A Proposal for a Bridge Method for the Calibration of Geomagnetic Sensors

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It is shown that an induction-type, geomagnetic sensor can be simply calibrated by using it as one arm of a balanced Wheatstone bridge, and by driving the bridge with an oscillator. The theoretical development of the calibration procedure takes into account the possibility that the sensor properties are not linear and therefore the calibration procedure can be used to detect nonlinearities. The theory does not demand that the input signal be sinusoidal.

Certain sensor configurations lead to exact analytical expressions for the calibration parameters. These include the various ellipsoids of revolution, for which special cases give results for the long, slender solenoid and for the flat disc. Expressions are also obtained for air-core cylinders of finite length, but these latter expressions assume that the sensors are linear.

The theory is extended to frequencies comparable with the sensor resonance frequency, and above, for the cases in which the sensor capacity can be represented by a linear, parallel capacitor.

The configuration suggested lends itself readily to feed-back circuits, including those incorporating digital elements.

1. Introduction

For research involving measurements of variations of the earth's magnetic field, it is a common procedure to use geomagnetic sensors of an induction type. These are air-core or iron-core coils of simple geometry, which can be aligned to observe the desired component of the earth's field variations. To interpret such measurements it is necessary to calibrate the coils used. Such coils can be calibrated by enclosing them within an artificially produced known field, for example such as would be produced by a long solenoid. This procedure has the practical disadvantage that the manufacture of the calibrating coils may be as expensive and complex as the manufacture of the sensors themselves. It has the philosophical difficulty
that the problem of calibrating one coil becomes the problem of calibrating
two coils, the sensor and the calibrating coil.

It is possible, as will be shown below, to avoid entirely the use of the
calibrating coil by making use of the magnetic properties of the sensor itself.
Apart from the calibrating oscillator, the additional components required are
few and simple, so that each sensor in an array may be permanently incor-
porated into a calibration configuration so that a calibration can be performed
at will, without rearranging the experimental setup.

Iron-core coils have the advantage of much greater sensitivity. Earlier
concern about nonlinearity (e.g. CAMPBELL, 1969) has been shown to have been
overstated (UEDA and WATANABE, 1975). Since the calibration procedure pro-
posed in this paper enables nonlinearities to be evaluated quantitatively, such
coils are of even greater potential. This possibility is particularly useful to
test whether non-linearities of iron-core sensors are sufficient to result in a
calibration constant dependent on the orientation with respect to the main
(essentially constant) magnetic field of the earth. Such nonlinear effects can
be expected to be small because of the fact that induction-type sensors nor-
mally have a substantial air gap, and the inclusion of an air gap in a mag-
netic circuit is a classical method for linearizing its behaviour. Nevertheless,
more sophisticated methods of data processing, as for example polarization
studies, require some quantitative knowledge of linearity of the measuring
system.

2. Theory

This paper proposes a calibration procedure for induction-type geomag-
netic sensors that is based on the connection of the sensor as one arm of a
balanced Wheatstone bridge. A voltage applied to the input of the bridge is
shown to be equivalent to immersing the sensor in a magnetic field of which
the strength $H$ is proportional to the applied voltage $v$. The proportionality
constant between $H$ and $v$ can be evaluated. That this equivalence is valid
can be shown easily in the case in which the properties of the sensor can be
assumed to be linear with respect to the sensor current and the applied field.
Precisely because the linearity of such sensors is suspect, particularly in the
case where ferromagnetic cores are used, any assumption of linearity is in-
appropriate and the following derivation seems preferable.

Of the various formulations we have considered, the use of Lagrange's
equations seems to us to be the most convenient. In this formulation the
reactive components appear in the Lagrangian variable in a form that makes
it possible to avoid assumptions of linearity. In the derivation that follows,
we are following the formulation used for solution of special relativistic problems in mechanics (e.g. Goldstein, 1959). In that case the mass is a function of velocity, a close parallel to the present problem in which the inductance is permitted to be a function of current.

Suppose that the sensor is incorporated into a bridge as shown in Fig. 1. The figure defines the symbols used for the various component values and sources of e.m.f. Later it will be assumed that the bridge is balanced at d.c. and that the sizes of some resistors are negligible with respect to others. For now, they will be completely arbitrary. The component $e$ is an external e.m.f. appearing at the output terminals in series with the load resistor. It is included to provide an appropriate back-e.m.f. in the case that the circuit loading the bridge contains reactive as well as resistive components.

All components except for the sensor itself are assumed to be linear. We have taken advantage of this assumption to use conventional linear circuit theory to evaluate the currents labelled on the diagram.

For the sensor, we identify an equivalent resistance $R$ that may be a function of the sensor current $i$. We define the resistance by the following equation

$$R(i) = \frac{V(i)}{i}$$

For the linear case this is the expected definition, but for the nonlinear case there are various possible definitions for resistance that are not equivalent (cf. Harnwell, 1949).

We also identify electrical properties of the sensor that result from its ability to interact with a magnetic field, both an external magnetic field and the magnetic field resulting from the current flowing in its own windings. For now we will assume that the magnetic properties can be provided for by a suitable Lagrangian $\mathcal{L}(i)$. The form of the Lagrangian will be con-

![Fig. 1. Construction of a bridge using the magnetic sensor R-L as one arm. The distributed capacitance of the sensor is ignored in this figure applicable to low frequency applications.](image)
sidered more explicitly, below. For now it is sufficient to note that assuming
dependence only on $i_2$ has the effect of assuming that the frequency of in-
terest is sufficiently low so that the self-capacitance of the sensor can be
neglected.

For electrical problems, Lagrange's equations take the form

$$\frac{d}{dt} \left( \frac{\partial L}{\partial i_k} \right) - \frac{\partial L}{\partial q_k} + \frac{1}{2} \frac{\partial F}{\partial i_k} = e_k, \quad k = 1, 2. \quad (1)$$

The quantities $q_1$ and $q_2$, and their derivatives $i_1$ and $i_2$, are the funda-
mental independent variables for this formulation. The quantities $e_1$ and $e_2$
represent external e.m.f.'s equal to the energy transfer per unit charge when
charges $q_1$ and $q_2$ are transferred. For the circuit in Fig. 1 they become

$$e_1 = e + v \frac{R_C}{R} \frac{\partial}{\partial i_1} \left( R_i \right),$$

$$e_2 = e + \frac{d}{dt} (N \phi). \quad (2)$$

In Eq. (1), the symbol $F$ represents a dissipation function. In the case
of linear resistors $F$ is simply the power dissipated. For nonlinear compo-
nents the form of $F$ must be such that the partial derivative with respect to
the corresponding current gives an appropriate e.m.f. To be consistent with
the above definition for the resistance of the (possibly) nonlinear resistance
of the sensor, we write $F$ for the bridge circuit as follows.

$$F = 2 \int_0^{i_2} i R(i) di + R_R \left( \frac{v + R_C i_1}{R_C + R} \right) + R (i_1 + i_2)^2 + R_C \left( \frac{v - R_C i_2}{R_C + R} \right) + R_i i_2. \quad (3)$$

Substituting for the Lagrangian

$$L = T - V = T(i_2) \quad (4)$$

it is found that Eq. (1) reduces to

$$\frac{d}{dt} \left( \frac{\partial T}{\partial i_k} \right) + \frac{1}{2} \frac{\partial F}{\partial i_k} = e_k. \quad (5)$$

Substituting Eqs. (3) and (4) into Eq. (5), there results the pair of equations

$$i_1 (R + R_C || R) + R_R i_2 = v \frac{R_C}{R_C + R} + e \quad (6)$$

$$\frac{d}{dt} \left( \frac{\partial T}{\partial i_2} - N \phi \right) + R_R i_1 + (R + R) i_2 = v. \quad (7)$$

Combining Eqs. (6) and (7), it is possible to eliminate the explicit de-
pendence on $i_2$ (although $i_2$ still remains in the derivative).
The result is as follows

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial i_2} - N \phi \right) = \left\{ R + \frac{R_B}{R_B + R_L + R_C} \right\} i_1 + \left( \frac{R_B R_R - R_C R}{R_B R_B + R_L + R_C} \right) \left( 1 + \frac{R}{R_B} \right) e .
\]  \hspace{1cm} (8)

To this point the derivation is quite rigorous. In order to proceed we need to make a number of assumptions, as follows:

1. The bridge is balanced at (an appropriate) d.c. so that
   \[ R_B R_R = R_C R \]
2. \[ R_B \ll R_C \]
3. \[ R_B \gg (R_L + R_C) \]
4. \[ R_B \gg R \]
5. Either \( e \) is negligibly small or \( e \) is a function only of \( i_1 \).

The first of these assumptions requires discussion, if \( R \) is not constant. In that case the relationship shown cannot be satisfied exactly. But we expect \( R \), unlike \( L \), to be very nearly constant and in many cases the relationship can be satisfied closely enough to introduce negligible error into the calibration. For greater nonlinearities of \( R \), it is in principle possible to choose values for the other bridge resistors that satisfy this equation in a least squares sense for any particular frequency. Then the calibration must be carried out with sinusoidal drive signals. For gross nonlinearities of \( R \) the validity of the theory would have to be reexamined. The first four of these assumptions enable us to simplify Eq. (8) to give

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial i_2} - N \phi \right) = (R + R_L + R_R)i_1 + e .
\]  \hspace{1cm} (9)

At this point we must return to the question of the appropriate form of the quantity \( T(i_2) \). By analogy with accepted Lagrangian formulations for relativistic mechanics, we write

\[
T = \int_{0}^{i_2} i L(i) di
\]

and

\[
\frac{\partial T}{\partial i_2} = i_2 L(i_2) .
\]  \hspace{1cm} (10)

From Eq. (6), and taking advantage of assumptions 1 to 4, we are left with the result

\[
i_2 = \frac{v}{R_B} + \frac{e}{R_B} - i_1 .
\]  \hspace{1cm} (11)

Substituting Eq. (11) into Eq. (7) we have the relationship
Now the calibration procedure depends on the following considerations. We ask whether there is a drive voltage \( v \) that will give an equivalent result to an external field \( H \) that provides in the sensor a flux \( \phi \). To do this we put in turn \( v \) equal to zero and \( \phi \) equal to zero. Since we assume that these arrangements will produce identical outputs, we can safely assume that \( i \) (and from assumption 5, \( e \)) will be the same for the two configurations. The resulting pair of equations will be the following

\[
\frac{d}{dt}\left\{\left(\frac{v}{R_L} + \frac{e}{R_B} - i_l\right)L(i_l) - N\phi\right\} = (R + R_L + R_B)i_l - e.
\] (12)

these are precisely equivalent in the linear case if we choose \( v \) so that

\[
v = -\frac{NR_B\phi}{L}.
\] (14)

In the nonlinear case one has to proceed more carefully, for the current flowing through \( L \) (and hence \( L \) itself) is different in the two modes. In the absence of nonlinearities a sinusoidal drive \( v \) will produce a sinusoidal current \( i_l \). Therefore the presence of nonlinearities can be tested by looking for harmonic distortion accompanying a sinusoidal drive or, better, by superimposing two sinusoidal drive voltages not harmonically related and searching for intermodulation distortion in \( i_l \). If the purpose of the calibration is to determine nonlinearities resulting from the superposition of a strong steady field on the varying field to be measured, then the drive voltage can simply be kept small enough so that the resulting sensor current is insufficient to create a significant change in \( L \). Such a test is useful, for example, to test the possibility that the calibration of the sensor depends on orientation relative to the earth's main magnetic field. The remainder of this paper assumes that the nonlinearity is small enough, or the drive voltage small enough, so that the use of Eq. (14) is justified.

Equation (14) is the essential result of the theoretical development and upon it rest all the various forms of the calibration procedures.

3. Practical Considerations

In carrying out an actual calibration, care must be taken not to violate the assumptions of the theoretical development. Some points are worth emphasis.
The theory as presented does not assume that $R$ is constant, but permits it to be a function of current. If so, then the balance condition will depend on the current through the sensor. This possibility should be checked as part of the calibration procedure. If the variation of $R$ with current is significant, then the resistor values should be chosen so that the d.c. balance condition is met for the average current values that will be used in the actual calibration experiment.

The sensor properties may be a function of an ambient d.c. magnetic field. Care should therefore be taken to make sure that the calibration is carried out with the same ambient field that will be met during the actual observations, unless the calibration procedures show that the dependence on field strength is negligible. In particular, it is important to check that the calibrations of an iron-core sensor are not different for orientations parallel to, and tangential to, the earth’s main field.

The success of the calibration is to evaluate the factors in the equality of Eq. (14). Both $\phi$ and $L$ are proportional to the effective permeability of the core material. Therefore the quantity $v/H$ is only weakly dependent on the magnetic properties of the core. Therefore it is best to determine the calibration constants for the coil in terms of $H$.

4. Ellipsoidal Sensor Geometries

For ellipsoids of revolution, assumed to be wound with a thin compact winding of low pitch so that the winding resembles a current sheet, there is an exact analytical solution for the parameters of the calibrating equation. If $\phi_e$ is the equivalent flux due to a sensor current $i_s$, then we can write the pair of equations

$$L = No_e i_s$$
$$\phi_e / \phi = H_e / H.$$  

Substituting these into Eq. (14), there results an alternate form of the calibration equation, namely

$$v = - R_B \left( \frac{i_s}{H_e} \right) H$$

where $H_e$ is the field equivalent to a current $i_s$. For example, in the case of an ideal cylindrical sensor of infinite length, whether iron-core or air-core, it is well known that $H_e = n i_s$, where $n$ is the number of turns per unit length. This leads to a very simple calibration equation, namely

$$v = - \frac{R_B H}{n}.$$
Calibration of particular sensors now is reduced to finding, instead of expressions for $\phi$ in terms of $H$ and for $L$ in terms of the coil geometry, the constant relating the coil current, $i_2$, to the equivalent external, uniform field, $H_e$. (Note that in this section for simplicity we henceforth will drop the subscript 2 on $i_2$ and the subscript 'e' on $H_e$).

The case of ellipsoidal sensors is of particular interest because the induced flux density in the presence of a uniform external field is uniform, and because, as we will show, a uniform field of the same form can be obtained by winding the sensor with a current carrying wire. Therefore, there is an exact equivalence between current flow in the winding and an external field and the equivalence is independent of the magnetic properties of the material contained by the winding. For real ellipsoidal sensors, the result will differ because of the finite dimensions of the windings, but the results for the idealized ellipsoidal sensor give a good approximation for the form factor for the practical case.

The procedure applied here for our solution is to determine the field $H$ and the current $i$ that when simultaneously applied will give an identically zero field everywhere inside the winding. The principle involved is that if two stimuli combine to give an identically zero response, they would individually give equal and opposite responses.

Consider a prolate ellipsoid immersed in a uniform field directed parallel to its axis of symmetry. Consider further that the ellipsoid is wound with a winding of $N$ turns of wire arranged so that the number of turns per unit length measured in the direction of the axis of symmetry is a constant, $n$. If the current $i$ through the winding is chosen so that the magnetic field inside the ellipsoid vanishes, then we can presume that the negative of that current will provide a flux equivalent to the field $H$. This is the equivalence that we require in order to apply the calibration procedure suggested in this paper.

Fig. 2. A coil, with an iron-core in the shape of a prolate ellipsoid, immersed in a uniform magnetic field. The field is parallel to the axis of symmetry of the core. The coil is wound with $n$ turns per unit length in the direction of the core's axis.
The symmetry of the problem suggests the use of ellipsoidal coordinates, $\xi$ and $\eta$, defined by the equations

$$\frac{x^2}{\eta^2} - \frac{r^2}{1-\eta^2} = f^2, \quad \frac{x^2}{\xi^2} + \frac{r^2}{\xi^2 - 1} = f^2, \quad x^2 + y^2 = z^2. \quad (18)$$

The lines $\xi = \text{a constant}$ define a set of confocal ellipsoids of revolution and the surfaces $\eta = \text{a constant}$ define a set of confocal hyperboloids of revolution. The particular surface $\xi = \xi_0$ represents the surface of the ellipsoidal sensor. At the poles of the ellipsoid $\eta = \pm 1$ and at the equator, $\eta = 0$. In terms of these variables the magnetostatic potential $V$ outside the sensor can be written in terms of a Legendre polynomial of the second kind in the form

$$V = -Hx + A\eta Q_1(\xi). \quad (19)$$

It can be shown that at very large distances from the focal points (i.e. $\xi$ is very large) $\eta$ approaches the value $\cos(\theta)$ and $Q_1(\xi)$ approaches $1/r^2$, where $\theta$ and $r$ are the usual polar coordinates. Thus, for $\xi$ large the potential becomes that of a dipole superimposed on that of a uniform field, as is to be expected.

To complete the solution the usual boundary conditions must be applied. These are that the normal component of the flux density $B$ must be continuous across the surface and the tangential component of the field $H$ must differ inside and outside by tangential component of the surface current density.

Inside the ellipsoid the potential vanishes everywhere because of the cancellation of the effects of $i$ and $H$. At the surface, the normal component of the total field outside the ellipsoid must vanish, in order to match the internal value. Therefore,

$$\nabla \cdot \nabla \bigg|_{\text{pole}} = H - A\eta \frac{\partial Q_1(\xi)}{\partial \xi} \bigg|_{\xi = \xi_0} = H - \frac{A}{f} Q_1'(\xi_0) = 0 \quad (20)$$

For the second boundary condition we must ensure that at the surface, the tangential component of the field must be less than the internal value (zero) by the tangential component of $ni$. Therefore

$$\nabla \cdot \nabla \bigg|_{\text{equator}} = H - \frac{fHQ_1(\xi)}{Q_1'(\xi_0)} \bigg|_{\xi = \xi_0} = H - \frac{HQ_1(\xi_0)}{\xi_0 Q_1'(\xi_0)} = H - \frac{HQ_1(\xi_0)}{\xi_0 Q_1'(\xi_0)}$$

For the second boundary condition we must ensure that at the surface, the tangential component of the field must be less than the internal value (zero) by the tangential component of $ni$. Therefore
\[
H = \frac{-ni}{1 - \frac{Q_1(\xi_0)}{\xi_0 Q'(\xi_0)}} \quad \text{(21)}
\]

Values for the Legendre polynomial of the second kind, \(Q_1(\xi)\), and of its derivative, can be obtained from tables of functions, or evaluated from the following relationship valid for \(\xi\) greater than unity
\[
Q_1(\xi) = \frac{1}{2} \xi \ln \left(\frac{\xi + 1}{\xi - 1}\right) - 1
\]
and the relationship valid for \(\xi\) less than unity
\[
Q_1(\xi) = \frac{1}{2} \xi \ln \left(\frac{1 + \xi}{1 - \xi}\right) - 1.
\]

Typical values for the ratio, \(g\), of \(H\) to \(ni\) are tabulated in Table 1.

<table>
<thead>
<tr>
<th>length/diameter</th>
<th>(f = \frac{L}{I_a}) air-core cylinder(^1)</th>
<th>(g = \frac{H}{ni}) ellipsoid(^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>.5251</td>
<td>.3722</td>
</tr>
<tr>
<td>1</td>
<td>.6884</td>
<td>.6667</td>
</tr>
<tr>
<td>2</td>
<td>.8181</td>
<td>.7688</td>
</tr>
<tr>
<td>3</td>
<td>.8722</td>
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<td>4</td>
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</tr>
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</tr>
<tr>
<td>500</td>
<td>.9992</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

\(^1\) From NAGAOKA'S (1909, p. 31) table as reproduced in ROSA and GROVER (1912).

\(^2\) This work, valid for air-core and iron-core sensors.

The form factor \(g\) for the ellipsoid appears in the overall calibration expression in the form
Equation (22) ignores the effects of multi-layer windings and the finite pitch of the windings. In the case that these are not negligible effects, the equations will have to be correspondingly refined. We do not know of any calculation of the magnitudes of the appropriate corrections, but we would expect them to be second order corrections.

5. Air-Core Solenoids

In practice, of course, coils are rarely spheroidal, but are more likely to take the form of finite cylinders. We have not attempted to calculate the cylindrical formulae for any but the air-core case, for which linearity can be assumed and for which inductance formulae are well known. In the examples that follow, it will be assumed that sensor nonlinearity is small enough to use conventional formulae to estimate values for $\phi$ and $L$.

The most simple example is that of a solenoid of length $l$ and cross section area $A$ such that end effects are negligible, and closely wound with $N$ turns of thin wire so that the winding resembles a current sheet. For this case the applicable equations are

$$L_s = \frac{\mu_0 k N^3 A}{l}$$

$$\varphi = \mu_0 k A H$$

$$\frac{N \phi}{L_s} = \frac{l H}{N} = \frac{H}{n}$$

(23)

($k$ is the relative permeability).

To simulate immersion of the sensor in a magnetic field $H$, it is only necessary to drive the bridge with a voltage equal to $H \cdot R_b/n$, where $n$ is the number of turns per unit length.

Note that Eq. (23) is valid for either air-core or iron-core sensors.

In the case of real sensors, the simple treatment indicated above is inadequate because the end effects are significant, because the wire with which the sensor is wound has a finite thickness for which the pitch is significant, and because a multi-layer winding is likely to be used. Fortunately, the inductances of real air-core solenoids have been thoroughly worked out, and for precisely determined geometries can often be calculated with a precision of a few parts per million. A very complete summary of the relevant equations is provided by Rosa and Grover (1912). A practical guide to
the application of such formulae is given by Grover (1946). In general, the inductance of a solenoid can be written

\[ L = fL_0 + \Delta L_1 + \Delta L_2. \]  

When calculating the inductance of an air-core solenoid, in the case of windings of finite thickness, the area \( A \) is calculated from the average coil diameter and the length \( l \) is the overall length of the coil including the thickness of the insulation (Rosa and Grover, 1912).

Several different equations have been proposed to provide numerical estimates for the quantity \( f \). For many purposes values interpolated from Table 1 will suffice. Tables for calculating \( \Delta L_1 \) and \( \Delta L_2 \) are given in Rosa and Grover (1912). In the case where the latter quantities are negligible

\[ L = \frac{\mu_0 N^2 A}{l}. \]  

The other necessary relationship is

\[ N\phi = \sum_{j=1}^{N} \mu_0 HA_j = \mu_0 HN\bar{A} \]  

for which Eq. (10) becomes

\[ v = \frac{HR_B\bar{A}}{fN} = \frac{R_BH}{fn}. \]  

The approximation in Eq. (22) arises because the mean area may not be the same as the area appropriate for Eq. (25).

6. Theory for Higher Frequencies

The theoretical development outlined above explicitly assumes that the frequencies are well below the natural resonance of the sensor so that sensor

![Fig. 3. Formation of a bridge for higher frequency applications. The distributed capacitance of the sensor is included as an equivalent parallel capacitor. The self-inductance \( l \) is added to the arm of \( R_c \) to compensate for the parallel capacitance \( C \). The time constant \( CR \) is equal to the time constant \( l/R_c \).](image)
capacitances may be neglected. At higher frequencies such capacitances are significant. Assume that they can be adequately represented by a single linear capacitor connected in parallel with the sensor (Fig. 3). It will be shown that the calibration procedure remains valid if a compensating inductor is connected into the opposite arm of the bridge as shown in the figure, and if the magnitude of the oscillator voltage is adjusted with frequency in an appropriate way.

The concept of the 'balance' condition has to be adjusted in this case. As for low frequencies the bridge is assumed to be balanced at d.c. In addition, the time constants of the reactive bridge arms, excluding the sensor inductance must be equal. Together these two conditions may be written

$$\frac{R_B}{R_C} = \frac{R_C}{CR}.$$  \hspace{1cm} (28)

In the Laplace transform domain, in which much of the following proof will be formulated, Eq. (28) become

$$\frac{R}{R_B} = \frac{R_B(1 + sCR)}{(sL + R_C)}.$$  \hspace{1cm} (29)

Using normal electric circuit theory for the linear components, and taking advantage of the first four assumptions listed for the low-frequency derivation, the new bridge circuit can be re-drawn in the equivalent form shown in Fig. 4. Note that re-drawing the circuit does not require the assumption of linearity of either the sensor resistance or inductance. The voltages $v_1$ and $v_2$ in Fig. 4 are easily shown to be, in the time domain

$$\frac{dv_1}{dt} + R_C v_1 = R_B v,$$

$$v_2 = \frac{1}{CR_B} \int v dt.$$  \hspace{1cm} (30)

Two more equations can be written representing Lagrange's equation for the variables $q_1$ and $i_1$, and for $q_2$ and $i_2$. These are

![Fig. 4. An equivalent circuit of the bridge of Fig. 3. Refer to the text for an explanation.](image)
\[
\frac{-\left(q_1 - q_2\right)}{C} + i_1(R_L + R_R) = v_1 + e - v_2
\]

\[
\frac{d}{dt}\left[i_2L(i_2) - N\phi\right] + \frac{\left(q_1 q_2\right)}{C} + i_2R = v_2.
\]

In the Laplace transform domain, the last four equations become
\[
(s + R_c)\hat{v}_1 = R_R \hat{v}
\]
\[
sC R_R \hat{v}_2 = \hat{v}
\]
\[
Y_L + \left(R + \frac{1}{sC} + R_L + R_R\right) = \hat{v}_1 + \hat{e} - \hat{v}_2
\]
\[
s\left[i_2 \ast \hat{L}(i_2) - N\phi\right] + \frac{1}{sC} + R\right) = \hat{v}_1 = \hat{v}_2.
\]

Following procedures quite analogous with the low-frequency theory, we are left with the equation
\[
s\left[i_2 \ast \hat{L} - N\phi\right] + \frac{1}{sC} + R\right) = \hat{v}_1 = \hat{v}_2.
\]

Making use of the balance condition and substituting for the factor \(i_2\), there results the equation
\[
\frac{s}{C} + \frac{R}{R_B} - \frac{1}{1 + sCR} \right) = \hat{e}(1 + sCR).
\]

By putting in turn \(v=0\) and \(\phi=0\) we find the equivalence
\[
\frac{\hat{v}}{R_B(1 + sCR)} \ast \hat{L} = N\phi
\]

which in time domain becomes
\[
v \equiv \frac{R_B N}{L} \phi
\]

where
\[
v + CR \frac{dv}{dt} = \phi.
\]

The frequency-dependent factor of Eq. (35) can be handled in one of two ways. If the calibrating voltage is sinusoidal then the magnitude of the corresponding frequency-domain factor \(\sqrt{1 + (\omega CR)^2}\) can be calculated for each frequency and included in the calibrating formula. If the bridge drive voltage is not sinusoidal, as would be the case when the bridge was used as
a part of a negative feedback circuit, then the input voltage to the bridge would have to be modified by an appropriate active filter constructed to simulate Eq. (36). In the case of the ellipsoidal sensor, the required drive voltage is

$$v = -\frac{R_B}{\gamma n} \left( H + CR \frac{dH}{dt} \right).$$  \hspace{1cm} (37)

7. Conclusions

It has been shown above that a bridge configuration is appropriate for the calibration of induction-type geomagnetic sensors. Specifically, a voltage applied to the input terminals of the bridge is shown to be precisely equivalent to a magnetic flux contained by its coils. The relationship is expressible in terms of the coil inductance, the bridge parameters and the number of turns with which the sensor is wound. Alternatively it is expressible in terms of the ratio of current through the sensor windings to the equivalent external field. For certain configurations the relationship is valid and exact even if the properties of the sensor are nonlinear. These configurations include long solenoids and ellipsoidal sensors. For other configurations the formulation should give a very close approximation to the correct calibration.

For other configurations it is necessary to be able to relate the inductance to the geometry of the sensor and the flux contained by the coils to the magnetic field in which the sensor is immersed. In the case of air-core sensors, equations are available for determining the inductance with great precision; however the nature of the winding and the finite diameter of the wire may introduce uncertainties about the relationship between flux and field. In the case of iron-core sensors there are, as shown above, exact solutions for sensors of an ellipsoidal shape. These apply, as a limiting case, to the long solenoid. Although this configuration is not a likely one for practical sensors, it does provide a baseline to which real sensors can be compared and it does show that, at least for this geometry, the shape of the hysteresis curve for the core material is immaterial.

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