Tidal Dynamo in the Upper Atmosphere, Validity and Limit of the Conventional Theory

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The conventional two-dimensional approach of tidal dynamo in the ionosphere is proved to be highly accurate provided that the upper boundary condition is relevant, the condition which requires no vertical electric current at the upper boundary of the ionosphere. However, the condition can be replaced by other one which requires only no field-aligned current at the upper boundary. Under such novel situation an ionospheric electric field can as well be generated by tidal dynamo but different from the conventional one. Details of the process by which the ionospheric electric field is produced is elucidated by considering the vertical current in a simple model ionosphere.

1. Introduction

Atmospheric tides produce a regular global wind system in the ionosphere. The wind drives ions and electrons across the geomagnetic field, thereby producing ionospheric electric currents. These electric currents generate magnetic daily variations observed as S_q variations on the ground. Tidal dynamo, as this mechanism is called, is now well accepted (e.g., RICHMOND *et al.*, 1976). However, the conventional theory of the tidal dynamo is based on several assumptions which are necessary to remove the dependency of the basic equation on heights. Then, the question arises as to whether the assumptions are all valid or there are any alternatives which lead us to different electric current and fields from conventional ones, once given a wind system. The present work would answer this question.

2. Assumption of the Conventional Theory

There are some assumptions which are fundamental to any dynamo theory, either conventional or novel. One of these assumptions is

$$\text{div } \mathbf{J} = 0 \quad (1)$$

where $\mathbf{J}$ is the electric current density. One can clearly allow (1) for S_q variations. Another assumption is electrostatic approximation as

$$\mathbf{E} = -\text{grad } S \quad (2)$$

where $\mathbf{E}$ is the electric field and S the potential. This can also be allowed for S_q variations.

The third assumption is Ohm's law as

$$\mathbf{J} = [\sigma] \hat{\mathbf{E}} \quad (3)$$

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yz} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \quad (3')$$

where $[\sigma]$ is the electric conductivity tensor and x , y , and z are the southward, eastward and upward Cartesian coordinates, respectively and

$$\begin{aligned} \sigma_{xx} &= \sigma_0 \cos^2 I + \sigma_1 \sin^2 I \\ \sigma_{xy} &= -\sigma_{yx} = \sigma_2 \sin I \\ \sigma_{xz} &= \sigma_{zx} = (\sigma_0 - \sigma_1) \cos I \sin I \\ \sigma_{yy} &= \sigma_1 \\ \sigma_{yz} &= -\sigma_{zy} = \sigma_2 \cos I \\ \sigma_{zz} &= \sigma_0 \sin^2 I + \sigma_1 \cos^2 I \end{aligned} \quad (3'')$$

where $\sigma_{0,1,2}$ are the parallel, Pedersen and Hall conductivities and I is the magnetic dip angle (e.g., MAEDA and KATO, 1966). And $\hat{\mathbf{E}} = \mathbf{E} + \mathbf{W} \times \mathbf{B}_0$ where $\mathbf{W}$ is the wind velocity and $\mathbf{B}_0$ the geomagnetic field: $\hat{\mathbf{E}}$ is called the total electric field. Equation (3) is valid under quasi-stationary states under which the ion and electron inertia can be neglected and the conductivities can be defined. In addition to (1)–(3) the following three conditions i.e., (4), (7), and (13), are necessary in the conventional dynamo theory.

The first assumption is

$$J_z = 0 \quad \text{at} \quad z = z_{1,2} \quad (4)$$

where $z_{1,2}$ denote the lower and upper boundaries, respectively. The condition is necessary to obtain the following basic equation of the theory:

$$\frac{\partial I_x}{\partial x} + \frac{\partial I_y}{\partial y} = 0 \quad (5)$$

where $I_{x,y}$ are the total current as

$$I_{x,y} = \int_{z_1}^{z_2} J_{x,y} dz. \quad (6)$$

Equation (5) is derived from (1) under condition (4). If the physical state of both hemispheres is symmetric with respect to the equator, this boundary condition (4) may be valid. This assumption is clearly *not valid in general*, however.

In order to determine $\mathbf{E}$, actually S , by (5), one faces such a difficulty to have an integrated E_z in (6); E_z can vary so rapidly with height that one cannot assume it to be as constant as $E_{x,y}$. The difficulty is due to anisotropic conductivity in (3''). However, MARTYN and BAKER (1953) were successful to remove the difficulty by the following approximation (the second condition):

$$J_z = 0 \quad z_1 < z < z_2 \quad (7)$$

which gives by (3)

$$\hat{E}_z = \frac{-\sigma_{zx}E_x - \sigma_{zy}\hat{E}_y}{\sigma_{zz}}. \quad (8)$$

After substituting (8) into (3), one obtains

$$\begin{bmatrix} J_x \\ J_y \end{bmatrix} = \begin{bmatrix} \tilde{\sigma}_{xx} & \tilde{\sigma}_{xy} \\ \tilde{\sigma}_{yx} & \tilde{\sigma}_{yy} \end{bmatrix} \begin{bmatrix} \hat{E}_x \\ \hat{E}_y \end{bmatrix} \quad (9)$$

where

$$\begin{aligned} \tilde{\sigma}_{xx} &= \sigma_{xx} - \frac{\sigma_{xz}\sigma_{zx}}{\sigma_{zz}} \simeq \frac{\sigma_1}{\sin^2 I} \\ \tilde{\sigma}_{xy} &= -\tilde{\sigma}_{yx} = \sigma_{xy} - \frac{\sigma_{xz}\sigma_{zy}}{\sigma_{zz}} \simeq \frac{\sigma_2}{\sin^2 I} \\ \tilde{\sigma}_{yy} &= \sigma_{yy} - \frac{\sigma_{yz}\sigma_{zy}}{\sigma_{zz}} \simeq \sigma_1. \end{aligned} \quad (10)$$

Substitution of (9) into (6) reduces (5) to a two-dimensional relation independent of z . It is readily found that, in general, $J_z \neq 0$ and (9) contains an additional term in the right-hand side as

$$\frac{1}{\sigma_{zz}} \begin{bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} J_z. \quad (11)$$

In (11) $(\sigma_{xz}/\sigma_{zz})$ and $(\sigma_{yz}/\sigma_{zz})$ amount to 20 at the most near the equator and $|J_z| \simeq (L_z/L_{x,y})|J_{x,y}|$ by (1) where $L_{x,y,z}$ are scale size along the x, y, z axes. Thus one can conclude (11) to be $10^{-1}|J|$ at the most and

$$\mathbf{J} = [\tilde{\sigma}]\hat{\mathbf{E}} + O(0.1\mathbf{J}) \quad (12)$$

where $[\tilde{\sigma}]$ is the tensor with elements as (10).

Another assumption is necessary to make (5) solvable. In the process of integration in (6), $E_{x,y}$ must be factored out the integral in order to make (5) a second order differential equation either for current function ψ or static potential S where

$$\mathbf{I} = \text{curl}_H(\psi \mathbf{i}_z) \quad (12')$$

where $\mathbf{i}_z$ is the unit vector along the z axis and

$$\mathbf{E} = -\text{grad}_H S \quad (12'')$$

curl_H and grad_H being two-dimensional operators. The necessary relation (the third condition) is

$$\left| \frac{\partial E_{x,y}}{\partial z} \Delta z \right| \ll |E_{x,y}| \quad (13)$$

where $\Delta z \leq |z_2 - z_1|$.

Consider the difference of $E_{x,y}$ between two points $A(x, y, z)$ and $C(x, y, z + \Delta z)$

Fig. 1. The paralleloid (left) is in a magnetic meridian, and the rectangle (right) is orthogonal to the meridian and parallel to B_0 . The length along B_0 is the same for both quadrangles.

where $|\Delta z| \ll a$. In Fig. 1 the left diagram indicates $\overline{ABCD}$ in a magnetic meridian plane; $\overline{AB}$ and $\overline{CD}$ are parallel to B_0 ; $\overline{AB} = \overline{CD} = \Delta z \operatorname{cosec} I$ and $\overline{AD} = \overline{BC} = \Delta z \cot I$; $I \neq 0$. Then, contour integration of E along the parallelogram vanishes because of $\operatorname{curl} E = 0$. This gives $[E(\overline{DA})\Delta z \cot I + E(\overline{AB})\Delta z \operatorname{cosec} I + E(\overline{BC})\Delta z \cot I + E(\overline{CD})\Delta z \operatorname{cosec} I] = 0$, where $E(\overline{DA})$ etc. denote the average electric field parallel to $\overline{DA}$ etc. Neglecting a very small electric field $E(\overline{AB})$ and $E(\overline{CD})$ along the highly conductive magnetic field lines, $E(\overline{DA}) = E(\overline{CB})$. And

$$\begin{aligned}
 |E_x(A) - E_x(C)| &= |E_x(A) - E_x(\overline{AD}) + E_x(\overline{AD}) - E_x(\overline{BC}) + E_x(\overline{BC}) - E_x(C)| \\
 &\leq |E_x(A) - E_x(\overline{AD})| + |E_x(\overline{AD}) - E_x(\overline{BC})| + |E_x(\overline{BC}) - E_x(C)| \\
 &= \left\{ \left| \frac{\partial E_x(A)}{\partial x} \right| + \left| \frac{\partial E_x(C)}{\partial x} \right| \right\} |\Delta z| \cot I \\
 &\simeq \{ |E_x(A)| + |E_x(C)| \} \frac{|\Delta z|}{a} \cot I
 \end{aligned} \tag{14}$$

where a is the earth's radius.

The right-hand side diagram in Fig. 1 is a rectangle $\overline{ABEF}$ parallel to B_0 and $\overline{AB}$ is common to the left parallelogram; $\overline{AF} = \overline{BE} < \Delta z$. Similar operation on $\overline{ABEF}$ as that on $\overline{ABCD}$ gives

$$\begin{aligned}
 |E_y(A) - E_y(B)| &\leq \{ |E_y(A)| + |E_y(B)| \} \frac{|\Delta z|}{a} \cot I \\
 |E_y(C) - E_y(B)| &\leq \{ |E_y(C)| + |E_y(B)| \} \frac{|\Delta z|}{a} \cot I.
 \end{aligned} \tag{14'}$$

So, we have

$$|E_y(A) - E_y(C)| \leq \{ |E_y(A)| + 2|E_y(B)| + |E_y(C)| \} \frac{|\Delta z|}{a} \cot I.$$

Therefore, by (14) and (14') and $|\Delta z| \ll a$

$$|E(A) - E(C)| \simeq \left| \frac{\partial E_{x,y}}{\partial z} \cdot \Delta z \right| \simeq \left(|E_{x,y}| \cdot \frac{|\Delta z|}{a} \right) \ll |E_{x,y}|. \tag{14''}$$

At the equator, which is excluded in the above argument i.e., $I \neq 0$, one has to consider the curved field-lines (KATO, 1973). Then, a similar conclusion is readily obtained with a modification as $(|\Delta z|/a)^{1/2}$ instead of $|\Delta z|/a$ in the parenthesis in (14''). Note, however, that even a small field along B_0 , as is neglected here, can produce a considerable field-aligned current.

3. Vertical Distribution of Electric Current and Field

In the preceding section we have found that the conventional approach of tidal dynamo is acceptable as far as the electric current is perfectly restricted inside the ionosphere i.e., (4) is valid. One may ask, then, if there are any alternatives to (5), alternatives which produce the observed electric current and electric field. To solve the problem one must understand the vertical structure of $\mathbf{J}$ and $\mathbf{E}$, because (4) as well as other conditions (7) and (13) is related to the vertical structure. While the conventional approach puts an emphasis on the horizontal structure of $\mathbf{J}$ and $\mathbf{E}$, the present attempt does so on the vertical structure. Thus, the ionosphere model is made extremely simple: The plane atmosphere with a constant dip, I . The wind velocity, $\mathbf{W}$, is of a sinusoidal wave about x , y , and z ; about y the wavelength corresponding to diurnal; about x the Hough function can be approximated by a combination of the two waves traveling oppositely. This simple ionospheric model will be found to be still informative on the present rudimental stage of our knowledge.

The first case (Case I) assumes $\sigma_{0,1,2}$ to be all constant vertically as well as horizontally. We can get the rigorous analytic solution in this case and an insight on the problem will be possible. However, any quantitative discussion will be possible only in considering realistic conductivities which vary vastly in the ionosphere. The solution will be numerically obtained in this second case (Case II). In these two cases, $J_z = 0$ at $z = z_{1,2}$ as in the convention. In the third case (Case III) the ionosphere model is equal to that of Case I, but the boundary condition is on the field-aligned current as $J_{\parallel} = 0$ at $z = z_2$ and $J_z = 0$ at $z = z_1$, a condition which is very new. The rigorous analytic solution will be found. The last case will consider height-dependent conductivities and a numerical result is obtained for this new boundary condition. Substitution of (2) and (3) into (1) gives

$$\operatorname{div} \{[\sigma] \operatorname{grad} S\} = \operatorname{div} \{[\sigma](\mathbf{W} \times \mathbf{B}_0)\} \quad (14''')$$

which is the equation to be solved below.

Case I. $d\sigma_{0,1,2}/dz = 0$.

Equation (14''') is

$$\begin{aligned} \epsilon^2 \left(\sin^2 I \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} - 2 \sin I \cos I \frac{\partial^2 S}{\partial x \partial z} + \cos^2 I \frac{\partial^2 S}{\partial z^2} \right) + \cos^2 I \frac{\partial^2 S}{\partial x^2} \\ + 2 \sin I \cos I \frac{\partial^2 S}{\partial x \partial z} + \sin^2 I \frac{\partial^2 S}{\partial z^2} = \frac{\operatorname{div} \mathbf{J}_0}{\sigma_0} \end{aligned} \quad (15)$$

where

$$\mathbf{J}_0 = [\sigma](\mathbf{W} \times \mathbf{B}_0)$$

$$\varepsilon = \sqrt{\frac{\sigma_1}{\sigma_0}} \ll 1 \quad (16)$$

$$\mathbf{W} = \begin{bmatrix} \tilde{W}_x \\ \tilde{W}_y \\ 0 \end{bmatrix} \exp[-i(\alpha x + \beta y + kz)]$$

$$\tilde{W}_{x,y} = \text{const.}$$

$$\alpha, \beta, k = \text{wave number, } \alpha = \beta = 10^{-7} \text{ m}^{-1}, k = 5 \times 10^{-4} - 5 \times 10^{-5} \text{ m}^{-1},$$

$$\mathbf{B}_0 = \text{centered dipole.}$$

Since we are interested in S such as $S \propto \exp[-i(\alpha x + \beta y)]$, (15) reduces to an ordinary second order linear differential equation about z . The solution consists of a particular solution, S_p , and the general solution, S_g , of the homogeneous equation which is obtained by putting the right-hand side of (15) zero. It is straightforward to get S_p as

$$S_p = \tilde{S}_p \exp[-i(\alpha x + \beta y + kz)] \quad (17)$$

where

$$\begin{aligned} \tilde{S}_p = & \frac{i\varepsilon^2 B_0 \{(-\alpha \sin I + k \cos I)[\tilde{W}_y - (\sigma_2/\sigma_1) \sin I \cdot \tilde{W}_x] + \beta[\sin I \cdot \tilde{W}_x + (\sigma_2/\sigma_1) \tilde{W}_y]\}}{(\alpha \cos I - k \sin I)^2} \\ & \times (1 - \varepsilon^2 \cot^2 I) \simeq i\varepsilon^2 \left(\frac{B_0}{k}\right) \left(\frac{\cos I}{\sin^2 I}\right) \left(\tilde{W}_y - \frac{\sigma_2}{\sigma_1} \tilde{W}_x\right). \end{aligned} \quad (18)$$

The electric field $\mathbf{E}_p$ by S_p is of the order of $\varepsilon^2[|W_y B_0| + (\sigma_2/\sigma_1)|W_x B_0|]$ and mainly along $\mathbf{k}$ by $|\mathbf{k}| \gg |\alpha|, |\beta|$. Note, however, that the small $\mathbf{E}_p$ is very effective to produce a field-aligned current $\mathbf{J}_p''$ as

$$\begin{aligned} \mathbf{J}_p'' = \sigma_0 \mathbf{E}_p'' = -\sigma_1 \mathbf{B}_0 \\ \times \frac{\{(-\alpha \sin I + k \cos I)[W_y - (\sigma_2/\sigma_1) \sin I \cdot W_x] + \beta[\sin I \cdot W_x + (\sigma_2/\sigma_1) W_y]\}}{(\alpha \cos I + k \sin I)} \\ \times (1 - \varepsilon^2 \cot^2 I) \end{aligned} \quad (19)$$

$$|\mathbf{J}_p''| \simeq |\mathbf{J}_0|. \quad (20)$$

The vertical component of $\mathbf{J}_p''$ and $\mathbf{J}_0$ tends to cancel each other and produces a small vertical current f_0 as

$$\begin{aligned} f_0 = \mathbf{J}_{pz}'' + \mathbf{J}_{0z} = & \left(\frac{\alpha}{k}\right) \left(\frac{\sigma_1 B_0}{\sin I}\right) \left[\left(1 - \frac{\sigma_2}{\sigma_1} \frac{\beta}{\alpha} \sin I\right. \right. \\ & \left. \left. + \varepsilon^2 \frac{\cos^4 I - \cos^2 I \sin I - (\sigma_2/\sigma_1)(\beta/\alpha) \cos^2 I}{\sin^2 I}\right) W_y - \sin I \right. \\ & \left. \times \left(\frac{\sigma_2}{\sigma_1} + \frac{\beta}{\alpha} \sin I + \frac{\sigma_2}{\sigma_1} \varepsilon^2 \frac{\cos^4 I - \cos^2 I \sin I - (\sigma_2/\sigma_1)(\beta/\alpha) \cos I}{\sin^2 I}\right) W_x \right]. \end{aligned} \quad (21)$$

Although $|f_0|$ is of the order of $(\alpha/k)|\mathbf{J}_0|$, it is f_0 which accumulates charges on the boundary at $z=z_{1,2}$ to produce $\mathbf{E}_g$ which is obtained as the general solution of the homogeneous equation S_g where

$$S_g = [\tilde{S}_1 \exp(\varepsilon \hat{\gamma} z) - \tilde{S}_2 \exp(-\varepsilon \hat{\gamma} z)] \exp[-i(\alpha x + \beta y - \alpha z \cot I)] \quad (22)$$

where

$$\hat{\gamma} = \frac{\sqrt{\alpha^2 + \beta^2 \sin^2 I}}{\sin^2 I}. \quad (23)$$

In (22), $\tilde{S}_{1,2}$ are arbitrary constants to be determined by the boundary conditions at $z=z_{1,2}$.

Consider $J_z=0$ at $z=z_{1,2}$. Then, by tedious but elementary algebra, we get

$$\tilde{S}_1 = \frac{1}{2}(\tilde{S}_1 + \tilde{S}_2), \quad \tilde{S}_2 = \frac{1}{2}(\tilde{S}_1 - \tilde{S}_2)$$

where

$$\begin{aligned} \tilde{S}_1 = & \frac{\varepsilon \alpha \mathbf{B}_0}{k \hat{\gamma} \sin^2 I} \left\{ \frac{\alpha \cos I [1 + (\sigma_2/\sigma_1)(\beta/\alpha) \sin I]}{\hat{\gamma}^2 \sin^4 I} \right. \\ & \times \frac{[\exp(-ikz_2) - \exp(-ikz_1)]}{(z_2 - z_1)} - \frac{1}{\sin I} \frac{[z_1 \exp(-ikz_2) - z_2 \exp(-ikz_1)]}{(z_2 - z_1)} \Big\} \\ & \times \left[\left(1 - \frac{\sigma_2}{\sigma_1} \frac{\beta}{\alpha} \right) \tilde{W}_y - \sin I \left(\frac{\sigma_2}{\sigma_1} + \frac{\beta}{\alpha} \sin I \right) \tilde{W}_x \right] \end{aligned} \quad (24)$$

$$\tilde{S}_2 = i \frac{\alpha \mathbf{B}_0}{\hat{\gamma}^2 \sin^3 I} \left[\sin I \left(\frac{\beta}{\alpha} \sin I + \frac{\sigma_2}{\sigma_1} \right) \langle W_x \rangle - \left(1 - \frac{\sigma_2}{\sigma_1} \frac{\beta}{\alpha} \sin I \right) \langle W_y \rangle \right] \quad (25)$$

$$\begin{aligned} \langle W_{x,y} \rangle &= \frac{1}{(z_2 - z_1)} \int_{z_1}^{z_2} \tilde{W}_{x,y} \exp(-ikz) dz \\ &= \frac{[\exp(-ikz_2) - \exp(-ikz_1)] \tilde{W}_{x,y}}{-ik(z_2 - z_1)} \end{aligned}$$

$$\propto \text{mean of } W_{x,y} \text{ over } (z_2 - z_1).$$

The horizontal electric field is given as

$$\begin{aligned} E_{gx} &= -\frac{\partial S_g}{\partial x} = i\alpha \tilde{S}_2 \exp\{-i[\beta y + \alpha(x - z \cot I)]\} \\ E_{gy} &= -\frac{\partial S_g}{\partial y} = i\beta \tilde{S}_2 \exp\{-i[\beta y + \alpha(x - z \cot I)]\}. \end{aligned} \quad (26)$$

The wave number of S_g in (22) is orthogonal to the field lines which are given by

Fig. 2. Dynamo mechanism. J_0 is the applied current due to only winds orthogonal to B_0 . J_p is the field-aligned current due to a small polarization field in the absence of boundary. J_g is the current due to the polarization field by $(J_p + J_0)$ in the presence of boundary which completely impedes the vertical current. J is the resultant horizontal current.

$(x - z \cot I = \text{const})$ except for a very small variation due to $(\hat{\gamma}z)$ in the exponent of the terms in the bracket. Therefore, E_g is almost orthogonal to the field lines as is expected. The field aligned electric field is

$$E_g'' = \varepsilon \hat{\gamma} \sin I \tilde{S}_1 \exp[-i(\alpha x + \beta y - \cot I \cdot \alpha z)] \quad (27)$$

and $|E_g''| \simeq \varepsilon^2 |E_g|$.

Note that S_g is originally produced by W , the wave number of which is *not* orthogonal to the field lines. It is the boundary which has converted the mode of W to that of S_g . Thus, the boundary is essential to the production of the electric field, which must always be orthogonal to B_0 in magnetoactive plasma. Of course, this cannot be true to E_p by S_p in (17); E_p is the same in mode i.e., of the same wave number as W as in (16), but E_p is extremely small.

The overall picture of the electric field production by tidal dynamo is as follows (Fig. 2): At first an initial current J_0 in (16) tends to flow, and a small field E_p is set up due to divergence of J_0 i.e., $\text{div } J_0 \neq 0$. Then, E_p drives a field-aligned current J_p'' (in (19)) which tends to cancel J_{0z} as in (21) where $|J_{0z} + J_p''| \ll |J_{0z}|$. However, the small resulting vertical current accumulates charges on the boundary at $z_{1,2}$. Eventually, the electric current due to the polarization field inhibits the total vertical current to flow, resulting in setting-up E_g , which can be as large as that really observed.

Next, the above result will be compared to that of the conventional approach in which the total horizontal current only is considered. By (5), (6), (9), and (10), we have generally, for any distribution of $\sigma_{0,1,2}$

$$\begin{aligned} \frac{1}{\sin^2 I} \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} = & -\frac{i}{\Sigma_1} \left(\frac{\alpha}{\sin^2 I} \int_{z_1}^{z_2} \sigma_1 \bar{E}_{0x} dz + \frac{\alpha}{\sin I} \int_{z_1}^{z_2} \sigma_2 \bar{E}_{0y} dz \right. \\ & \left. + \beta \int_{z_1}^{z_2} \sigma_1 \bar{E}_{0y} dz - \frac{\beta}{\sin I} \int_{z_1}^{z_2} \sigma_2 \bar{E}_{0x} dz \right) \end{aligned} \quad (28)$$

where $\Sigma_{1,2} = \int_{z_1}^{z_2} \sigma_{1,2} dz$ (total conductivity). Equation (28) gives the following solution:

Fig. 3. (a) $\sigma_{0,1,2}$ versus z , the solid, broken and chain curves, respectively. Multiply by 10^{11} to obtain mho m^{-1} . (b) $(d\sigma_{0,1,2}/dz)$ versus z ; $\sigma_{0,1,2}$ varies with z as in (a). The broken and chain curves denote the magnitude of $(d\sigma_{1,2}/dz)$ which become negative above the maximum of $\sigma_{1,2}$.

$$S = \tilde{S} \exp [-i(\alpha x + \beta y)]$$

$$\tilde{S} = \frac{i\alpha}{\Sigma_1(\alpha^2 + \beta^2 \sin^2 I)} \left[\int_{z_1}^{z_2} \sigma_1 \bar{E}_{0x} dz - \sin I (\beta/\alpha) \int_{z_1}^{z_2} \sigma_2 \bar{E}_{0x} dz \right. \\ \left. + \sin^2 I (\beta/\alpha) \int_{z_1}^{z_2} \sigma_1 \bar{E}_{0y} dz + \sin I \int_{z_1}^{z_2} \sigma_2 \bar{E}_{0y} dz \right] \quad (29)$$

where

$$\bar{E}_0 = (\tilde{W} \times B_0) \exp(-ikz) = (-\tilde{W}_y, \tilde{W}_x) B_0 \sin I e^{-jkz}.$$

When $\sigma_{1,2}$ are constant as in the present case, we can take $\sigma_{1,2}$ out of the integration. And by $\Sigma_1 = (z_2 - z_1)\sigma_1$, it is found that $\tilde{S}$ in (29) and $\tilde{S}_2$ in (25) are *identical*. Thus, the resulting horizontal fields are also identical except for a factor as $e^{-\alpha \cot I z}$ which amounts to almost unity (0.99 for $I = 45^\circ$ and $\alpha z = 0.01$). Even if we consider height-dependent $[\sigma]$, the situation remains unchanged in essence, as will be shown below.

Case II. $d\sigma_{0,1,2}/dz \neq 0$.

In this case (Fig. 3) the basic equation corresponding to (15) contains additional

terms in the left-hand side due to $(d\sigma_{0,1,2}/dz) \neq 0$ as follows:

$$-\left(\frac{d\sigma_2}{dz}\right) \cos I \frac{\partial S}{\partial y} - \left(\frac{d\sigma_1}{dz} - \frac{d\sigma_0}{dz}\right) \sin I \cos I \frac{\partial S}{\partial x} + \left(\frac{d\sigma_0}{dz} \sin^2 I + \frac{d\sigma_1}{dz} \cos^2 I\right) \frac{\partial S}{\partial z}. \quad (30)$$

$\text{div } \mathbf{J}_0$ in the right-hand side also suffers a change corresponding to the above terms replacing $(\text{grad } S)$ by $(\mathbf{W} \times \mathbf{B}_0)$. Since (28) is valid to any distribution of conductivities which are all integrated with respect to z , the result by the conventional method is readily obtained by calculating (29) for given $\sigma_{0,1,2}$ as in Fig. 3. The result of calculation is illustrated in Fig. 4. As both conductivities and winds vary with z , $j_{x,y,z}$ vary in complicated way. But $\mathbf{E}$ remains almost constant as is expected and is very close to that obtained by the conventional two-dimensional approach. j_z is much smaller than $j_{x,y}$ as is expected. Linear combinations of the solid and dotted lines of $j_{x,y,z}$ in Fig. 4 present general solutions for the $\sin(\alpha x)$ mode in (a) and the $\cos(\alpha x)$ mode in (b). These modes are equivalent with Hough modes, though specific correspondence is not given. Of course, only numerical integration of the basic equation is now possible. In the numerical approach we have followed the steps of the analytical approach; at first a particular solution has been numerically obtained for arbitrary boundary condition, and then two solutions for two independent initial conditions at $z=z_2$ have been obtained for the homogeneous equation. Combine the particular solution and the two independent solutions multiplied by an arbitrary constants on each of them. The boundary $J_z=0$ and $z=z_{1,2}$ determines the constants. Probably because of wide variation of parameters especially σ_0 , the relevant solution has required a very high accuracy in computation. The wind velocity amplitude is normalized as 10^2 m/s.

The boundary condition $J_z=0$ at $z_{1,2}$ has been used implicitly in the conventional approach of the dynamo theory, but the real meaning of the condition seems ambiguous. Mathematically, any quantity, as $\mathbf{J}$ in the present case, consists of both symmetric and anti-symmetric parts with respect to the equator. It is expected that the symmetric part of $\mathbf{J}$ is larger than the anti-symmetric part in the equinoctial time, and perhaps even in the solstitial period. For the symmetric part of $\mathbf{J}$, the above boundary condition may be relevant. However, the condition is sufficient and not necessary. It is possible to consider those situations in which $J_z \neq 0$ at z_2 , and yet no current flows between the symmetric points across the equator (Fig. 5). In Cases III and IV we shall adopt the vanishing field-aligned current as $J_{\parallel}=0$ at z_2 , and $J_z=0$ at z_1 using the same ionosphere model in Cases I and II, respectively. This situation can really occur.

Fig. 4. Electric field and current for the condition $J_z(z_2)=0$. (a) $J_{x,y,z}$ and E_x : The vertical wavelength of the equivalent tidal wave is 50 km and the amplitude of $W_{x,y}$ is 100 m sec^{-1} . Along the north-south direction, the wave is a standing wave proportional to $(\sin \alpha x)$. The solid lines are the result for $\sin(\omega t - \beta y - kz)$. The dotted lines, except in the righthand side on the bottom where the broken line is used, are the result for $\cos(\omega t - \beta y - kz)$. $\oplus$ and $\ominus$ denote positive and negative. The arrows in E_x denote the value obtained by the conventional approach (see text). (b) Similar to (a) except for $\cos(\alpha x)$. Multiply by 10^5 to obtain, in m.k.s. units, $\mathbf{J}$ in A m^{-2} and by 10^{-6} to obtain $\mathbf{E}$ in V m^{-1} .

Fig. 5. Schematic diagram about the new upper boundary condition. The non-zero vertical current may flow between the nighttime and daytime hemispheres.

Case III. $d\sigma_{0,1,2}/dz=0$.

$$E_x = - \frac{\left[B_0 \sin I \left(\frac{\sigma_2}{\sigma_1} \tilde{W}_x - \tilde{W}_y \right) \exp(-ikz_2) \right]}{\left[1 + \left(\frac{\sigma_2}{\sigma_1} \right) \left(\frac{\beta}{\alpha} \right) \sin I \right]} \exp[-i(\alpha x + \beta y - \alpha z \cdot \cot I)]$$

$$E_y = - \left(\frac{\beta}{\alpha} \right) \frac{\left[B_0 \sin I \left(\frac{\sigma_2}{\sigma_1} \tilde{W}_x - \tilde{W}_y \right) \exp(-ikz_2) \right]}{\left[1 + \left(\frac{\sigma_2}{\sigma_1} \right) \left(\frac{\beta}{\alpha} \right) \sin I \right]} \exp[-i(\alpha x + \beta y - \alpha z \cdot \cot I)]. \quad (31)$$

E in (31) can be larger or smaller than that in (26) depending on (σ_2/σ_1) and I .

Case IV. $d(\sigma_{0,1,2})/dz \neq 0$ (Fig. 3).

The same numerical integration as in Case II leads us to the result illustrated in Fig. 6. As in Case II, $j_{x,y,z}$ vary widely with z whilst E remains almost constant. However, unlike in Case II this electric field is much different from that obtained by the conventional two-dimensional approach. The result is yet consistent with that of Case III. Thus, it is concluded that the electric field is considerably dependent on the boundary condition.

4. Discussion

In Section 2 it has been demonstrated that the assumptions in the conventional two-dimensional approach of dynamo theory are valid as far as the electric current is completely confined inside the ionosphere i.e., $J_z=0$ at $z=z_{1,2}$. In Section 3 simple models have been introduced to show how these assumptions hold valid in the exact three dimensional solutions, in which the vertical distribution of $\mathbf{J}$ and $\mathbf{E}$ are considered. In the present model the wind corresponds to positive modes traveling vertically. However, evanescent modes can be introduced at will by considering imaginary k 's without changing the present main conclusion. For instance, f_0 in (21) may remain as small as

Fig. 6. Solution of the new upper boundary condition as $J_{\parallel}=0$ ($J_z \neq 0$) at z_2 (a) Similar to Fig. 4(a) except for the boundary condition. (b) Similar to Fig. 4(b) except for the boundary condition.

$\alpha/|k|$; $\alpha/|k| \ll 1$ is valid also for evanescent modes. If one consider $[\sigma]$ which varies with height, the smaller vertical scale between those of the conductivity and the wind may determine the magnitude of j_z . This is just inferred from the equation of continuity as is noted in relation to (11) in Section 2. It is also found that the upper boundary condition $J_z=0$ at $z=z_2$ can be replaced by $J_{\parallel}=0$ at $z=z_2$, the condition which says the inhibition of field-aligned current. The replacement results in producing different electric fields which can be larger or smaller than those by the conventional condition $J_z=0$ at $z=z_2$, the intensity dependent on ionosphere parameters as $\sigma_{1,2}$. The novel condition implies that certain currents flow either out or in the ionosphere, orthogonally to $\mathbf{B}_0$, possible currents which flow between day and night hemispheres (Fig. 5) because of different conductivities. These currents may be very weak. Nevertheless, the currents may change the distribution of polarization and the electric field set up. Note that it is a weak vertical current f_0 in (21) which causes the electric field to generate as discussed in Section 3. The present simple model is based on a plane ionosphere and a constant conductivity between day and night, so one can tell nothing about where these currents are really flowing in or from. However, the present implication suggests a new situation which has not been considered. It is open to question which condition is real or whether others may exist. These possibility must be checked by observations, probably in future. The conventional approach can hardly be compatible with these new situations, because (5) is not true any more. One must admit that the electric field model which has been obtained through the conventional approach can be improper.

While the main conclusion reached here would generally be valid, the present model is so simple that each result obtained might not be very accurate and somewhat different from that which is based on more realistic models with horizontal nonuniformity, curved field lines and the earth's sphericity.

An intuitive picture (Fig. 2) of wind dynamo would help understand how the electric field is set up; the existence of boundaries at $z=z_{1,2}$ is essential in the electric field set-up. This is not the case for isotropic conductivity which was usually assumed (e.g., CHAPMAN and BARTELS, 1940). In the isotropic conductivity (14) reduces to

$$\nabla^2 S = \text{div } \mathbf{E}_0 \quad (32)$$

where a uniform as well as isotropic conductivity is assumed as $\sigma = \sigma_0 = \sigma_1$. Following a similar treatment as in Case I, one gets, with no difficulty,

$$S = S_p + S_g \quad (33)$$

where

$$S_p = \frac{i\beta}{\alpha^2 + \beta^2} = E_0 e^{-i(\alpha x + \beta y)} \quad (34)$$

and by $J_z=0$ at $z=z_{1,2}$

$$S_g = \left[-z + \frac{z_1 + z_2}{2} \right] (\cot I) E_0 e^{-i(\alpha x + \beta y)} \quad (35)$$

$$E_0 = W_y B_0 \sin I. \quad (36)$$

The wind in (35) is assumed vertically uniform blowing only along the east-west direction just for simplification. Then,

$$\begin{aligned} E_x &= -\frac{\partial S}{\partial x} = -\frac{\partial S_p}{\partial x} = -\frac{\alpha\beta}{\alpha^2 + \beta^2} E_0 e^{-i(\alpha x + \beta y)} \\ E_y &= -\left(\frac{\beta}{\alpha}\right) E_x \\ E_z &= -\frac{\partial S}{\partial z} \simeq -\frac{\partial S_g}{\partial z} = (\cot I) E_0 e^{-i(\alpha x + \beta y)} \end{aligned} \quad (37)$$

which shows that the *horizontal electric field* is generated in the absence of the boundary at $z=z_{1,2}$; the vertical field is set-up by the boundary. Thus, it should be pointed out here that the isotropic conductivity approximation is, in general, seriously misleading in the approach of dynamo theory.

In the present approach as well as the conventional one, the hydromagnetic effect is disregarded. If this effect is considered, $\mathcal{W}$ becomes dependent on $\mathbf{J}$. However, the hydromagnetic consideration would not be very important in the determination of global electric field and current distribution, though certain interesting dynamic effects appear (e.g., VOLLAND and GRELLMAN, 1978).

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