar{\mathbf{B}}^{(+)} / \partial r)_{r=a} = -(s/\eta_a)^{1/2} \mathbf{r} \times \bar{\mathbf{B}}_a. \quad (3.2)$$

Now, by continuity of $\mathbf{r} \times \mathbf{E}$, we have [*cf.* Eq. (2.8)]

$$\eta_a \mathbf{r} \times \text{curl} \mathbf{B}_{r=a}^{(+)} = -(\mathbf{r} \cdot \mathbf{B}_a) \mathbf{u}. \quad (3.3)$$

On combining with Eq. (3.2), recalling that the gradient operator is asymptotically radial in the boundary layer approximation, we find

$$\mathbf{r} \times \bar{\mathbf{B}}_a = -(s\eta_a)^{-1/2}(\mathbf{r} \times \bar{\mathbf{B}}_{ar}\mathbf{u}), \quad (3.4)$$

so that, by the convolution theorem,

$$\mathbf{r} \times \mathbf{B}_a = - \int_0^t B_{ar}(t-t')[\mathbf{r} \times \mathbf{u}(t-t')] \frac{dt'}{\sqrt{(\pi\eta_a t')}}. \quad (3.5)$$

Substituting into Eq. (2.20), we have

$$\mathbf{\Gamma} = \frac{1}{\mu} \int_0^t dt' \int dS \frac{B_{ar}(t)B_{ar}(t-t')}{\sqrt{(\pi\eta_a t')}} [\mathbf{r} \times \mathbf{u}(t-t')]. \quad (3.6)$$

Consider, for example, a situation in which $\mathbf{B}$ is axially symmetric and a perturbation $\delta\mathbf{u}$ in $\mathbf{u}$ is initiated at $t=0$ which has the form of a solid body rotation with axis parallel to the symmetry axis. Then we have

$$\delta\mathbf{u} = \delta\boldsymbol{\omega} \times \mathbf{r}, \quad (3.7)$$

and the change $\delta\mathbf{\Gamma}$ in the couple $\mathbf{\Gamma}$ has only a z -component which is given by

$$\delta\Gamma_z(t) = \int_0^t G(t')\delta\omega(t-t')dt', \quad (3.8)$$

where

$$G(t) = K/(\pi\tau_\eta^a t)^{1/2}, \quad (3.9)$$

and τ_η^a is now defined not by Eq. (1.4) but by

$$\frac{K}{\sqrt{\tau_\eta^a}} = \frac{1}{\mu\sqrt{\eta_a}} \int (rB_{ar}\sin\theta)^2 dS > 0. \quad (3.10)$$

The equations of motion of the mantle and core (assumed solid) give

$$I \frac{d}{dt} \delta\omega = - \int_0^t G(t')\delta\omega(t-t')dt' + \delta\Gamma(t), \quad (3.11)$$

where $\delta\Gamma$ is the applied couple (which, it is supposed, tends to $\delta\Gamma_0$ as $t \rightarrow \infty$). Equivalently, in Laplace transformed variables, Eq. (3.11) becomes [for $\delta\omega(0)=0$]

$$\bar{\delta\omega} = \bar{\delta\Gamma}/[Is + \bar{G}(s)], \quad (3.12)$$

which, when $t \ll \tau_\eta^a$ and Eq. (3.9) applies, gives

$$\frac{\bar{\delta\omega}}{\bar{\delta\omega}_0} = \frac{(s\tau_\eta^a)^{1/2}}{1 + \chi(s\tau_\eta^a)^{3/2}} \left[\frac{\bar{\delta\Gamma}}{\bar{\delta\Gamma}_0} \right], \quad (3.13)$$

where $\chi = \tau_I / \tau_\eta^a$ and $\delta\omega_0 = \delta\Gamma_0 / K$ is the final steady state value of $\delta\omega$.

We considered in Section 1 two examples. In the first, $\delta\Gamma$ is a step function torque, *i.e.*, a sudden constant couple applied at $t=0$. Then $\delta\bar{\Gamma} = \delta\Gamma_0/s$, and Eq. (3.13) inverts at once to give Eq. (1.24) with $\tau_c = \tau_I^{2/3} \tau_\eta^{a/3}$. It may be seen that, if $\tau_\eta^a \gg \tau_I$ ($\chi \ll 1$), $\delta\omega$ rises in a time of order τ_c to a peak of order $\delta\omega_0/\chi^{1/3} \gg \delta\omega_0$. According to Eq. (1.24), it then returns to zero. This corresponds to the fact that, for $t \ll \tau_\eta^a$, the mantle conductivity is essentially infinite and the lines of force are wound up until the Maxwell stresses with which they are associated balance $\delta\Gamma_0$; the mantle therefore acquires a net angular displacement but not a net angular velocity. As t becomes large compared with τ_c , diffusion becomes potent, and $\delta\omega$ approaches $\delta\omega_0$ in a time of order τ_η^a .

For the second example, we suppose that $\delta\Gamma$ increases on a diffusive time-scale, τ_E , taking

$$\delta\Gamma = (t/\pi\tau_E)^{1/2} \delta\Gamma_0, \quad (3.14)$$

for the times $t \ll \tau_\eta^a$ of interest. Now $\delta\bar{\Gamma} = \delta\Gamma_0/2s(\pi\tau_E)^{1/2}$, and we obtain

$$\delta\omega = \frac{\delta\omega_0}{3} \left(\frac{\tau_\eta^a}{\tau_E} \right)^{1/2} \left[3 - e^{t/\tau_c} \operatorname{erfc} \sqrt{\frac{t}{\tau_c}} - e^{\omega^2 t/\tau_c} \operatorname{erfc} \omega \sqrt{\frac{t}{\tau_c}} - e^{\omega t/\tau_c} \operatorname{erfc} \omega^2 \sqrt{\frac{t}{\tau_c}} \right]. \quad (3.15)$$

For the case $\tau_\eta \gg \tau_I$, it is still true that $\delta\omega$ grows on the τ_c time-scale, but the peak value is now only of order $\delta\omega_0$.

We have now developed a theory good for both short times and (Section 2) for long. The intermediate case is intricate and we again confine ourselves to the case of axisymmetry. The poloidal parts of $\mathbf{B}$ are not shielded by the mantle and we may write (*cf.* Eq. 2.4)

$$B_r = \sum_{n=1}^{\infty} (n+1) g_{n0} (r_e/r)^{n+2} P_n(\theta), \quad r \geq a. \quad (3.16)$$

By Eqs. (2.2) and (3.3), we now have

$$B_r^{(+)}(a, \theta, t) = \int_0^t \mathcal{H}(\theta, t') \delta\omega(t-t') dt', \quad (3.17)$$

where the Green's function $\mathcal{H}(\theta, t)$ may be obtained by recourse to standard induction theory for spherical conductors (BULLARD, 1949; HERZENBERG and LOWES, 1957). We readily find that

$$\mathcal{H}(\theta, s) = \frac{a^2}{\eta} \sum_{n=1}^{\infty} \frac{(n+1)}{(2n+1)} g_{n0} \left[\frac{1}{\alpha_{n-1}} \frac{\partial P_{n-1}}{\partial \theta} - \frac{1}{\alpha_{n+1}} \frac{\partial P_{n+1}}{\partial \theta} \right] \left(\frac{r_e}{a} \right)^{n+2}, \quad (3.18)$$

where

$$\alpha_n(s) = -\frac{1}{2} - ka \left[\frac{I_{n+1/2}'(ka)K_{n+1/2}(kb) - I_{n+1/2}(kb)K_{n+1/2}'(ka)}{I_{n+1/2}(ka)K_{n+1/2}(kb) - I_{n+1/2}(kb)K_{n+1/2}(ka)} \right], \quad (3.19)$$

and $k^2 = s/\eta_a$, where $R(k) > 0$ for $-\pi < \arg s < \pi$, the s plane being cut along the negative real axis. We may note two limiting forms of α_n :

$$\alpha_n(0) = q_n, \quad (3.20)$$

$$\alpha_n(s) \sim ka, \quad s \rightarrow \infty. \quad (3.21)$$

Also, if $L \ll a$ and $n \ll a/L$, we have (for all except large values of s)

$$\alpha_n = a/L + O(nL/a), \quad (3.22)$$

a result independent of n to leading order.

If we now use Eq. (3.17) and expression (2.20) for Γ_z , we find that

$$\begin{aligned} \bar{G}(s) = \frac{4\pi a^5}{\mu\eta} \sum_{n=1}^{\infty} \left\{ \frac{(n+1)^2 g_{n,0}^2}{(2n+1)} \left[\frac{n(n-1)}{(2n-1)\alpha_{n-1}} + \frac{(n+1)(n+2)}{(2n+3)\alpha_{n+1}} \right] \right. \\ \left. - \frac{2n^2(n+1)(n+2)g_{n-1,0}g_{n+1,0}}{(2n-1)(2n+1)(2n+3)\alpha_n} \right\} \left(\frac{r_e}{a} \right)^{2n+4}. \end{aligned} \quad (3.23)$$

From Eqs. (3.9) and (3.10) we see that $\bar{G}(0)$ is a constant (equal to K , say) which agrees with that predicted for the theory of Section 2. For $s \rightarrow \infty$, Eq. (3.21) reduces Eq. (3.23) to

$$\bar{G}(s) = K(s\tau_\eta^a)^{-1/2}, \quad (3.24)$$

where τ_η^a is given by Eq. (3.10). In other words, Eq. (3.9) is recovered precisely. The superfix on τ_η^a is from now onwards omitted.

We now see that an expression for $G(t)$ which is exact for small and large times, and is presumably qualitatively correct for all t , is given by

$$\bar{G}(s) = K/[1 + \sqrt{(s\tau_\eta)}]. \quad (3.25)$$

This leads to the expression (1.17). To solve the equation of motion (3.11) for this more general form of $G(t)$, we again proceed by Laplace transformation, obtaining from Eqs. (3.12) and (3.25) the result

$$\frac{\bar{\delta\omega}}{\bar{\delta\omega}_0} = \frac{[1 + \sqrt{(s\tau_\eta)}]}{\{1 + \chi s\tau_\eta[1 + \sqrt{(s\tau_\eta)}]\}} \left(\frac{\bar{\delta\Gamma}}{\bar{\delta\Gamma}_0} \right), \quad (3.26)$$

in place of Eq. (3.13). For the case of a step function torque, $\bar{\delta\Gamma}/\bar{\delta\Gamma}_0 = 1/s$, and, being a rational function of $\sqrt{s}$, Eq. (3.26) can be inverted as a sum of error functions. Let x_1 , x_2 and x_3 be the roots of

$$x^3 + \chi x - \chi = 0, \quad (3.27)$$

where x_1 is real and positive, and $x_2 = x_3^*$. Then Eq. (3.26) yields

$$\delta\omega = \delta\omega_0 \left[1 + \frac{x_1(1-x_1)}{(x_1-x_2)(x_1-x_3)} W\left(\frac{t}{x_1^2\tau_\gamma}\right) + \frac{x_2(1-x_2)}{(x_2-x_3)(x_2-x_1)} W\left(\frac{t}{x_2^2\tau_\gamma}\right) + \frac{x_3(1-x_3)}{(x_3-x_1)(x_3-x_2)} W\left(\frac{t}{x_3^2\tau_\gamma}\right) \right], \quad (3.28)$$

where

$$W(z) = e^z \operatorname{erfc} \sqrt{z}, \quad (3.29)$$

If $\tau_I \gg \tau_\gamma$, then $\chi \gg 1$, and the roots of Eq. (3.27) are approximately $x_1 = 1$, $x_2 = i\sqrt{\chi}$ and $x_3 = -i\sqrt{\chi}$. We may now use the result that (for real y) $\operatorname{erf} iy = i \operatorname{erf} y$ to show that, except for small and large t , Eq. (1.23) follows. If $\tau_I \ll \tau_\gamma$, then $\chi \ll 1$, and the roots of Eq. (3.27) are approximately $\chi^{1/3}$, $\omega\chi^{1/3}$ and $\omega^2\chi^{1/3}$, and Eq. (1.24) follows at once. The expansions (1.21) and (1.22) for small and large t follow most readily from, respectively, the large and small s expansions of Eq. (3.26).

Suppose now that, in place of an initial step function in $\delta\Gamma$, this couple grows at the rate defined by the electromagnetic diffusion time, τ_γ , itself. Then in place of $\delta\bar{\Gamma} = \delta\Gamma_0/s$ we have

$$\delta\bar{\Gamma} = \delta\Gamma_0 / \{s[1 + \sqrt{(s\tau_\gamma)}]\}, \quad (3.30)$$

and by Eq. (3.26) we have

$$\frac{\delta\omega}{\delta\omega_0} = \frac{1}{s[1 + \chi s\tau_\gamma[1 + \sqrt{(s\tau_\gamma)}]]}. \quad (3.31)$$

This expression is readily inverted as before and, in place of Eq. (3.28), we find

$$\delta\omega = \delta\omega_0 \left[1 - \frac{x_1^2}{(x_1-x_2)(x_1-x_3)} W\left(\frac{t}{x_1^2\tau_\gamma}\right) - \frac{x_2^2}{(x_2-x_3)(x_2-x_1)} W\left(\frac{t}{x_2^2\tau_\gamma}\right) - \frac{x_3^2}{(x_3-x_1)(x_3-x_2)} W\left(\frac{t}{x_3^2\tau_\gamma}\right) \right]. \quad (3.32)$$

Results (1.23) hold for large χ , while when $\chi \ll 1$ we must amend Eq. (1.24) to Eq. (1.25). The expansions for small t and large t which replace Eqs. (1.21) and (1.22) are

$$\delta\omega = \frac{4\delta\omega_0}{3\sqrt{\pi}} \left(\frac{t}{t_c}\right)^{3/2} + O(t^2), \quad t \rightarrow 0, \quad (3.33)$$

$$\delta\omega \sim \delta\omega_0 \left[1 + \frac{1}{2} \pi \chi \left(\frac{\tau_\eta}{\pi t} \right)^{3/2} + O(t^{-5/2}) \right], \quad t \rightarrow \infty. \quad (3.34)$$

4. A Shellular Core Model

It has already been noted in Section 1 that the solid core model of Section 3 may be over simplified, since it excludes what may be a significant physical process, namely Alfvén radiation in the core. As far as the author is aware, he was the first (ROBERTS 1954, 1955a, reported by RUNCORN 1954) to infer the existence of Alfvén waves in the core from a study of the secular variation. The Alfvén wave was then seen as a means of resolving a troublesome paradox: if a solid has its conductivity of the core, no electromagnetic phenomenon varying with secular variation rapidly can traverse more than a few kilometres without severe attenuation by the ‘skin effect.’ It seemed, therefore, that the sources of secular variation must be sought in the upper few kilometres of the core (LOWES and RUNCORN, 1951). In order to produce secular variation of the right magnitude in such a thin ‘skin,’ however, currents of implausible intensity were required to flow within it (*cf. e.g.*, HERZENBERG and LOWES, 1957). Roberts observed that, if the fluidity of the core were restored and a poloidal field of only a few gauss were postulated, the Alfvén waves then present could carry the disturbance over many multiples of the radius of the earth without serious attenuation. For example, the Alfvén velocity $V = B_r / (\mu\rho)^{1/2}$ for a poloidal field B_r of 3.5 gauss is about $3.5 \cdot 10^{-3}$ m/s (taking the core density, ρ , to be 10^4 kg/m³). Such a wave will pass from geocentre to core surface in about $\tau_A = 35$ years. Assuming a core diffusivity, η , of $3 \text{ m}^2/\text{s}$ (corresponding to a conductivity of $3 \cdot 10^5$ mho/m), a disturbance of wave length $l = 500$ km could travel on the wave for a time of order $l^2/\eta \simeq 2500$ years before being reduced in amplitude by a factor of $1/e$, *i.e.*, for a time of order 35 core transits. (Indeed, dissipation in the mantle during reflection of the wave at the core surface may well be a more significant attenuation process; see below.) It seems clear today that, in its neglect of Coriolis forces on wave propagation, Roberts’s argument left much to be desired (as indeed, HIDE and ROBERTS, 1961, recognized). It is nevertheless valid for a significant class of motions, the so-called ‘geostrophic modes,’ or ‘torsional oscillations’. These were discussed briefly in Section 1. Here we will exclude rotational effects.

It is well known that the surface currents, which featured on the solid core of the model of Section 3, cannot be supported by a fluid conductor (except in the irrelevant case of zero B_r ; see ROBERTS, 1955b). On the reasonable assumption of zero magnetic Prandtl number, all components

of $\mathbf{B}$ must be continuous across $r=a$, and it is no longer necessary to distinguish between values of $\mathbf{B}^{(+)}$ and $\mathbf{B}^{(-)}$. The surface current which would form on a solid core is radiated away as it is created by Alfvén waves (STEWARTSON, 1957). There is a discontinuity in $\mathbf{u}$ on S however, and the demand of no slipping is resolved by a Hartmann layer (*cf. e.g.*, ROBERTS, 1967, Ch. 6).

To examine the effect of Alfvén radiation, we adopt a crude model (ROBERTS, 1954) in which the fluidity of the core is only partially allowed for; the core is supposed to be structured as an infinite number of concentric rigid shells rather like an onion. Each shell is free to move relative to its neighbour without friction, and with perfect electrical contact. The angular velocity of the shell of radius r is taken to be $\omega_c(r, t)$ and

$$\omega(r, t) = \omega_m(t) - \omega_c(r, t). \quad (4.1)$$

Two consequences of the remaining departure from full fluidity should be noted. First, the rigidity of the shells is a constraint which reduces the effective Alfvén velocity across the core; second, some remnants of the surface currents present in Section 3 still remain, but the net torque they produce is no longer non-zero.

This shellular model bears a superficial resemblance to a concept of ELSASSER and TAKEUCHI (1955). They have, however, explicitly excluded Alfvén wave motion, as may be seen from their Eq. (6), and the excitations of the upper core which they consider therefore have the characters of (turbulent) Hartmann and Ekman layers.

To examine the effect of core motion, we specialize the model further; as in Section 2, we suppose axisymmetry with respect to Oz . We introduce

$$M_c(r, t)dr = \frac{8\pi\rho r^4}{3}\omega_c(r, t)dr, \quad (4.2)$$

which is the angular momentum of the shell of radius r and thickness dr . The couple on the entire interior of this shell is

$$\Gamma_F(r, t) = \frac{2\pi r^3}{\mu} \int_0^\pi B_r(r, \theta) B_\theta(r, \theta, t) \sin^2\theta d\theta, \quad (4.3)$$

so that the equation of conservation of moment of momentum is

$$\frac{\partial M_c}{\partial t} = \frac{\partial \Gamma_F}{\partial r} - g(r, t), \quad (4.4)$$

where $g(r, t)$ is the applied couple per unit thickness of the shell. On differentiating (4.3) with respect to time using the induction equation

$$\frac{\partial B_\theta}{\partial t} = r B_r \frac{\partial \omega_c}{\partial r},$$

for a perfectly conducting core, we obtain

$$\frac{\partial \Gamma_F}{\partial t} = V^2(r) r^4 \frac{\partial}{\partial r} \left(\frac{M_c}{r^4} \right), \quad (4.5)$$

where

$$V^2 = \frac{3}{4\mu\rho} \int_0^\pi B_r^2(r, \theta) \sin^3 \theta d\theta \quad (4.6)$$

is a mean Alfvén velocity; if B_r is provided by a uniform field (B_0) as we will suppose below, V is a constant ($= B_0 / \sqrt{5\mu\rho}$). The resemblance between Eqs. (4.4) and (4.5) to the familiar equations relating the transverse components of field and flow in an Alfvén wave is striking. We may note that the general solution to Eqs. (4.4) and (4.5) in the case $g=0$ is composed of a linear combination of outgoing (lower signs) and ingoing (upper signs) waves of the form

$$M_c = \left(r^2 \frac{\partial^2}{\partial r^2} - r \frac{\partial}{\partial r} \right) h \left(t \pm \frac{r}{V} \right), \quad (4.7)$$

$$\Gamma_F = \pm V \left(r^2 \frac{\partial^2}{\partial r^2} - 3r \frac{\partial}{\partial r} + 3 \right) h \left(t \pm \frac{r}{V} \right). \quad (4.8)$$

The linear combination used below ensures that $M_c = O(r^4)$ and $\Gamma_F = O(r^5)$ as $r \rightarrow 0$, as required. The quantity,

$$\Gamma(t) = \int_0^a g(r, t) dr, \quad (4.9)$$

is the net applied couple to the mantle necessary to preserve the total momentum of the core-mantle system.

In our first application of the theory, we suppose that the accelerating couple on the core is applied only to a thin layer near its surface. This picture is not unattractive if we believe that the changing electromagnetic couple between core and mantle is created by a local disturbance near the core boundary as, for example, would occur in the upwelling of toroidal field in the formation of "core spots" (ALLAN and BULLARD, 1966). Mathematically, we may describe the system most conveniently in terms of a limit. It is supposed that g vanishes except in the layer $a - \varepsilon \leq r \leq a$; later the limit $\varepsilon \rightarrow 0$ is taken. We may denote by $S^{(+)}$ the true top surface of the core, and by $S^{(-)}$ the lower surface of the layer in which $\delta\Gamma$ acts. Since, as $\varepsilon \rightarrow 0$, the inertia of the layer is negligible, the electromagnetic

couple $\delta\Gamma_F^{(+)} - \delta\Gamma_F^{(-)}$ must balance the applied couple $\delta\Gamma$, *i.e.*,

$$I_m \frac{d}{dt} \delta\omega_m = -\delta\Gamma_F^{(+)} + \delta\Gamma = -\delta\Gamma_F^{(-)}. \quad (4.10)$$

Also, the argument which led to Eq. (1.15) is still valid provided we replace the term on the left by $I_m d\delta\omega_m/dt$ (to which it was equal in that section): in this way, we obtain

$$\delta\Gamma_F^{(+)}(t) = \int_0^t \left[\delta\omega_m(t-t') - \frac{3\delta M_c(a, t-t')}{8\pi\rho a^4} \right] G(t') dt'. \quad (4.11)$$

Finally since g is zero for $r < a$, δM_c and $\delta\Gamma_F$ are given by Eqs. (4.7) and (4.8).

On taking Laplace transforms and assuming a step-function applied torque $\delta\Gamma_0$, we obtain

$$\overline{\delta\omega_m} = \delta\Gamma_0 / [s\{sI_m + \overline{G}[1 + I_m q(sa/V)/5I_c]\}], \quad (4.12)$$

$$\overline{\delta\omega} = [1 + I_m q(sa/V)/5I_c] \overline{\delta\omega_m}, \quad (4.13)$$

where

$$q(z) = z^2 [z \cosh z - \sinh z] / [(3 + z^2) \sinh z - 3z \cosh z], \quad (4.14)$$

and $\delta\omega$ denotes the angular velocity difference between the top surface of the core and the base of the mantle, a quantity related to the westward drift. The zeros of $q(z)$ are associated with the normal modes of magneto-hydrodynamic oscillation. These increase the difficulties of inversion. It is clear enough however that, since $q(0) = 5$, $\delta\omega_m$ and $\delta\omega$ ultimately ($s=0$; $\overline{G}=K$) settle down to the correct steady state values. It is also easy to make progress for small values of t , by expansion about $s=\infty$. We obtain

$$\delta\omega_m = \frac{I\delta\omega_0}{I_m} \left(\frac{t}{\tau_I} \right) \left[1 - \frac{4I_m}{15\sqrt{\pi}(I_m + I_c)} \left(\frac{\tau_A t^{1/2}}{\tau_I \tau_\eta^{1/2}} \right) + O(t) \right], \quad (4.15)$$

$$\delta\omega = \frac{I\delta\omega_0}{I_c} \left(\frac{\tau_A}{\tau_I} \right) \left[1 - \frac{2I_m}{5\sqrt{\pi}(I_m + I_c)} \left(\frac{\tau_A t^{1/2}}{\tau_I \tau_\eta^{1/2}} \right) + O(t) \right]. \quad (4.16)$$

The rapid response of the upper layers to the applied torque is evident. If we suppose (Section 1) that $\tau_A^2 = 0(\tau_I \tau_\eta)$, we see that the ratio of second to first terms in each series is of the order $\sqrt{(t/\tau_I)}$, and both $\delta\omega$ and $\delta\omega_m$ grow on the τ_I time-scale. The larger τ_A , the more rapid the initial westward drift Eq. (4.16).

As an alternative to supposing that the excitation occurs in the uppermost layers, we may suppose that the mantle is set into motion by a transient Alfvén wave travelling outwards from a source deep in the core.

The highly accelerated surface layer of the last model is absent and, in a first treatment, we will ignore mantle motion entirely, simply asking how the field, B_ϕ , and the couple it creates will grow with time. A particularly clear example is provided by the interaction of a plane Alfvén wave impinging normally on the plane boundary $z=0$ of a fixed uniform mantle $z>0$. The tangential components of flow and field in the fluid are

$$u = V \left[f \left(t - \frac{z}{V} \right) + p \left(t + \frac{z}{V} \right) \right], \quad (4.17)$$

$$b = B_0 \left[-f \left(t - \frac{z}{V} \right) + p \left(t + \frac{z}{V} \right) \right]. \quad (4.18)$$

Here f is supposed known (the incoming wave) and p is to be determined (the reflected wave). The field at the base of the mantle is clearly

$$b(0, t) B_0 [-f(t) + p(t)] = B_0 h(t), \text{ say.} \quad (4.19)$$

Solving the induction equation, we have within the mantle

$$b^{(+)}(z, t) = \int_{-\infty}^t h(t-t') \frac{ze^{-z^2/4\eta t'}}{2\sqrt{(\pi\eta t'^3)}} dt', \quad (4.20)$$

whence

$$\left(\frac{\partial b^{(+)}}{\partial z} \right)_{z=0} = \int_{-\infty}^t \frac{h(t-t')}{2\sqrt{(\pi\eta t'^3)}} dt'. \quad (4.21)$$

Continuity of the electric field, E_y , at $z=0$ now requires

$$u(0, t) = \eta \left(\frac{\partial b^{(+)}}{\partial z} \right)_{z=0} = \int_{-\infty}^t \left(\frac{\eta}{4\pi t'^3} \right)^{1/2} h(t-t') dt'. \quad (4.22)$$

We can now determine p and h from f , using Eqs. (4.17), (4.18) and (4.22). For definiteness, suppose a sharp-fronted incoming wave, *i.e.*,

$$f(z-Vt) = k_0 [1 - H(z-Vt)] \quad (4.23)$$

where H is Heaviside's unit function. Then, taking Laplace transforms, we obtain

$$\bar{p}(s) = \frac{k_0 [\sqrt{(\eta s/V^2)} - 1]}{s [\sqrt{(\eta s/V^2)} + 1]}, \quad \bar{h}(s) = -\frac{2k_0}{s [\sqrt{(\eta s/V^2)} + 1]}. \quad (4.24, 4.25)$$

It is remarkable that the induced field $B_0 h(t)$ at the base of the mantle, and therefore the stress on the mantle also, obeys precisely the law governing H in the theory of Section 3, provided the τ_η of that expression is replaced by η/V^2 ; compare Eq. (4.25) with Eq. (3.13). Thus the form of $h(t)$

is that shown for H in Fig. 1, with the scale of the abscissa amended appropriately. Moreover, the theory of Section 3 can be applied with little error. For example, when $\tau_\eta = \eta/V^2$ (i.e., when the Lundquist number VL/η is of order 1), the evolution of $\delta\omega$ is that shown in Fig. 2 for the gradually applied torque. As V is increased, the response will increasingly resemble that of the step-function torque. The time-scale η/V^2 is essentially τ_A^2/τ_η , and is equal to τ_η , when Eqs. (1.26) to (1.28) are adopted. If we take $\eta = 3 \times 10^3 \text{ m}^2/\text{s}$ ($\sigma = 300 \text{ mho/m}$) and $V = 3.5 \times 10^{-3} \text{ m/s}$ ($B_0 \simeq 3.5 \text{ gauss}$), we find that $\eta/V^2 \simeq 7 \text{ years}$ and $VL/\eta \simeq 1.2$. The results just reported may be extended with some difficulty to the full shellular model.

The fact that τ_η^{-1} is of the same order as the frequency, τ_A^{-1} , of the Alfvén waves at the core boundary implies that the mantle damps these waves extremely efficiently. An example of the absorption of normally incident Alfvén waves at plane conducting boundaries has been considered by HIDE and ROBERTS (1962); see also ROBERTS (1967, p. 140). They showed that, when aV/η is small, the surface is essentially a node of b and an anti-node of u , and that the dissipation rate is small [$O(V^2/\eta)$]; when aV/η is large, the surface is a node of u and an anti-node of b , and the dissipation rate is again small [$O(\eta/a^2)$]. Between these two extremes, a state of maximum dissipation occurs for which* $aV/\eta \simeq 1.275$ and the e -folding time of the field is then about $3.115 V/a$.

Similar considerations hold for the geostrophic mode of BRAGINSKIĬ (1970), and it seems that mantle conduction provides the most potent means by which these torsional oscillations are damped. Presumably similar results will follow for the shellular model.

* Hide and Roberts did not consider the fundamental mode, for which the unnumerical results stated here apply.

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