Price suggested two iterative methods of calculating the currents induced in the oceans by a given primary magnetic field. For a sub-storm field the actual earth lies outside the ranges of convergence of both methods. These ranges can be extended both by applying different iteration algorithms to the continents and oceans, and by using accelerators, but convergence is still not rapid enough to give useful results.

1. Introduction

Although it has been realised for many years (Chapman and Whitehead, 1922) that the high conductivity of the oceans may play an important part in generating secondary geomagnetic fields, very little progress has been made towards calculating the effect of the actual oceans. Price (1949) discussed induction in thin spherical shells and suggested two algorithms for calculating the eddy currents in a conducting shell of non-uniform resistivity. Most calculations made since then have been based on these algorithms. A number of analytic solutions have been obtained for simplified models of the oceans. They have been summarised by Ashour (1973). The only solution to the problem of induction in the actual oceans appears to be that of Bullard and Parker (1970). This applies to low-frequency time variations with a period of the order of 24 hours or longer.

An understanding of the role played by currents induced in the oceans is important for two reasons. First, any determination of the conductivity of the interior of the earth must take into account their effect. The calculations of Bullard and Parker were directed towards evaluating this. Secondly, many anomalies in the behaviour of the time varying geomagnetic field occur at the edges of oceans. In fact the coast effect seems to be an almost universal feature of continental margins (Parkinson, 1962; Schmucker, 1970; Everett and Hyndman, 1967). The implications of this effect in terms of mantle conductivity under oceans and continents must remain in doubt until more is known about the influence of the highly conducting sea water.
2. Theory

Price (1949) derived an equation, which is a special case of Ohm's law, relating the current function on a spherical conducting shell to the vertical components of the time varying magnetic field on the shell. The 2-d vector current density can be described by a scalar current function $\phi$ (Chapman and Bartels, 1940) such that the current density $J$ is

$$J = -n \times \text{grad } \phi$$

where $n$ is the unit outward normal to the spherical surface. Price's equation is

$$\text{div} (R \text{grad } \phi) = -\frac{\partial Z_t}{\partial t}$$  \hspace{1cm} (1)

where $Z_t$ is the vertical component of the total magnetic field at the surface of the sphere and $R$ is the surface resistivity. The right hand side can be expressed in terms of the normal derivative of the external (primary) and internal (induced) magnetic potentials. Equation (1) can then be used as the boundary conditions of a potential that obeys Laplace's equation either within or outside the spherical surface. However the approach here will be to regard $\phi$ itself as the dependent unknown. If this can be found the induced field follows.

Price (1949) also suggested two methods of successive approximations to

![Diagram](image.png)

Fig. 1. Cyclic iteration processes (a) for Price's method I, (b) for Price's method II, (c) for Hobbs' combined method.
obtain $\phi$ given an external primary field and the distribution of $R$ over the surface. In the first of these (method I, see Fig. 1a) we start with an approximation for $\phi$. From this, the vertical component of the field caused by the eddy currents in the conducting shell can be calculated. This will be called the induced field, and its vertical component will be written $Z_i$. Now adding the primary field (given) to this, we get the total vertical component $Z_t$ used in (1), so a better approximation is obtained by solving Eq. (1) with $\phi$ as unknown. For the case of a sinusoidal time varying field of the form $S_n \cdot e^{i\omega t}$ ($S_n$ is a spherical harmonic of order $n$) and a uniform conducting shell of surface resistivity $R$ and radius $a$, Price showed that the successive approximations converge to the analytic solution if

$$R(a\omega)^{-1} > 4\pi(2n+1)^{-1}$$

The second successive approximation method (method II, see Fig. 1b) suggested by Price is the converse of the first. From an approximation to $\phi$, $Z_t$ is calculated by (1). Then the primary vertical field is subtracted to give $Z_i$, from which a better approximation to $\phi$ is obtained. The criterion for convergence of this method is

$$R(a\omega)^{-1} < 4\pi(2n+1)^{-1}$$

Price assumed that the interior of the earth was non-conducting. Hobbs (1971) discussed the effect of a conductosphere on these criteria. If the conductosphere is a perfect conductor with a radius $p$ times that of the earth its effect is that of a system of image currents. There are four fields whose vertical components we must consider, the primary external field, the field of currents induced in the oceans and the fields of the images of each of these. Define $Z_0$ as the primary field plus its image, $ZZ$ as the field of ocean currents and $Z_m$ as the field of the image of the $Z_i$ field.

The criterion for method I to converge becomes

$$R(a\omega)^{-1} > 4\pi(1 - p^{2n+1})(2n+1)^{-1}$$

The criterion for method II depends on how $Z_m$ is estimated. After the first step of method II there are two ways of obtaining $Z_m$. The total field used in Price's equation is then

$$Z_i = Z_0 + Z_i + Z_m$$

Now it can be shown that (see Appendix)

$$Z_m = -Z_i \cdot p^{2n+1}$$

i.e.

$$Z_i = (Z_i + Z_m)(1 - p^{2n+1})^{-1} = (Z_i - Z_0)(1 - p^{2n+1})^{-1}$$

The next approximation to $\phi$ can be calculated as the current function required
to give this value of $Z_t$. For a uniform shell the region of convergence is
\[ R(\alpha\omega)^{-1} < 4\pi(1 - p^{2n+1})(2n + 1)^{-1} \] (4)
so the division between the regions of convergence of the two methods is changed, but there is still no overlap.

$Z_m$ can also be derived from $\phi$ (Hobbs, 1971). In this case the criterion for convergence is
\[ |p^{2n+1} + i(2n + 1)R(4\pi\alpha\omega)^{-1}| < 1 \]
i.e.
\[ R(\alpha\omega)^{-1} < 4\pi(1 - p^{2n+1})^{1/3}(2n + 1)^{-1} \] (5)
which gives a slightly larger region of convergence than (4).

The main difficulty in dealing with a model incorporating realistic ocean and land resistivities and primary fields is the large range of $R$ and $n$. For frequencies of interest in conductivity anomaly studies (4) and (5) are satisfied by the deep oceans and (2) by the continents. The range of spherical harmonics involved in sub-storm fields also increases the difficulty of defining a converging process.

Hobbs (1971) suggested that some of these difficulties might be removed if a mixed method were used in which method I is used for the ocean regions and method II for the continents (see Fig. 1c). The main purpose of this paper is to investigate the possibilities of this combined method.

3. Details of Calculations

In the calculation of the current function the following assumptions are made:
1. The continents are non-conducting.
2. The oceans can be represented by shells of infinitesimal thickness whose surface resistivity is inversely proportional to the water depth. The volume conductivity of sea water is taken as 3 mho/m (Bullard and Parker, 1970).
3. The interior of the earth is insulating above a depth equal to 1/10 of the radius, at which depth there is a perfectly conducting sphere, i.e. $p=0.9$.
4. The fields have a time variation of the form $e^{i\omega t}$, where $\omega=2\pi/T$ and $T$ is taken as 40 minutes. Thus $\phi$, $Z_t$ and $Z_m$ are complex. The origin of time is taken so that $Z_0$ is real. Equation (1) can then be written
\[ Z_t = (i/\omega)\nabla \cdot (R\nabla\phi) \] (1A)
5. The difference between geographic and geomagnetic coordinates is ignored.
Because the convergence criteria (4) and (5) depend on the form of the primary field, it is desirable to investigate two different primary fields, one of a simple form, the other of a form more like that of an actual sub-storm. The simple primary field was taken as

$$Z_0 = \sin 2\theta \cos \phi$$

(6)

where $\theta$ and $\phi$ are colatitude and longitude. The more realistic primary field is that suggested for a sub-storm by KISABETH and ROSTOCKER (1971).

In any digital calculation, space variables can be specified only at a finite number of points. Some compromise must be made between resolution and economy of computing. A grid spacing of ten degrees was chosen. This gives rather poor resolution near coastlines but a good representation of the global current pattern. Some of the earlier iterations were done with a grid spacing of 20 degrees. Most of the computing time is taken up with surface integrations, so it decreases with the fourth power of the grid spacing.

Surface integral formulae developed by HOBBS and PRICE (1970) were used to relate the variables $\psi$, $Z_t$, $Z_m$ and $U$, the magnetic potential, to one another. For convenience the formulae used are listed here:

$$\psi = (1/8\pi^2a) \int \int [\log (1 + \sin \gamma/2) - (\sin \gamma/2)^{-1}]Z_0 dS$$

(7)

$$Z_t(A) = \int \int (1 + \sin^2 \gamma/2)(\psi(B) - \psi(A))(8a^3 \sin^3 \gamma/2)^{-1}dS$$

(8)

$$Z_m = -(p^3/a^3) \int \int \left[ \frac{p^3 \cos^3 \gamma - 2(1+p^4) \cos \gamma + 3p^4}{(1-2p \cos \gamma + p^2)^{3/2}} \right] \psi dS$$

(9)

$$U(A) = (1/4a^2) \int \int [\psi(B) - \psi(A)](\sin \gamma/2)^{-1}dS$$

(10)

$$Z_m = \int G(\gamma)(Z_t + Z_m)dS$$

(11)

Formulas (7), (8), (9) and (10) are equivalent to Eqs. (63), (66), (91) and (56) of Hobbs and Price. The kernel of (11) cannot be obtained in closed form but the function $G$ can be calculated as shown in the Appendix. In the above formulae, $A$ indicates the point on the sphere at which the variable on the left hand side is being evaluated, $B$ indicates the point at which the surface element $dS$ is located, and $\gamma$ is the angular distance between $A$ and $B$. Where the arguments $A$ and $B$ have been omitted the function on the left hand side refers to point $A$ and those on the right hand side to point $B$.

The integrations are performed by dividing the sphere into tesserae centred at each grid point (generally multiples of 10° in latitude and longitude, and the poles). The kernels for each integral are calculated as a function of $\gamma$ for every
hundredth of a radian, and stored. Then the integral at $A$ is calculated by summing the product of the integrand variable by the kernel for the appropriate value of $\gamma$. This gives quite good results in most cases. Equation (8) is the least satisfactory. However this is not important in calculating $\psi$.

The kernels of Eqs. (7), (8) and (10) have singularities at $\gamma=0$. The contribution to the integral of the tessera containing $A$ is therefore estimated by integrating the kernel over a circle of the same area as the tessera and multiplying by the value of the integrand variable at $A$. At high latitudes the tessera defined by adjacent meridians and parallels is very different in shape to a circle. In this case the integral is taken over several adjacent tesserae with the same colatitude.

In the case of Eq. (8) the integrand variable $\psi(B)-\psi(A)$ vanishes at $B=A$ where the kernel has a singularity. It can be shown that a small circle of radius $\epsilon$ centred on $A$ contributes an amount

$$2\pi\epsilon c/a^3$$

to the integral, where $c$ is the coefficient of $\gamma^2$ in a Taylor series expansion of $\psi(B)$ about the point $A$.

The iteration system is essentially that proposed by Hobbs (1971). Initially $Z_2$ is assumed to be zero over continents and equal to $-Z_0$ over oceans (as it would be over perfectly conducting oceans). This induced field is used in Eq. (7) to give the first approximation for $\psi$. Equation (1) is used to calculate $Z_t$ on the oceans, Eq. (8) to calculate $Z_l$ on land (as in Price's method I) and Eq. (9) to calculate $Z_m$ over the whole sphere. $Z_l$ on the oceans can then be obtained from Eq. (3) and combined with the land values used in Eq. (7) to get a better approximation to $\psi$. The iteration cycle is shown in Fig. 1c. The last step is the most doubtful. The current function obtained is that of the currents flowing over the whole sphere which would give rise to $Z_t$. What we want is the current pattern confined to the oceans that would give rise to $Z_l$. To ensure that there is no current over the continents the current function $\phi$ is averaged over each continent and set equal to its average value throughout that continent. For this purpose four continents are recognised, Eurasia-Africa, America, Australia and Antarctica. This averaging probably exaggerates the current density along some coastlines. However it has been observed that if there is no averaging in subsequent iterations the calculated current density remains low over the continents. It appears therefore that no great distortion of the actual current flow is caused.

Equation (9) is used rather than (11) because of the slightly more favourable convergence inferred by Eqs. (4) and (5). However (11) is used below in another context.
4. Convergence of Iteration Systems

Even if the primary field contains only one harmonic, the current function will contain a wide range of harmonics when the surface resistivity is non-uniform. This range will be more curtailed the fewer the number of points used to represent the field. Therefore it might be expected that the convergence could depend on the grid spacing. Convergence can be investigated by plotting the complex number representing current flow across a line joining two fixed points on the sphere. This is simply the difference between the values of $\phi$ at the two points. As an example Fig. 2 shows these complex numbers for the current between 50° N, 180° E and the continent of Asia. The primary field was of the form given by Eq. (6). The first 9 iterations are performed with a grid spacing of 20° and iterations 10 to 16 with a spacing of 10°. It is clear that the process converges with the coarser grid but diverges with the finer one.

Hutson et al. (1972) pointed out that it is possible to change a divergent sequence of operations into a convergent one by adding a term to each side of the iteration equation. These terms become equal in the limit. The process is illustrated by a simple example. Consider the equation

$$x = 3x - 1$$

Fig. 2. Successive approximations to the complex value of current between the point at 50° N and 180° E and the continent of Asia, after various numbers of iterations. For the first 9 iterations the field was specified on a 20° × 20° grid; for iterations 10 to 16 on a 10° × 10° grid.
which has the solution
\[ x = \frac{1}{2} \]

Now set it up as a successive approximation problem
\[ x_n = 3x_{n-1} - 1 \]

Starting with \( x = 0 \), we get the successive values
\[ -1, -4, -13, -40, \ldots \]

However if we write the equation
\[ x_n - \frac{5}{2} x_{n-1} = 3x_{n-1} - \frac{5}{2} x_{n-1} - 1 \]
then starting with \( x = 0 \), we get the sequences of values
\[ \frac{2}{3}, \frac{4}{9}, \frac{14}{27}, \frac{40}{81}, \ldots \]

The quantity \( \frac{5}{2} \) in this example is called an "accelerator". Geometrically this amounts to shifting the origin of the coefficients of \( x_n \) and \( x_{n-1} \). If these are complex the requirement for convergence is that the circle centred at the origin containing the coefficient of \( x_{n-1} \) should exclude the coefficient of \( x_n \). Generalizing this to the case where
\[ x_n = L(x_{n-1}) + Z \]  
(12)

where \( L \) is a linear operator and \( Z \) is independent of \( x \), the requirement for convergence is that the circle containing all the eigenvalues of \( L \) should exclude the coefficient of \( x_n \).

The combined approximation method of Hobbs can be stated in the form of Eq. (12), where the operator \( L \) contains a double surface integration. A much simpler formulation is possible if the integrations are taken over the whole spherical surface. This can be achieved without appreciably distorting the problem by assigning a high, but finite resistivity to the continents. In this case a resistivity equivalent to 10 m of sea water was assigned at all continental points. This enables the order of integration to be reversed as shown in the derivation of Eq. (14). For the calculation of eigenvalues it is simpler if \( L \) is purely imaginary rather than complex. To ensure this, \( Z_m \) was calculated using Eq. (11) instead of (9).

For convenience let us write kernels of Eqs. (7), (8) and (11) so that

(7) becomes
\[ \phi = \int BZ dS \]  
(7A)

(8) becomes
\[ Z_t = \int F[\phi(B) - \phi(A)] dS_t \]  
(8A)

(11) becomes
\[ Z_m = \int G(Z_t - Z_0) dS \]  
(11A)
the last making use of Eq. (3). Starting with Eq. (7A) and using Eq. (3)
\[
\phi = \int B(Z_t - Z_m - Z_0) dS
\]  
(13)

Eliminating \(Z_m\) with (11A) the middle term becomes
\[
\int B(\gamma_{AB}) \left[ \int G(\gamma_{BC}) [Z_i(C) - Z_0(C)] dS_c \right] dS_B .
\]

\(A\) is the point on the sphere at which \(\phi\) is evaluated and \(B\) and \(C\) are points on the sphere at which the integration variable \(dS\) is taken. Because both integrals are over the whole spherical surface the order of integration can be changed to give
\[
\int BZ_m dS = \int B_i(C) \left[ \int B(\gamma_{AB}) \cdot G(\gamma_{BC}) \cdot dS_B \right] dS_C - \int Z_0(C) \left[ \int B(\gamma_{AB}) \cdot G(\gamma_{BC}) \cdot dS_B \right] dS_C .
\]

By symmetry the expression in square brackets is a function only of the angle \(\gamma_{AC}\) and can be written \(K(\gamma_{AC})\). Thus Eq. (13) becomes
\[
\phi = \int (B - K) \cdot Z_t \cdot dS - \int (B - K) \cdot Z_0 \cdot dS .
\]

If the effect of the conductosphere were ignored the kernel in these integrals would be \(B\) instead of \(B - K\). Now substituting from Eq. (1A) we get
\[
\phi = \int (B - K)(i/\omega) \text{div} (R \text{grad} \phi) \cdot dS - \int (B - K) \cdot Z_0 \cdot dS
\]  
(14)

\[
= L(\phi) - Z' \quad (15)
\]

The convergence now depends on the eigenvalues of \(L\). If \(\phi\) is determined at \(N\) points on the sphere, \(L\) can be expressed as a \(N \times N\) matrix. However it is impractical to describe the operator \(L\) with the resolution used in the surface integrations. This would require \(N = 614\). The actual field is reasonably smooth over most of the sphere, and a much smaller matrix should give a useful approximation to the actual accelerator. The matrix can be set up in terms of any set of basis vectors, and one possibility is to take as basis vectors the Kronecker functions which are zero at every point but one. In this case either the order of the matrix becomes uncomfortably large or the differentiations involved in Eq. (14) are very rough. It is better to take spherical harmonics as basis vectors. Thus the columns of the matrix \(L\) are determined by substituting for \(\phi(\theta, \phi)\) successively the values \(P_1(\theta), \ldots, P_n(\theta) \cdot \sin 4\phi\) into
\[
L(\phi) = \int (B - K)(i/\omega) \text{div} (R \text{grad} \phi) dS
\]  
(16)

The eigenvalues of the matrix \(L\) were determined by a standard program. In
Fig. 3 are plotted the eigenvalues of $L$ together with the coefficient of $\phi$ in (15) (i.e., unity). It will be seen that any circle centred on the origin enclosing all the eigenvalues would also enclose the coefficient of $\phi$. But it is possible to find circles satisfying the criterion. Intuitively it appears that the circle must pass between the origin and the point 1 with a tangent making an angle of approximately $45^\circ$ to the positive real axis. This requires that the centre of the circle be on a line making an angle of $45^\circ$ to the negative real axis.

There is an advantage in using the accelerator with the smallest modulus. Writing the approximation equation, with accelerator $\alpha$ in the form

$$\phi_n = [L(\phi_{n-1}) - \alpha \phi_{n-1}](1-\alpha)^{-1} - Z'(1-\alpha)^{-1}$$
we can see that for a sufficiently large value of $\alpha$

$$\phi_n \approx \phi_{n-1}$$

regardless of the value of $\phi_{n-1}$. The optimum value of $\alpha$ therefore appears to be $-12+i12$. This value was used for the computations described below. Without an excessively large number of iterations it is difficult to be sure that the process has converged. The consistency of successive iterations is not necessarily an indication that a good approximation to the correct current function has been obtained.

5. Results

Several current patterns were calculated for the primary fields of the form of Eq. (6) and of a sub-storm, using the above iteration process with an accelerator of $-12+i12$. Figure 4 shows a typical result. The primary field was a sub-storm centred on longitude $30^\circ$E. The first seven iterations were computed with a grid spacing of $20^\circ$. The current function on a $10^\circ \times 10^\circ$ grid was then obtained by interpolation and a further two iterations computed. Only slight changes to $\phi$ resulted from the latest two iterations.

Fig. 4. Flow lines of in-phase current induced in the oceans by a sub-storm field symmetrical about $30^\circ$E meridian. Current between light contours is 500amps, between heavy contours 2500amps.
Also shown in Fig. 4 is the "ramp function" coastline drawn half-way between adjacent land and ocean grid points. This is the coastline used by the program.

In principle the complete field can be calculated from these values of current function. However until a better indication of convergence can be obtained, this does not seem to be worth while.

The concentration of current near coastlines may not be real. It may have been introduced by the method of adjusting the calculated current function so that currents are confined to the oceans. Whether this effect is real or not is an important question. Unfortunately this method of iteration does not seem capable of answering it. More work on the reliability of iterative calculations must be done before the influence of the oceans can be evaluated with any certainty. The feasibility of direct inversion of the matrix in Eq. (15) should be investigated. Preliminary work suggests that this may lead to useful results.

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REFERENCES

The vertical component of the magnetic field, \( Z_m \), due to image currents is wanted in terms of the total vertical component \( Z_t \) and the primary field \( Z_0 \). Using Eq. (3) it is sufficient to express \( Z_m \) in terms of \( Z_t + Z_m \). Let \( a \) be the radius of the earth and \( b \) the radius of the perfectly conducting conductosphere, and define

\[
p = \frac{b}{a}.
\]

Now if \( c \) is the radius of the sphere on which the image currents can be considered to flow then, in the region between \( r = a \) and \( r = c \), we can express the magnetic potential due to ocean currents by

\[
U_z = \sum_{n=1}^{\infty} U_{z_n} (r/a)^{n+1}
\]

and that due to the image of ocean currents by

\[
U_m = \sum_{n=1}^{\infty} U_{m_n} (c/r)^{n+1}.
\]

The vertical components of these fields are then

\[
Z_t = \sum_{n=1}^{\infty} U_{z_n} (r/a)^{n+1}
\]

\[
Z_m = -\sum_{n=1}^{\infty} U_{m_n} (n+1)(c/r)^{n+1}
\]

Several different choices of \( c \) are possible, but the most convenient is that which gives the image currents the same form as the induced currents, i.e.

\[
c = \frac{b^2}{a} = \frac{bp}{2}
\]

In this case to ensure that \( Z_t + Z_m \) vanishes at \( r = b \) we must have

\[
U_{m_n} = U_{z_n} n(n+1)^{-1} p^{-3}
\]

Notice that it is not necessary to make \( Z_t + Z_m + Z_0 \) vanish at \( r = b \) because \( Z_0 \) has been defined to be the external primary field plus its image, so it automatically vanishes at the conductosphere surface.

Let the contributions of the harmonic of degree \( n \) on the surface \( r = a \) be

\[
Z_{i_n} = nU_{i_n}
\]

\[
Z_{m_n} = -nU_{i_n} p^{2n+1}
\]

So corresponding to Eq. (34) of Hobbs and Price we get

\[
Z_{m_n} = k_n (Z_{i_n} + Z_{m_n})
\]

with

\[
k_n = -p^{2n+1}(1-p^{2n+1})^{-1}
\]

\[
= -(p^{2n+1} + p^{2n+3} + p^{2n+5} + \ldots)
\]
The required kernel is

\[ G(\gamma) = \sum_{n=1}^{\infty} (2n+1)k_n P_n(\cos \gamma) \]

Now define

\[ f(p) = \sum_{n=1}^{\infty} (2n+1)p^{2n}P_n(\cos \gamma) \]

\[ = \frac{(1-p^4)a^3}{a^3 - 1} \]

where

\[ \rho^3 = a^3(1 + p^4 - 2p^5 \cos \gamma) . \]

This follows from the relation

\[ \sum_{n=0}^{\infty} (2n+1)p^{2n}P_n(\cos \gamma) = a(a^2 - c^2)\rho^{-3} . \]

So the expression for the kernel becomes

\[ G(\gamma) = -(p^2f(p) + p^3f(p^2) + p^4f(p^3) + \cdots) . \]

This series converges quite rapidly. An expansion to 20 terms gives sufficient accuracy.