Hydromagnetic Propagation in the Magnetosphere: A Mechanical Analogue for the Study of the Initial Value Problem of Coupled Modes

Henry R. RADOSKI

Air Force Cambridge Research Laboratories, Bedford, Massachusetts, U.S.A.

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Several alternative analyses of the steady state solutions of coupled modes have been proposed. In determining which, if any, steady state solution is of physical significance, the manner in which the hydromagnetic system evolves from an initial disturbance would be decisive. The basic initial value problems are those of field line excitation and relaxation. These two cases correspond to the energy being initially in the poloidal or toroidal modes. These difficult problems are investigated by establishing the relevance of a simple mechanical analogue which is studied in detail. This is accomplished in three stages: (a) a rectangular model of the plasmasphere is described and its physical characteristics are compared with those of a dipole field magnetized plasma; (b) a mechanical system is derived consisting of an elastic string with each point of which is associated an oscillator. For the symmetric modes with weak ion-cyclotron coupling, the wave equations for the hydromagnetic and mechanical systems are essentially identical; (c) the mechanical system is reduced to an elementary wave-oscillator model representing the coupling between resonant poloidal and toroidal modes. The characteristic motion of the simplified system is independent of the strength of the coupling. The solutions suggest that resonant field lines tend to reflect an incident poloidal wave and transmit only an evanescent wave. The toroidal mode does not become singular but attains a finite amplitude in performing this reflection. In general, it is concluded that the behavior of coupled and uncoupled modes is essentially different. When any coupling is present, steady state solutions, as normally understood, are not possible.

1. Introduction

Because of the geometric complexity of the magnetosphere, the only feasible way to study its characteristic oscillations is analytically. In other words, the large scale vibrations of the magnetosphere must be considered as an interaction of the oscillations of its fundamental structural elements, principally the inner magnetosphere or plasmasphere, the outer magnetosphere and the geomagnetic tail. The tail is the portion most amenable to study, since as a good first approximation the ambient magnetic field is uniform (McCLAY and RADOSKI,

1967; PATEL, 1968; MCKENZIE, 1970; ERSHKOVICH and NUSINOV, 1972). The outer magnetosphere is the most difficult section for study. Its ensemble of both open and terminated, deformed dipole-field lines makes the construction of simplified models for quantitative results difficult. This region may require further subdivision if any advance is to be made beyond qualitative estimates. At present, the plasmasphere is the most interesting region for quantitative study. The approximate axisymmetry of the inner magnetosphere lends itself to modeling. For example, one can begin with a plasma magnetized by a pure dipole field and then idealize to simpler geometries retaining the essential physical characteristics of the plasmasphere. Each magnetic field line in this region can have a characteristic period associated with it, essentially proportional to its length. Hence, there will be a spatially dependent continuous spectrum with periods ideally ranging from a maximum at the plasmopause to zero at the equator. For a given mode, these field lines can be considered to be a set of harmonic oscillators. As long as these oscillators do not interact no problems arise. Then the hydromagnetic wave equation would describe a set of discrete, spatially independent, cavity or poloidal modes and a set of continuous, spatially dependent, field line or toroidal modes. A unique situation develops when the toroidal oscillators are made to interact with each other through coupling to the poloidal mode which then can propagate energy between field line oscillators at the Alfvén speed. Although this interaction is certainly extraordinary in nature, it is likely to be common in occurrence: the wave equations in general couple the poloidal and toroidal fields and non-uniformities in the magnetosphere would tend to do the same. Since two fundamentally different types of modes are coupled, it is not immediately evident that a steady state solution can exist. If there should be eigenmodes, the most obvious question concerns the nature of the resulting modes when continuous and discrete spectra are blended together. If there are no steady state solutions, the problem becomes one of defining the basic temporal response of the system to some realistic initial disturbance.

Some comment should be made on the question of a non-existent steady state in a system without damping. Of course, even for coupled hydromagnetic waves, it is possible to assume a harmonic factor of the form $e^{i\omega t}$ to separate the time dependence in the wave equation. By there being no steady state is meant that the only *physically realizable* solutions are those for which the various Fourier components must be integrated over all possible frequencies. For example consider, $\Psi(x, \omega) = \delta(\omega \pm x/A)$, as the solution of the Fourier component of a particular field variable. In fact, such a singular distribution is the only possible solution of the toroidal electric field for the case of completely uncoupled axisymmetric modes (RADOSKI and McCLAY, 1967). In the above expression, x , is analogous to the parameter specifying a given field line oscillator, such as L for a dipole field, and A is the characteristic propagation speed; in this case, the Alfvén speed. Then the physical solution, completely specified by the initial

conditions is given in terms of an integral of the singularities over the continuous spectrum.

$$\Psi(x, t) = \int_{-\infty}^{+\infty} (C_1 \delta(\omega - x/A) + C_2 \delta(\omega + x/A)) e^{i\omega t} d\omega = C_1 e^{i(x/A)t} + C_2 e^{-i(x/A)t}$$

where $C_{1,2}$ are constants determined by the conditions

$$\Psi(x, t_0) \quad \text{and} \quad \partial \Psi(x, t_0) / \partial t$$

at the initial time t_0 . The integration of an infinite set of singular delta functions renders the behavior in space and time non-singular and non-pathological. The above result is not peculiar to problems of hydromagnetic wave propagation. An essentially similar result occurs for a special case of the problem of plasma oscillations in a non-uniform, unmagnetized plasma (BARSTON, 1964; SEDLÁČEK, 1971a, b). Both problems involve the study of plasmas characterized by spatially-dependent frequencies. In the hydromagnetic case, there occur spatially dependent field line frequencies produced by field lines of changing length and/or changing Alfvén speed along the lines. In the plasma oscillation case, there occur spatially dependent plasma frequencies produced by a non-uniform electron number density. In both cases there is the possibility of localized resonances at those points in the plasma where the characteristic frequencies are the same as an impressed frequency. It is shown in Appendix 1 that as the limit of field line resonance is approached the hydromagnetic wave equation becomes identical in form to the plasma oscillation equation. This is an interesting result but too specialized for direct application to the plasmasphere problem.

As a first approximation the ambient field in the plasmasphere can be considered to be a simple dipole field. An orthogonal coordinate system based on the geometry of the ambient field would appear to offer the most suitable means of writing the wave equation. Although the dipole field is the simplest magnetic multipole field, unfortunately the hydromagnetic wave equation in dipole coordinates does not share in its simplicity. The principal difficulty is that the equation is non-separable in the two coordinates specifying the direction along the field lines and at right angles to them in a meridian plane. Because the dipole field has axial symmetry the azimuthal dependence can be factored out of the coupled wave equation but the resulting two dimensional problem remains intractable. Hence, only limited success has been obtained in manipulating the equations in dipole coordinates for very special cases such as: assuming incompressibility (GAJEWSKI, 1964; INFELD, 1964), perfect field line guidance (CUMMINGS *et al.*, 1969), neglecting coupling between the two modes (RADOSKI, 1972), or considering only axisymmetric solution (RADOSKI and CAROVILLANO, 1966; CAROVILLANO *et al.*, 1966). Therefore, it has become expedient to seek an elucidation of the phenomena associated with plasmaspheric oscillations by developing simplified models of the plasmasphere. A cylindrical model has received the most study (RADOSKI, 1970; MCCLAY, 1970) and a rec-

tangular model has been proposed (RADOSKI, 1971a). In these models the mathematics describing wave propagation and boundary conditions are more conducive to treatment and interpretation. The eigenfunctions for variations along the field lines can be specified and separated, reducing the problem to a single dimension. Thus far, only steady state solutions, assuming a harmonic time dependence, have been sought. If it is assumed that there is no mechanism for damping in the system, all the solutions have singularities or discontinuities in those regions where the poloidal and toroidal modes are in resonance with each other. Physically, these regions are field lines or shells of field lines which have the same period as the cavity oscillation. At present, it is not known how, or even whether, such singularities or discontinuities can develop in time; nor is it clear what sort of phenomena are associated with an integration of such modes over their associated spectra. To resolve these uncertainties it would be necessary to study the initial value problem for the hydromagnetic wave equation to discover what forms the temporal development of the system is able to take, starting from a realistic initial state. To perform such a calculation directly is difficult since the eigenfunctions in closed form are not known, at present, for any of the idealized models no matter how simplified they have been made. Therefore, an indirect method of attacking the problem has been sought. A promising approach may be found in the similarity that has been noted between the coupled hydromagnetic wave equation and an interesting mechanical system (RADOSKI, 1971b). This system is composed of an elastic string loaded at each point with a mass-spring harmonic oscillator. The infinite set of oscillators have resonant frequencies which vary from point to point along the string. These oscillators represent the set of field line oscillators of the toroidal mode with its spatially dependent characteristic frequencies. The elastic string represents the poloidal mode which couples all the oscillators together and can also propagate its own type of waves. This mechanical analogue has several positive features: it presents a system of coupled modes that is easy to visualize; the equations describing the wave propagation are a good representation of the hydromagnetic equations; and, most important, it affords an obvious means of further simplifying the analysis. If the equations governing the mechanical system merely were almost identical to those in the hydromagnetic system, there would be no essential advantage in trying to solve the former since they would be just as difficult. However, since it is possible to simplify the mechanical analogue, a basic physical structure can be isolated which is representative of the resonant interaction of the poloidal and toroidal modes. The behavior of this distillate of the original mechanical system can be predicted exactly and can be conceptualized and understood easily. The following paragraphs of this paper are to contain: the formulation and explanation of the mechanical analogue; the solution of basic initial value problems; and the conclusions to be drawn from these solutions with respect to plasmaspheric waves and resonances. Our hope is to

develop a suggestive hypothesis and an approach in sharp contrast to direct attacks on the problem, whose success has been, at best, limited.

2. Basic Equations

2.1 The hydromagnetic system

In this section we shall present the fundamental hydromagnetic wave equation. In order to treat the case of weak coupling, a small correction term of first order in the ratio of the wave frequency to the ion-cyclotron frequency is retained. Without this term, the symmetric modes would be completely uncoupled and the asymmetric modes would be strongly coupled. In the case of a dipole field, symmetric modes are those which are independent of longitude. In other geometries they are independent of that variable, which is the obvious analogue of longitude. With a small ion cyclotron term, the symmetric modes remain weakly coupled. In the following analysis rectangular coordinates are used so that the differential operators and the geometry will be at their simplest. Nothing of importance should be lost by choosing this geometry. The essential problem involves the exchange of energy between the poloidal and toroidal mode. The physics of the exchange should be the same independent of rectangular, cylindrical or, even, dipole coordinates. Furthermore, although one certainly expects the rate of energy exchange to depend on whether the coupling is weak or strong the nature of that energy transfer should not change. Whether a singularity eventually does or does not develop in time at a resonant field line should be independent of the fine details of the coupling between the poloidal and toroidal modes. The hydromagnetic wave equation in terms of the electric field, E , in a fully ionized plasma of infinite conductivity, can be expressed (e.g. RADOSKI, 1968) as

$$\begin{aligned} [\nabla \times (\nabla \times E)]_{\perp} - (\omega/A)^2 E_{\perp} &= i(\omega/A)^2 \beta_+ \boldsymbol{\tau} \times E \\ E_{\parallel} &= 0 \end{aligned} \quad (1)$$

The fields are assumed to vary as $e^{i\omega t}$, A is the Alfvén speed; $\perp, \parallel$ refer to components perpendicular and parallel to the ambient magnetic field $\mathbf{B} = B_0 \boldsymbol{\tau}$, where $\boldsymbol{\tau}$ is a unit vector along a field line; $\beta_+ = \omega/\omega_+$ is weak coupling parameter, where ω_+ is the ion-cyclotron frequency and it has been assumed that $\beta_+^2 \ll 1$.

The rectangular geometry to be used is shown in Fig. 1 along with the dipole geometry to indicate the correspondence of the coordinates and the boundaries. The z direction is along the field lines; x is normal to the field lines and is the analogue of ν which is chosen opposite to the principal normal in the meridian planes of the dipole field; y is the analogue of longitude, ϕ . In order to represent the periodicity of ϕ , a periodic boundary condition may be assumed for the y variable. The vectors $\boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{\phi}$ and z, x, y are right handed triads. Although it requires some imagination to picture the cutting and stretching

Fig. 1. Correspondence between a section of an ideal plasmasphere in dipole coordinates and a section of the hydromagnetic box in rectangular coordinates B and D are northern and southern sections of the boundaries at the surface of the earth. A is the field line marking the inner extent of the plasmasphere; and C is the field line at the plasmopause.

required for the topological transformation of the plasmasphere into the hydromagnetic box, the two models can be made to coincide in the essential feature of characteristic period. The characteristic period of a field line is defined as twice the round trip travel time along the field line at the Alfvén speed. In the case of the plasmasphere the field lines have characteristic periods which change principally because the field lines increase in length from the equator towards the polar regions. In the box model the field lines are all the same length. Therefore, to produce the same physical effect of spatially dependent field line periods the Alfvén speed is made a function of position. For a dipole field, where for simplicity the Alfvén speed is assumed independent of radial distance, i.e., $A = (1 + 3 \cos^2 \theta)^{1/2} A_0$ and A_0 is a constant, the period T is

$$T = 2 \int_{(\text{field line})} ds/A = (4/A_0)[L(L-1)]^{1/2}$$

where L is the equatorial distance in earth radii; for the hydromagnetic box,

$$T = 4z_0/A(x)$$

Therefore, if we choose $A = A_0[x(x+1)]^{-1/2}$, where x is in units of z_0 , the periods agree when we associate $x = L - 1$. In fact, a simple but suitable choice for the Alfvén speed in the box would be $A = A_0/x$. Then $x = 0$ would represent the zero period field line at the equator. The dipole result approaches this form at large L values.

We assume the simplest boundary conditions of perfect reflection at the walls: $E_x = E_y = 0$ at $z = \pm z_0$, which corresponds to reflection at the ends of the field lines at the earth; and $E_y = 0$ at $x = 0, x_0$ which corresponds to reflection from the plasmasphere boundaries. For the earth, the inner plasmasphere boundary may be brought down to the surface without any loss of generality.

The electric field components are written as

$$\begin{aligned} E_x(x, z, t) &= T(x) \sin[k_z(z - z_0)]e^{i\omega t} \\ E_y(x, z, t) &= iP(x) \sin[k_z(z - z_0)]e^{i\omega t} \end{aligned}$$

where $T(x)$ is the toroidal or guided mode amplitude and $P(x)$ is the poloidal or cavity mode amplitude. In order to study the case of weak coupling, the fields are taken to be independent of the y coordinate. The coupled "symmetric" modes are obtained from the components of the wave Eq. (1) as a pair of one dimensional equations

$$[(\omega/A)^2 - k_z^2]T = -\beta_+(\omega/A)^2P \quad (2)$$

$$[d^2/dx^2 + (\omega/A)^2 - k_z^2]P = -\beta_+(\omega/A)^2T \quad (3)$$

where $2k_z z_0 = n\pi$. It is important to understand the nature of the completely uncoupled results. If we ignore the weak β coupling for the moment, Equation (2) exemplifies the usual toroidal mode solution with spatially dependent frequencies $\omega = k_z A(x)$ or periods $T = 4z_0/A(x)$ as calculated above. This is a guided mode since there is no magnetic field component, parallel to the ambient field, associated with it. With both the parallel electric and magnetic variations zero, the Poynting vector only can be directed along the ambient magnetic field lines. The poloidal mode is represented by the wave Eq. (3). This is a cavity mode since the eigenvalues depend on both boundaries, x as well as z , and, hence, on the size of the cavity. If we choose $A = A_0(z_0/x)$ with A_0 a constant then the differential equation can be put in the form of Weber's equation (MILLER, 1955). By scaling x as $\xi = (2\omega/z_0 A_0)^{1/2}x$, the poloidal equation becomes

$$[d^2/d\xi^2 + (\xi^2/4) - n^2 t]P(\xi) = 0 \quad (4)$$

where $t = (\pi^2 A_0 / 8\omega z_0) = T/T_0$, $T_0 = (4/\pi)(4z_0/A_0)$. Corresponding to the boundary condition $P(x) = 0$ at $x = 0$ and $x = x_0 = Lz_0$, where L is a constant, $P(\xi)$ is subject to $P(\xi) = 0$ at $\xi = 0$ and $\xi = L(\pi/2)t^{-1/2}$. For a particular case of the fundamental mode $n=1$, with $L=5$ the eigenperiod would be approximately $t=2.5$. With the scale length $z_0 = 1R_e$ and $A_0 = 10^8$ cm/sec, the period would be about 81 seconds which is not unreasonable for the poloidal mode (CAROVILLANO *et al.*, 1966).

As can be seen above, the eigenvalue, t , appears in both the differential equation and the boundary condition. Therefore, this equation cannot be solved as clearly as one would want for an ideal example. This is the principal reason why the problem in rectangular coordinates has not received much attention. Other forms of the Alfvén speed function $A(x)$ might give a more tractable uncoupled poloidal mode but then the analogy with the dipole results would be weakened. For this paper such mathematical complications are academic since we do not intend to solve the coupled wave equations directly.

When coupling is considered the problem becomes much more interesting. This is due to the toroidal resonances now appearing as singularities in the poloidal equation. No matter how small the coupling parameter β_+ may be, as long as it is not identically zero, these singularities cannot be ignored. Nor can they be avoided by a judicious choice of ω ; e.g. by choosing frequencies lower

than the lowest field line frequencies. Physically such solutions are not possible since they would demand that the box resonate at frequencies less than any of its characteristic frequencies. Using the same values for the Alfvén speed and the scale length as in the poloidal example, the continuous toroidal spectrum covers the period range zero to 132 seconds. Since the uncoupled poloidal mode periods are 81 seconds or less, all of the poloidal characteristic frequencies have field lines with which they can be in resonance. What occurs in the resonant regions is not obvious. For the toroidal oscillator Eq. (2) it is seen that the poloidal amplitude acts as a driving force upon the field line oscillators. If the poloidal amplitude were a constant, then at resonance, the toroidal mode amplitude would increase without limit, in the absence of damping. But the poloidal amplitude may not be constant, since the toroidal amplitude reacts on it through the wave equation, Eq. (3). The ultimate effect of this feed-back may be to cause the poloidal amplitude to tend towards a null at the position of the resonating field line. Certainly, this is a possible solution to Eq. (2) when $\omega/A(x) = k_z$. However, this solution is also difficult to understand, since it implies that the poloidal solutions with very weak coupling are *fundamentally different* from the uncoupled poloidal solutions, which are well-behaved, continuous functions. It has been shown for a cylindrical geometry, the “hydromagnetic wedge”, that there are no poloidal solutions which have a null at the resonant field lines and also satisfy the boundary conditions (RADOSKI, 1970). If a null is required then only discontinuous solutions are allowed. It is very unlikely that this result is dependent on the choice of geometry. Such discontinuous solutions are as unsatisfactory as singular solutions to equations which have been assumed to be linear and to have small amplitude field variations.

Of course, if there were no simple harmonic solutions of the form $e^{i\omega t}$ then none of these difficulties would arise. But this generates the problem of determining just what class of temporal variations these basic, but non-uniform, hydromagnetic systems can sustain. However, it does seem unusual that the uncoupled poloidal and toroidal modes should oscillate very well separately, but not at all when coupled together.

Since the systems under discussion are lossless, it has been tacitly assumed that the frequency, ω , is real. If we include some sort of damping in the equations the frequency becomes complex and the resonance may be circumvented. But the insertion of a damping mechanism would be, at best, an ad hoc assumption which would tend to complicate a sufficiently difficult problem and mask the very resonance interaction we are trying to investigate. Furthermore, it is not reasonable to include a small damping term and neglect other physically meaningful corrections whose magnitude is commensurate or even larger than the damping term. For example: (a) The effect on the wave equation of damping from a finite conductivity $\sigma \simeq 10^{11} \text{ sec}^{-1}$ (McCLAY, 1970) is a term of order $(\omega/4\pi\sigma)(c/A)^2 = 4.5 \times 10^{-7}$, which is approximately the square of the ratio of skin-

depth to wave length for a wave of one second period and Alfvén speed 10^8 cm/sec. (b) The correction for the neglected gas pressure term is of order $(V_s/A)^2$ where V_s is the sound speed. For a temperature of $(10^3)^\circ\text{K}$ this ratio is 7.4×10^{-6} ; 16 times larger than the damping term. (c) For a wave of amplitude one gamma in an ambient field of one gauss the non-linear corrections such as the variation of the Alfvén speed itself, are of order 10^{-5} , 22 times larger than the damping term. It does not seem possible to attribute special significance to damping from finite conductivity to the exclusion of all other competing processes; nor is it practical to attempt to include all conceivable appendages to the wave equation. At the present time, there is no reason to doubt that the hydromagnetic wave equation given in Eq. (1) is sufficient in itself without further emendation.

2.2 The mechanical system

The several interlocking complications discussed in the paragraphs above should provide a sufficient rationale for the mechanical analogue. In this section we shall develop the mechanical model of the hydromagnetic system and compare its governing equations with the hydromagnetic wave equation. A special form of this model has been used previously (RADOSKI, 1971b). But in that case it was intuitively presented without any detailed justification of the analogy or comparison of the equations and with none of the relevant mathematics included. Since the relationships between the mechanical and hydromagnetic system is not manifestly obvious, substantial explanation is required especially with regard to boundary conditions and further simplifications.

By defining the spatially dependent, characteristic field line frequencies as $\omega_0(x) = k_z A(x)$, the coupled hydromagnetic wave Eqs. (2) and (3), can be rewritten in the proper form for analysis and comparison.

$$[\omega^2 - \omega_0^2(x)]T = -\beta_+ \omega^2 P \quad (5)$$

$$[A^2 d^2/dx^2 + \omega^2]P = \omega_0^2(x)P - \beta_+ \omega^2 T \quad (6)$$

As has been noted in the previous section, the toroidal Eq. (5) has the form of an oscillator, such as a simple mass and spring, which has a resonant frequency, $\omega_0(x)$, and the poloidal mode, P , acts as a driving force. The amplitude, T , represents the displacement of the oscillator from equilibrium. The effect of the poloidal amplitude is reduced by the small cyclotron coupling factor, β_+ . The left-hand side of Eq. (6) has the appearance of a one-dimensional wave equation, such as an elastic string, coupled to the set of toroidal oscillators. The coupling amplitude again is reduced by the factor β_+ .

Figure 2 illustrates a mechanical model which utilizes the above similarities with the hydromagnetic equations. A set of mass-spring oscillators is shown coupled to an elastic string stretched between the points 0 and x_0 . The number of oscillators is increased to the limit of a continuous system of oscillators with

Fig. 2. A mechanical system representing the case of weak interaction between the hydromagnetic toroidal and poloidal modes. The motion of the elastic string, $\mathcal{P}(x, t)$ is an analogue of the poloidal mode and the vibration of the set of oscillators, $\mathcal{T}(x, t)$, is an analogue of the toroidal mode. Coupling is through an attenuating system, a , described in the text.

distributed properties of mass per unit length, $\mu(x)$, and spring constant per unit length $\eta(x)$. The string is considered to have a tension τ and a variable density $\rho(x)$. The spring amplitudes are denoted by $\mathcal{T}(x)$ and the string amplitude by $\mathcal{P}(x)$ is analogy to the toroidal and poloidal amplitudes. The coupling is through an “attenuator”, a , which reduces the effect of the oscillators’ motion on the string and the string’s motion on the oscillators by a factor a , analogous to β_+ . In other words, the force per unit length on the string is $-\eta(\mathcal{P} - a\mathcal{T})$ and on the springs $-\eta(\mathcal{T} - a\mathcal{P})$. When $a=1$, the oscillators are directly attached to the string. The use of an ad hoc attenuator avoids a discussion of the possible mechanisms of the coupling system. In the weak coupling case, it is not considered of critical importance. For the sake of illustration, one type of coupling mechanism is detailed in Appendix 2.

The force equations governing the motion of the string and oscillators are

$$\mu(x)d^2\mathcal{T}/dt^2 = -\eta(x)(\mathcal{T} - a\mathcal{P}) \quad (7)$$

$$\rho(x)\partial^2\mathcal{P}/\partial t^2 = \tau\partial^2\mathcal{P}/\partial x^2 - \eta(x)(\mathcal{P} - a\mathcal{T}) \quad (8)$$

If a harmonic time dependence, $e^{i\omega t}$, is assumed with the definition $\tau/\rho(x) = A^2(x)$, for the square of the propagation speed along the string and $\eta(x)/\mu(x) = \omega_0^2(x)$, for the square of the natural frequency of the oscillators, these force equations can be rewritten as

$$[\omega^2 - \omega_0^2(x)]\mathcal{T} = -a\omega_0^2(x)\mathcal{P} \quad (9)$$

$$[A^2d^2/dx^2 + \omega^2]\mathcal{P} = (\mu/\rho)[\omega_0^2(x)\mathcal{P} - a\omega_0^2(x)\mathcal{T}] \quad (10)$$

When these mechanical equations are compared to the hydromagnetic Eqs. (5) and (6), the similarity in formalism becomes apparent. In fact, if the spring and string mass densities were equal, $\mu/\rho=1$, the only discrepancy between the two cases is that ω_0^2 appears instead of ω^2 in the coupling term. This difference in multipliers is totally insignificant, especially since these terms only become important near a resonance point, where $\omega_0^2 \simeq \omega^2$.

An important virtue of the mechanical model is also its main fault. The

model clearly is an accurate representation of the hydromagnetic system. Since it is an accurate representation, it follows that it must be as difficult to solve as the original problem. However, the principal virtue of the mechanical model is that it suggests a means of readily simplifying itself and the problem of coupled modes still further.

3. Model of the Resonant Interaction

The key to the final reduction of the problem is the realization that the essence of the problem is in the behavior of the interacting modes at the resonance points. The resonance points are defined as those spatial positions, x , such that $\omega = \omega_0(x)$, i.e., where the system's frequency coincides with a particular value of the continuous, spatially dependent spectrum of the springs' sub-system. Although these locations are referred to as "points", it should be remembered that for the hydromagnetic system an entire sheet of field lines can have the same value of x associated with it, analogous to the L value of a dipole field line. The importance of the resonance interaction clearly can be seen in the spring-string equations above. Equation (10) states that the propagation in the string (poloidal-type) mode is perturbed by a weak coupling to the spring (toroidal-type) mode. Equation (9) then shows that this perturbation can be only of importance near the resonance points. For example: if $\omega_0^2 \gg \omega^2$, then $\mathcal{T} \simeq a\mathcal{P}$, and the correction to the poloidal equation is of order a^2 ; if $\omega^2 \gg \omega_0^2$, the correction is even smaller. Therefore, if we can simplify the model but retain the resonance, the most important feature of the interaction still will be present in the modified model. This can be done by reducing the continuous spring system to a single resonating spring located at x_i . In this case ω_0 becomes a constant and the spring mass density is $\mu(x) = m_i \delta(x - x_i)$, where m_i is the mass loading the spring at x_i whose resonant frequency is ω_0 . The normal mode equations for this simplified mechanical system are

$$(\omega^2 - \omega_0^2)\mathcal{T} = -a\omega_0^2\mathcal{P}(x_i) \quad (11)$$

$$(A^2 d^2/dx^2 + \omega^2)\mathcal{P}(x) = (m_i/\rho)\omega_0^2\delta(x - x_i)(\mathcal{P} - a\mathcal{T}) \quad (12)$$

In the wave Eq. (12), nothing of critical importance to the physics of the problem is lost by considering the propagation speed, A , to be a constant. This assumption has the decided advantage of making the eigenfunctions of the string ordinary trigonometric functions. Changing from a continuum of oscillators to a point oscillator does introduce a discontinuity in the slope of the string at x_i , but this is not significant. Of course, such a major simplification as has been effected does involve some loss to the problem. Now it is not possible to retain the boundaries at $x=0$, x_0 and still easily study the analogue of a field line resonance. The reason for this is that the simplified problem *with boundaries* has no resonances of the type we wish to examine. The bounded spring-string

system can be solved in closed form. It has a discrete set of eigenfrequencies. But none of these frequencies can be equal to the original oscillator frequency, ω_0 . As is the case with the coupling of two simple oscillators, the coupling itself changes the original resonant frequencies. To be specific: with the appropriate boundary conditions of $\mathcal{P}=0$ at $x=0, x_0$, the eigenspectrum for the above mechanical system is determined readily by the continuity of the amplitude and the discontinuity of the slope of $\mathcal{P}$ at x_i .

$$\Phi_{\mathcal{S}} \cdot \Phi_{\mathcal{S}} = a^2$$

where $\Phi_{\mathcal{S}}=0$, is the uncoupled, ($a=0$), spring mode and $\Phi_{\mathcal{S}}=0$ is the uncoupled string mode

$$\begin{aligned} \Phi_{\mathcal{S}} &= (\omega/\omega_0)^2 - 1 \\ \Phi_{\mathcal{S}} &= \left[\frac{(\omega x_0/A)(\rho x_0/m_i)(A/\omega_0 x_0)^2 \sin(\omega x_0/A)}{\sin(\omega x_i/A) \sin[(\omega/A)(x_i - x_0)]} \right] - 1 \end{aligned}$$

The eigenperiods of the uncoupled string depend on several factors such as the length of the string, x_0 , the location of the spring, x_i , the mass ratio of string to spring, $\rho x_0/m_i$, and the ratio of characteristic string frequency, A/x_0 , to the spring frequency, ω_0 . For any value of these parameters and the coupling factor, a , the eigenvalues can be calculated. But for the principal problem before us these characteristic frequencies are not important since there is no spring of the same frequency to resonate with the system. In the general hydromagnetic or mechanical problem, no matter what the system frequency there always would be a location at which a field line or spring would have the same frequency, i.e., the physical parameters of the problem were such that $\omega_0(x)=\omega$ for some x in the domain of the system. The easiest way to restore this critical element of resonance, is to remove the boundaries. On an unbounded string waves of any frequency can propagate and, in particular, these waves may be of the same frequency as the resonant frequency of the coupled oscillator. In this manner, the resonant interaction can be studied completely isolated from and unencumbered by any extraneous features.

4. Initial Value Problems

In this section we explore two initial value problems for the mechanical system formulated above as a representation of the coupling between toroidal and poloidal hydromagnetic waves. The first case involves the behavior in time of the resonant interaction between a wave of frequency ω_0 and a weakly coupled oscillator of the same frequency. This is the problem of excitation. The second case to be developed is the complementary problem of the relaxation of the oscillator after the driving force has been removed.

The basic equations are obtained from the simplified mechanical model de-

scribed by Eqs. (11) and (12) with the time derivatives restored; i.e., $i\omega \rightarrow \partial/\partial t$ or d/dt . Since the string is now unbounded, the oscillator location is irrelevant and, for convenience, it is placed at the origin, $x_i=0$.

$$(d^2/dt^2 + \omega_0^2)\mathcal{T} = a\omega_0^2\mathcal{P}(x=0) \quad (13)$$

$$(A^2\partial^2/\partial x^2 - \partial^2/\partial t^2)\mathcal{P} = (m_i/\rho)\omega_0^2\delta(x)(\mathcal{P} - a\mathcal{T}) \quad (14)$$

It should be recalled that all the parameters in these equations, A , ω_0 , a , m_i , ρ are now constants, independent of the spatial variable, x . In the following analysis we shall not immediately impose the condition that the attenuation, a , is small. This is done to indicate that the physical nature of the result depends on the existence of coupling and is independent of the strength of this coupling.

4.1 The excitation problem

The physical situation for driving the oscillator is set up as follows. A wave of frequency ω_0 has been propagating along the string from the positive x direction to the left towards the oscillator at $x=0$. For $t < 0$,

$$\begin{aligned} \mathcal{P} &= \mathcal{P}_0 \sin \omega_0(t + x/A) & x > 0 \\ \mathcal{P} &\equiv 0 & x < 0 \end{aligned}$$

At the time, $t=0$, the wave just hits the motionless spring system. Therefore, the initial conditions are

$$\left. \begin{aligned} \mathcal{P}(x, t=0) &= \mathcal{P}_0 \sin (\omega_0 x/A) \\ \partial \mathcal{P}(x, t=0)/\partial t &= \mathcal{P}_0 \omega_0 \cos (\omega_0 x/A) \end{aligned} \right\} x \geq 0$$

$$\mathcal{P}(x, t=0) \equiv 0 \quad x < 0$$

$$\mathcal{T}(t=0) = d\mathcal{T}(t=0)/dt = 0$$

The basic questions to be answered are: (1) Does the wave drive the oscillator into resonance? (2) Towards what sort of steady state, if any, does the system evolve? It is shown below that the resonator's amplitude is finite. Its motion reradiates the incoming wave and tends to produce a total reflection of this wave. The steady state approached by the system consists of a standing wave to the right of the oscillator and an undisturbed string to the left. The motion of the point of contact between the spring and string is, in general, a *damped* sinusoid. Therefore, when there is any coupling at all between the wave and oscillator, this point *always* tends to become a *node*. It should be recalled that such a null was a possible, although unusual, solution for the steady state hydromagnetic system at the resonant field line.

This initial value problem is solved by using the Laplace transform on the time variable and imposing the boundary conditions on the spatial functions at the coupling point, $x=0$. The transforms of the wave amplitude $\mathcal{P}(x, t)$ and the oscillator amplitude $\mathcal{P}(t)$ are denoted by $\Pi(x, s)$ and $T(s)$. In terms of

these transforms the force Eqs. (13) and (14) become

$$\begin{aligned} (s^2 - A^2 d^2/dx^2) \Pi(x, s) + (m/\rho) \omega_0^2 \delta(x) [\Pi - aT] \\ = \begin{cases} s \mathcal{P}_0 \sin(\omega_0 x/A) + \omega \mathcal{P}_0 \cos(\omega x/A) & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (15) \\ (s^2 + \omega_0^2) T(s) = a \omega_0^2 \Pi(s, x=0) \quad (16) \end{aligned}$$

The boundary conditions at the oscillator, $x=0$, are the continuity of the amplitude $\Pi(s, x=0)$ and the discontinuity of the slope $d\Pi(s, 0)/dx$, which is obtained from the above equations as

$$A^2 d \Pi/dx \Big|_{x=0}^{\pm\epsilon} = (m/\rho) \omega_0^2 [s^2 + (1-a^2) \omega_0^2] \Pi(s, 0) / (s^2 + \omega_0^2) \quad (17)$$

The wave amplitude, $\Pi(x, s)$, must have the following spatial variation to the left and right of the coupling point.

$$\Pi(s, x) = \tau(s) e^{sx/A} \quad x \leq 0 \quad (18)$$

$$\begin{aligned} \Pi(s, x) = [\mathcal{P}_0 / (s^2 + \omega_0^2)] [s \sin(\omega_0 x/A) + \omega \cos(\omega_0 x/A)] \\ + \mathcal{R}(s) e^{-sx/A} \quad x \geq 0 \quad (19) \end{aligned}$$

In Eq. (18), $\tau(s)$ represents the transmitted amplitude and, in Eq. (19), the first term is the incident wave and $\mathcal{R}(s)$ represents the reflected amplitude. The algebraic expressions for the amplitudes $\tau(s)$ and $\mathcal{R}(s)$ are obtained by solving the two simultaneous equations produced by application of the boundary conditions to the above forms of the wave amplitude $\Pi(s, x)$.

$$\begin{aligned} \tau(s) &= [\omega_0 \mathcal{P}_0 / (s^2 + \omega_0^2)] + \mathcal{R}(s) && \text{continuity} \\ (s/A) [\omega_0 \mathcal{P}_0 / (s^2 + \omega_0^2) - \mathcal{R}(s) - \tau(s)] &= (m/\rho) (\omega_0/A)^2 \cdot \\ & \quad ([s^2 + (1-a^2) \omega_0^2] / (s^2 + \omega_0^2)) \tau(s) && \text{slope} \end{aligned}$$

The results for the transmitted and reflected amplitudes are

$$\tau(s) / \mathcal{P}_0 = \omega_0 s / \{s(s^2 + \omega_0^2) + \Omega[s^2 + (1-a^2) \omega_0^2]\} \quad (20)$$

$$\begin{aligned} \mathcal{R}(s) / \mathcal{P}_0 = -([s^2 + (1-a^2) \omega_0^2] / (s^2 + \omega_0^2)) \\ \cdot (\omega_0 \Omega / \{s(s^2 + \omega_0^2) + \Omega[s^2 + (1-a^2) \omega_0^2]\}) \quad (21) \end{aligned}$$

The rate, Ω , is defined as $\Omega = (1/2)(m/\rho)(\omega_0^2/A)$ and is the resonant frequency, ω_0 , multiplied by a factor proportional to the ratio of spring mass to the string mass contained in a wave length.

The behavior of the mechanical system is contained in the poles of the above transforms of the transmission and reflection coefficients. The most important results are governed by the zeroes of the cubic form

$$D(s) = s^3 + \Omega s^2 + \omega_0^2 s + \Omega(1-a^2) \omega_0^2 \quad (22)$$

which appears in both of the denominators. Before obtaining and interpreting the singularities for an arbitrary coupling coefficient, a , it is instructive to ex-

amine the extreme cases of no coupling, $a=0$, and strong coupling, $a=1$.

When there is no contact between the oscillator and the string, obviously the oscillator remains motionless, as indicated by its Eq. (16). At the point of contact, the wave encounters only a spring whose other end is fixed as if to a *rigid* support and, hence, has no resonant properties. For this case the zeroes of $D(s)$ are $s_1 = -\Omega$, $s_{2,3} = \pm i\omega_0$. These results indicate an oscillation at the wave frequency, ω_0 , and a transient, which decays at the rate, Ω . In particular the motion of the contact point is

$$\mathcal{P}(x=0, a=0, t)/\mathcal{P}_0 = [\omega_0/(\omega_0^2 + \Omega^2)][\Omega \cos(\omega_0 t) - \Omega e^{-\Omega t} + \omega_0 \sin(\omega_0 t)]$$

By applying these results to the hydromagnetic system, all we have shown is that if the poloidal and toroidal modes are not coupled, then exciting a poloidal wave results in a well behaved oscillation and has no effect on the toroidal resonators. This is neither a surprising nor an interesting result. However, it has some importance for the purpose of contrast. For the physics changes dramatically when we consider the limit of strong coupling, $a=1$, corresponding to a direct connection of the resonator system and the wave system. In this case, the cubic form simplifies to a quadratic and the transmission coefficient in Eq. (20) becomes

$$\tau(s)/\mathcal{P}_0 = \omega_0/(s^2 + \Omega s + \omega_0^2)$$

The poles are at:

$$2s_{1,2} = -\Omega \pm (\Omega^2 - 4\omega_0^2)^{1/2}$$

This result indicates that the transmitted wave and the motion of the contact point are *always damped*. If $\Omega < 2\omega_0$, the motion is a damped sinusoid; and if $\Omega \geq 2\omega_0$, there is damping only. For the former condition, the transmitted wave has the solution

$$\mathcal{P}(x \leq 0, t, a=1)/\mathcal{P}_0 = (\omega_0/\omega_1) \cdot e^{-(x/A+t)(\Omega/2)} \sin \omega_1(x/A+t)$$

for $x/A+t \geq 0$, where

$$\omega_1 = [\omega_0^2 - (\Omega/2)^2]^{1/2}.$$

For $x/A+t < 0$, $\mathcal{P}(x < 0, t) = 0$. This shows a damped wave propagating to the left of the contact point, at which a node is forming. To the right of the contact point the incident and reflected waves are

$$\begin{aligned} \mathcal{P}(x \geq 0, t, a=1)/\mathcal{P}_0 &= \sin \omega_0(x/A+t) + \varepsilon R(x, t) \\ R(x, t) &= \sin \omega_0(x/A-t) + (\omega_0/\omega_1) e^{-(t-x/A)(\Omega/2)} \sin \omega_1(t-x/A) \end{aligned}$$

where $\varepsilon=1$ for $t-x/A \geq 0$ and $\varepsilon=0$ otherwise. The first term in the reflected wave interacts with the incident wave to produce a standing wave, $2 \sin(\omega_0 x/A) \cos(\omega_0 t)$. Hence, if we had come upon scene long after the disturbance had started and had no prior knowledge of the existence of the oscillator, it would appear as if the string simply was held fixed at the contact point, $x=0$.

Fig. 3. Excitation of the 'toroidal' oscillator of frequency ω_0 by a 'poloidal' wave of the same frequency traveling from the right. The case shown corresponds to strong coupling, $a=1$, and $\Omega=0.5\omega_0$.

The motion of the resonator completes the picture.

$$\mathcal{T}(t \geq 0)/\mathcal{P}_0 = -(\omega_0/\Omega) \cos(\omega_0 t) + [(\omega_0/2\omega_1) \sin(\omega_1 t) + (\omega_0/\Omega) \cos(\omega_1 t)] e^{-(\Omega/2)t}$$

Figure 3 shows the response of the strongly coupled system for the particular case of underdamped motion, $\omega_0/\Omega=2$.

If we had anticipated the result that the oscillator would become a perfect reflector of the incident radiation and a standing wave would form to the right of the contact point the steady state amplitude of $\mathcal{T}$ as $-(\omega_0/\Omega)\mathcal{P}_0$ easily could have been computed from the wave Eq. (12). This equation defines the slope at the origin to be

$$Ad/dx\mathcal{P}(x=0) = -2a\Omega\mathcal{T}_0$$

With $\mathcal{P} = 2\mathcal{P}_0 \sin(\omega_0 x/A)$, where the factor of 2 indicates the standing wave has twice the amplitude of the incident wave, we find that

$$\mathcal{T}_0/\mathcal{P}_0 = -[\omega_0/(a\Omega)]$$

The coupling factor, a , has been retained to emphasize the difference between the case of no coupling, $a=0$, and finite coupling, $a \neq 0$, where a can be chosen as small as we wish. When there is absolutely no coupling, the oscillator amplitude must be zero since this mode can not be excited; but in the limit of weak, but *finite* coupling, the steady state oscillator amplitude becomes large. Thus, there is a discontinuity in the physical behavior of the system between the zero and non-zero coupling *steady state* results. The source of this behavior is the node which always forms at the contact point and reflects the incoming wave when any coupling is present. This type of discontinuity is not unusual; e.g., the value of Bessel functions $J_\nu(x)$ at the origin is zero for $\nu \neq 0$ and unity for $\nu=0$. For Bessel functions, the discontinuity arises from the function x^ν as $x, \nu \rightarrow 0$. Similarly, for coupled modes, it arises from damping

terms which, for small a , have the form $e^{-a^2\Gamma t}$ as $a, t^{-1} \rightarrow 0$ where $\Gamma = \Omega\omega_0^2/2(\Omega^2 + \omega_0^2)$.

For values of a between the extreme cases treated above, $0 \leq a \leq 1$, the zeroes of the cubic form $D(s)$ in Eq. (22) can be obtained explicitly in terms of the parameters Ω , ω_0^2 and a (e.g., BURINGTON, 1954). However, there is no conscientious reason for presenting the cumbersome algebra associated with the general solution to the cubic equation. It will more than suffice: (a) to establish the principal result of physical importance, that a node always occurs at the contact point; and, (b) to show the explicit results for a relatively tractable case, $(\Omega/\omega_0)^2 = 3$. The formation of the node, as well as the damping of the transmitted wave and the reflection of the incident wave, are indicated by the roots of $D(s)$, or their real parts, being negative. It is easy to establish the existence of damping for the one real root or if all three roots are real. This result follows immediately when the cubic Eq. (22) is rewritten as

$$s = \frac{-\Omega[s^2 + (1 - a^2)\omega_0^2]}{(s^2 + \omega_0^2)} \leq 0$$

for $0 \leq a^2 \leq 1$, where the zero value occurs at $a = 1$. The only other case has two of the roots as complex conjugates and the sign of the real part must be determined. The technique is essentially the same as for the real root. If the one real root is defined by s_1 and the complex roots as $s_{2,3} = s_0 e^{\pm i\varphi}$, with a real part, s_r , then the coefficients of the cubic equation give the relations:

$$\begin{aligned} 2s_r + s_1 &= -\Omega \\ 2s_1s_r + s_0^2 &= \omega_0^2 \\ s_0^2s_1 &= -\Omega(1 - a^2) \end{aligned}$$

By eliminating s_0^2 and s_1 a cubic equation governing s_r is obtained. This equation can be written in two useful forms.

$$(2s_r + \Omega) = \frac{\Omega[(2s_r + \Omega)^2 + \omega_0^2(1 - a^2)]}{[(2s_r + \Omega)^2 + \omega_0^2]} \geq 0$$

This result puts a lower bound on s_r , $s_r \geq -\Omega/2$, for $0 \leq a^2 \leq 1$.

$$2s_r = -\frac{a^2\Omega\omega_0^2}{[(2s_r + \Omega)^2 + \omega_0^2]} \leq 0$$

This result places an upper bound on s_r and shows that damping occurs for any non-zero value of the coupling factor, a .

For the particular choice of parameters $3\omega_0^2 = \Omega^2$, which corresponds to a mass ratio $m/\rho\lambda = \sqrt{3}/\pi$, where λ is the wavelength, the roots of $D(s)$ are

$$\begin{aligned} s_1 &= -(\Omega/3)(1 + 2\gamma) = -\Omega_1 \\ s_{2,3} &= -(\Omega/3)(1 - \gamma) \pm i\omega_0\gamma = -\Omega_2 \pm i\omega_1 \end{aligned}$$

where $\gamma = [1 - (9/8)a^2]^{1/3}$. It can be seen that the limiting cases are recovered: for $a=0$, $s = -\Omega$, $\pm i\omega_0$ and for $a=1$, $s=0$, $-(\Omega \pm i\omega_0)/2$. Since γ decreases monotonically with a , damping is always present for intermediate values of a .

With these roots the transmitted wave and the motion of the contact point are given by the inverse transform of Eq. (18) using the coefficient of Eq. (20).

$$\begin{aligned} \mathcal{P}(x \leq 0, t, a) / \mathcal{P}_0 = & \{ \omega_0 / [\omega_1^2 + (\Omega_2 - \Omega_1)^2] \} \\ & \cdot \{ -\Omega_1 e^{-\Omega_1 p} + \Omega_1 e^{-\Omega_2 p} \cos(\omega_1 p) + [(\omega_1^2 + (\Omega_2 - \Omega_1)\Omega_2) / \omega_1] \cdot e^{-\Omega_2 p} \sin(\omega_1 p) \} \end{aligned}$$

for $p = t + x/A \geq 0$. This form shows that the transmitted solution conforms to the initial conditions at the contact point, $\mathcal{P}|_{\mathcal{P}_0} = 0$, $\dot{\mathcal{P}}|_{\mathcal{P}_0} = \omega_0$ and describes a transient disturbance to the left of the contact point. The structure of the reflected solution presents no difficulty. As for the case, $a=1$, the wave form to the right of the contact point consists of the incident wave and a reflected wave of the same amplitude with an evanescent wave identical to the transmitted solution above except that $p = t - x/A \geq 0$. The one remaining result of interest, which is not readily apparent, is the steady state amplitude of the resonant oscillator itself. In the discussion following the case of strong coupling, $a=1$, this amplitude was determined from the wave equation to be inversely proportional to a , while the oscillator equation itself (16) seems to indicate a direct proportionality. The steady state amplitude can be obtained for any a by calculating the inverse transform of Eq. (16) for the poles representing undamped oscillations at the resonant frequencies, i.e. $s = \pm i\omega_0$.

$$[\mathcal{T}(t) / \mathcal{P}_0]_{t \rightarrow \infty} = a\omega_0^3 \text{Re}[e^{i\omega_0 t} / D(i\omega_0)]$$

From the definition of the cubic form in Eq. (22). $D(i\omega_0) = -a^2\Omega\omega_0^2$. Therefore, in accord with the previous result, we have

$$\mathcal{T}(t) / \mathcal{P}_0 \rightarrow [-\omega_0 / (a\Omega)] \cos(\omega_0 t)$$

In the above paragraphs we have treated the interaction between a wave, representing the poloidal hydromagnetic mode, and a resonant oscillator, representing the toroidal mode. The principal results of this interaction are: the reflection of the incident wave, the damping of the transmitted wave and the finite steady state amplitude of the oscillator. The fundamental character of these results is independent of the strength of the coupling.

4.2 The relaxation problem

In the previous section we modeled the poloidal mode excitation of the toroidal mode. In this section we investigate the converse problem in which the toroidal mode oscillator has the initial disturbance and loses its energy to the poloidal wave. We are trying to gain some insight into the manner in which a resonating field line relaxes from a disturbance by transmitting its

energy into the surrounding plasma through coupling to the cavity mode. The initial configuration of the mechanical analogue is set up as follows. At the time, $t=0$, the mechanical system is completely at rest except for an initial amplitude for the oscillator.

$$\begin{aligned}\mathcal{P}(x, t=0) &= \partial \mathcal{P}(x, t=0) / \partial t = 0 \\ \mathcal{T}(t=0) &= \mathcal{T}_0 \\ d\mathcal{T}(t=0)/dt &= 0\end{aligned}$$

With these initial conditions, the Laplace transforms of the basic mechanical Eq. (13) and (14) become

$$(s^2 - A^2 d^2/dx^2) \Pi(s, x) + 2\Omega A \delta(x) [\Pi - aT] = 0 \quad (23)$$

$$(s^2 + \omega_0^2) T(s) = a\omega_0^2 \Pi(s, x=0) + s\mathcal{T}_0 \quad (24)$$

where, as before, $\Omega = (1/2)(m/\rho)(\omega_0^2/A)$. Continuity at the contact point, $x=0$, is assured by choosing the wave functions to be

$$\begin{aligned}\Pi(s, x) &= R(s) e^{-sx/A} & x \geq 0 \\ \Pi(s, x) &= R(s) e^{sx/A} & x \leq 0\end{aligned}$$

The wave amplitude $R(s)$ is determined from the discontinuity of the slope as in Eq. (17).

$$A d \Pi(s, x)/dx \Big|_{x=-\varepsilon}^{x=+\varepsilon} = 2\Omega \{ [s^2 + (1-a^2)\omega_0^2] \Pi(s, 0) - a s \mathcal{T}_0 \} / (s^2 + \omega_0^2) \quad (25)$$

The result is

$$R(s) = a\Omega s / D(s) \quad (26)$$

where $D(s)$ is the same cubic form given in Eq. (22). Except for a constant factor, $a\Omega$ instead of ω_0 , this amplitude is identical to the transmitted wave of the excitation problem shown in Eq. (20). Therefore, because of our detailed analysis of the excitation problem, we already appreciate the physical nature of the wave generated by the oscillator. Not surprisingly, this wave represents a damped disturbance spreading symmetrically from the point of contact. As an illustration of this, we choose the particularly uncluttered case of the strong coupling, $a=1$, and characteristic frequencies $\Omega=2\omega_0$. This is the case of critical damping when both poles are at $-\Omega/2$. The radiated wave is simply

$$\mathcal{P}(x, t)/\mathcal{T}_0 = 2\omega_0 p e^{-\omega_0 p}$$

for $x \leq 0$, $p = t + (x/A) \geq 0$; for $x \geq 0$, $p = t - (x/A) \geq 0$. The oscillator amplitude is obtained from Eqs. (24) and (26).

$$T(s)/\mathcal{T}_0 = s(s + \Omega)/D(s) \quad (27)$$

Thus the oscillator is always damped when any coupling exists. It has a steady

Fig. 4. Relaxation of the 'toroidal' oscillator with initial amplitude $\mathcal{T}_0$ by radiation of 'poloidal' waves. The case shown corresponds to strong coupling, $a=1$, and critical damping, $\Omega=2\omega_0$.

state of zero amplitude except for the case of no coupling, $a=0$. Then, $T(s)/\mathcal{T}_0 = s/(s^2 + \omega_0^2)$ and $\mathcal{T}(t)/\mathcal{T}_0 = \cos(\omega_0 t)$. For the relaxation problem, it is easy to understand the important dichotomy between the case of uncoupled modes and finite coupling. If there is no coupling the energy obviously can not leave the oscillator. But if there is any coupling, all of the energy eventually leaks out. For strong coupling, $a=1$, the oscillator amplitude is $T(s)/\mathcal{T}_0 = (s + \Omega)/(s^2 + \Omega s + \omega_0^2)$. For the case treated above, $\Omega=2\omega_0$, the oscillator motion is $\mathcal{T}(t)/\mathcal{T}_0 = (1 + \omega_0 t)e^{-\omega_0 t}$. The results for this special case are shown in Fig. 4.

5. Summary

The properties of geomagnetic micropulsations are influenced by the hydro-magnetic resonances of the magnetosphere and its components as well as by the modes of low frequency propagation within the magnetosphere. Before micropulsations can be utilized as a diagnostic tool, the critical theoretical problem of coupled hydromagnetic modes must be solved. In the above paragraphs we have attempted to squeeze out the essence of the physical phenomena manifested by coupled hydromagnetic waves. After several stages of simplification the problem of oscillations of the magnetosphere has been reduced ultimately to the study of the motion of a spring and a string. This basic mechanical model was shown to be a representation of the resonant interaction between the poloidal or isotropic and toroidal or guided modes. The evolution of the mechanical system from configurations in which the initial energy was contained in the poloidal or toroidal modes was analyzed exhaustively. Such a detailed analysis was necessary because of the uncertainty and even antipodal opinions as to the existence and form of the steady state modes in the presence of coupling; and, in

any event, would have been justified by the pivotal importance of the problem. The resolution of these difficulties is found in determining the behavior of field lines driven into resonance and the effect of their reaction back on the driving force. The most important result of the present study is in specifying such behavior. In a resonant interaction of the type described by the hydromagnetic wave equations, the field lines tend to reflect poloidal disturbances whose frequency is the same as the characteristic frequency of the field lines. In accomplishing this reflection, the toroidal mode attains *finite* amplitude. Only an evanescent poloidal wave can be transmitted. Therefore, on one side of the field lines there can be no steady state solution. This is the reason for the impossibility of solving an ordinary boundary value problem with continuous eigenfunctions. As a corollary to this solution, it is evident that the uncoupled hydromagnetic modes can not possibly offer good physical insight into the phenomena to be expected when the cavity and field line modes interact. The reason is simply that the results are produced by and are completely dependent on the coupling itself. This conclusion in no way demeans the uncoupled modes since they provide the essential frame of reference and remain the basic concepts in any discussion of the coupled mode problem. As a subsidiary conclusion, the damped forms frequently observed in micropulsation records do not necessarily indicate the properties of the source or an actual loss mechanism but very possibly may be a manifestation of the evanescent waves initially transmitted beyond the resonant field lines or radiated by their relaxation.

At this point it can be appreciated that hydromagnetic propagation in the magnetosphere poses an excruciatingly difficult problem. One has only to examine the trail of logic that has led us to our present position. Hydromagnetic resonance of the plasmasphere is a portion of the general problem of magnetospheric resonance. The dipole approximation is an idealization of the plasmasphere. The cylindrical or rectangular geometries are models of the plasmasphere. The mechanical system is an analogue of these simplified geometries. The spring-string resonance is a simplification of the mechanical system. Therefore, after almost two decades since DUNGEY (1954) first attempted the problem, we have managed to solve a simplification of an analogue of a model of an idealization of a portion of the real problem of magnetospheric oscillations. The trip back to the magnetosphere is not expected to be less arduous but it is hoped that it may be more rapid because of this solution.

APPENDIX 1

Comparison of the equations describing plasma oscillations and hydromagnetic waves.

In Cartesian coordinates the equation describing the free oscillation of a cold inhomogeneous plasma is (SEDLÁČEK, 1971a)

$$d/dx[(\omega^2 - \Omega_p^2(x))d\Phi/dx] - \lambda^2(\omega^2 - \Omega_p^2(x))\Phi = 0 \quad (A1.1)$$

where Φ is the electric potential, Ω_p is the plasma frequency, which is non-uniform, $\lambda^2 = k_z^2 + k_y^2$, $k_{y,z}$ being wave numbers in the y, z directions in which the density is uniform. Factoring out the resonance to show the coupling produced by the x dependence of Ω_p , the equation may be rewritten as

$$(\omega^2 - \Omega_p^2(x))(d^2/dx^2 - \lambda^2)\Phi = (d\Omega_p^2/dx)d\Phi/dx \quad (\text{A1} \cdot 2)$$

The basic hydromagnetic wave equations can be written in terms of two harmonic oscillators coupled by $iC_z = (\nabla \times E)_z = iP/dx - i\lambda T = -c^{-1}\partial b_z/\partial t$.

$$(K^2 - k^2)T = -\lambda C_z \quad (\text{A1} \cdot 3)$$

$$(K^2 - k^2)P = -dC_z/dx \quad (\text{A1} \cdot 4)$$

where

$$E_x(x, y, z, t) = T(x)e^{ikz}e^{i\lambda y}e^{i\omega t}$$

$$E_y(x, y, z, t) = iP(x)e^{ikz}e^{i\lambda y}e^{i\omega t} \quad \text{and} \quad K = \omega/A(x)$$

A single equation for the parallel magnetic field or C_z is obtained by adding the toroidal equation multiplied by $-\lambda$ to the poloidal equation multiplied by d/dx .

$$[d^2/dx^2 + (K^2 - k^2 - \lambda^2)]C = -(dK^2/dx)P \quad (\text{A1} \cdot 5)$$

Finally, rewriting the poloidal field on the r.h.s. in terms of C_z from Eq. (A1.4) gives the low frequency analogue of the resonance coupling of Eq. (A1.2)

$$(K^2 - k^2)[d^2/dx^2 + (K^2 - k^2 - \lambda^2)]b_z = (dK^2/dx)db_z/dx \quad (\text{A1} \cdot 6)$$

In Eq. (A1.6) the differential equation, to which the hydromagnetic resonance is coupled, is a wave equation, in contrast to the plasma resonance equation, which does not oscillate. To make the similarity clearer an Alfvén frequency is defined as $\Omega_a^2(x) = k^2 A^2$. Then the hydromagnetic wave equation becomes

$$(\omega^2 - \Omega_a^2(x))[d^2/dx^2 + (K^2 - k^2 - \lambda^2)]b_z = -(\omega^2/\Omega_a^2)d\Omega_a^2/dx \quad (\text{A1} \cdot 7)$$

As resonance is approached, i.e., as $K^2 - k^2 \rightarrow 0$ or $\omega^2 - \Omega_a^2 \rightarrow 0$, the wave equation reduces to

$$(\omega^2 - \Omega_a^2(x))(d^2/dx^2 - \lambda^2)b_z = -(d\Omega_a^2/dx)db_z/dx \quad (\text{A1} \cdot 8)$$

Except for the sign of the coupling term this equation is identical to that governing plasma oscillations. The difference is inconsequential. The change of sign means that in one case the resonance is approached from the left and in the other case from the right. By choosing

$$\Omega_a^2 \approx \omega^2 + (d\Omega_a^2/dx)\delta x \quad \text{and} \quad \Omega_p^2 \approx \omega^2 - (d\Omega_p^2/dx)\delta x$$

the two equations become identical near resonance. The results for the high frequency plasma resonance case are not at all simple and it is not possible to directly transfer them to the low frequency Alfvén resonance case. However, some of the more general results are suggestive of what may be expected: (a) a steady state solution does not exist because of and at the resonance points; (b)

asymptotically in time, damping proportional to inverse powers of time is obtained.

APPENDIX 2

A mechanism for the production of weak coupling

In this section one possible form of the "attenuator" discussed in connection with Fig. 2 is derived.

In place of the attenuator, a , the $\mathcal{P}$ and $\mathcal{T}$ systems are coupled through an element of large mass per unit length, M . The motion of this element is indicated by y_0 . The force equations are

$$\rho \partial^2 \mathcal{P} / \partial t^2 = \tau \nabla^2 \mathcal{P} - \eta (\mathcal{P} - y_0) \quad (\text{A2.1})$$

$$\mu d^2 \mathcal{T} / dt^2 = -\eta (\mathcal{T} - y_0) \quad (\text{A2.2})$$

$$M \ddot{y}_0 = -\eta (y_0 - \mathcal{T}) - \eta (y_0 - \mathcal{P}) \quad (\text{A2.3})$$

For a harmonic time dependence the solution of the y_0 equation, Eq. (3) is

$$y_0 = \omega_1^2 [\mathcal{P} + \mathcal{T}] / (2\omega_1^2 - \omega^2)$$

where

$$\omega_1^2 = \eta / M.$$

The spring forces in Eqs. (A2.1) and (A2.2) become

$$\begin{aligned} -\eta (\mathcal{P} - y_0) &= -\eta \left[\left(\frac{\omega_1^2 - \omega^2}{2\omega_1^2 - \omega^2} \right) \mathcal{P} - \left(\frac{\omega_1^2}{2\omega_1^2 - \omega^2} \right) \mathcal{T} \right] \\ -\eta (\mathcal{T} - y_0) &= -\eta \left[\left(\frac{\omega_1^2 - \omega^2}{2\omega_1^2 - \omega^2} \right) \mathcal{T} - \left(\frac{\omega_1^2}{2\omega_1^2 - \omega^2} \right) \mathcal{P} \right] \end{aligned}$$

If the coupling element is massive and almost motionless such that $2\omega_1^2 \ll \omega^2$, the spring forces coupling $\mathcal{P}$ and $\mathcal{T}$ are approximately

$$\begin{aligned} -\eta (\mathcal{P} - y_0) &\simeq -\eta (\mathcal{P} - a\mathcal{T}) \\ -\eta (\mathcal{T} - y_0) &\simeq -\eta (\mathcal{T} - a\mathcal{P}) \end{aligned}$$

where

$$a = -(\omega_1/\omega)^2.$$

Undoubtedly, more clever methods can be imagined to bring about the same type of coupling between the wave and oscillator modes. However, it is not the manner in which the coupling is produced but the fact that coupling exists and the phenomena that it produces which are the essential elements in the analysis.

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