Harmonics Required in Main Field and Secular Variation Models

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I show that for varying dipole sources in the core, the mean-square value over the surface of the Earth of the main field falls off more rapidly with the spherical harmonic degree, n , than does the corresponding factor for the secular variation field. The same general conclusion was reached in a 1974 study by F. J. Lowes from a consideration of the observed westward drift in the field and from consideration of known spherical harmonic coefficients. This means that a higher degree secular variation model is required than for a main field model for equally good representation of both fields. If the only use of the secular variation model is for updating a main field model in predictive modeling, then the secular variation model should be carried to the same degree as the corresponding main field model if possible. This has not usually been done because of the difficulty in obtaining adequate secular variation data. Satellite data may help, but variable fields from external sources, which are not considered as part of the secular variation field, limit their use in secular variation models.

The use of a secular variation model in studying mantle conductivity or dynamo theory may require a higher degree model than for the corresponding main field model in order to obtain details of the secular variation that may be critical to relevant theories.

1. Introduction

The term main field (M.F.) as used here refers to the geomagnetic field that is assumed to originate in the core of the Earth from dynamo action. This is the usual definition of the term. Nearly all global models of the field, such as the various International Geomagnetic Reference Fields (IGRF) attempt to model the M.F. Anomalies caused by crustal sources are often delineated by subtracting an IGRF from observed values. The secular variation (S.V.) field is taken to be the rate-of-change of the field from the same sources that cause the M.F.

If there were as much good S.V. data as there are M.F. data, then direct spherical harmonic analyses (SHA) of each field would give the answer sought in this short report. Such is not the case, however, so that other less direct methods must be used to extend the discussion beyond the harmonic limit where reliable S.V. SHA stops.

Spherical harmonic models of the M.F. that are developed mainly for charting

purposes have traditionally contained higher degree harmonics than have been used in corresponding models of the S.V. field. This has not been done by choice, but was required because of the paucity of S.V. data compared with M.F. data. Although it may be impossible at the present time to change this situation, it is worthwhile to investigate the problem to determine whether or not it should be changed.

S.V. models have uses other than for the updating of predictive M.F. models in charting. Studies of the conductivity of the lower mantle, and the kinds of mechanisms that should be built into geomagnetic dynamos depend on observations of the S.V. and models derived therefrom. To capture S.V. details that may be important to developing theories may require higher harmonics than for corresponding M.F. models.

It has been shown that models consisting of dipoles in the core can give a reasonable representation of the geomagnetic field, and that these same dipoles along with rates-of-change of the dipole parameters can give a reasonable representation of the secular variation (ALLDREDGE and STEARNS, 1969). Because a number of varying dipoles in the core does closely produce the observed M.F. and S.V. field, it is assumed that trends predicted from the dipole fields for higher harmonics than are available for observed S.V. fields will be useful in a discussion of what harmonics are necessary to properly represent both the M.F. and its S.V.

2. Dipole Source with Varying Moment

The potential V , at the field point (r, θ, λ) , of a radial dipole with moment M at a distance r_0 from the center of a spherical Earth of radius a , in colatitude θ_0 and east longitude λ_0 is given by HURWITZ (1960) as

$$V = a \sum_{n=1}^{\infty} \sum_{m=0}^n \left(\frac{a}{r}\right)^{n+1} (g_n^m \cos(m\lambda) + h_n^m \sin(m\lambda)) P_n^m(\theta), \quad (1)$$

where

$$g_n^m = K R^{n-1} n P_n^m(\theta_0) \cos(m\lambda_0), \quad (2)$$

and

$$h_n^m = g_n^m \tan(m\lambda_0), \quad (3)$$

are the usual spherical harmonic coefficients (SHC),

$$K = M/a^3, \quad R = \frac{r_0}{a},$$

and P_n^m are the quasi-normalized associated Legendre polynomials.

If the moment M is the only parameter varying with time, it is quite obvious that the ratio of the M.F. SHC to the corresponding S.V. SHC will be constant, indicating that models of the M.F. and S.V. should be carried to the same harmonic degree for equally good representations.

The above conclusion is true for a single radial dipole source, but would not necessarily be true for a number of dipole sources varying with different periods. In this case, rapidly varying sources having very small moments may not contribute significantly to a M.F. model, but when they are differentiated with respect to time they are amplified relative to slowly varying sources and may tend to dominate the S.V. model. If the rapidly varying sources are located at a depth shallower than the slowly varying sources, they will be proportionately richer in high-degree harmonic terms than the deeper slower varying sources. This situation would require a higher harmonic degree for the S.V. model than for the M.F. model. The above line of reasoning holds true regardless of the mantle conductivity.

3. Dipole Source Changing Location

Consider Eqs. (2) and (3) with θ_0 and λ_0 having time derivatives $\dot{\theta}_0$ and $\dot{\lambda}_0$. The case for $\dot{r}_0$ was also considered, but was found to be negligible for $\dot{r}_0$ values found by ALLDREDGE and STEARNS (1969). Set $K = 1$ because only relative values will be of interest. It follows that

$$\dot{g}_n^m = R^{n-1}n \left[\cos(m\lambda_0) \frac{dP_n^m(\theta_0)}{d(\theta_0)} \dot{\theta}_0 - P_n^m(\theta_0)m \sin(m\lambda_0) \dot{\lambda}_0 \right] \quad (4)$$

and

$$\dot{h}_n^m = R^{n-1}n \left[\sin(m\lambda_0) \frac{dP_n^m(\theta_0)}{d(\theta_0)} \dot{\theta}_0 + P_n^m(\theta_0)m \cos(m\lambda_0) \dot{\lambda}_0 \right]. \quad (5)$$

To determine the relative value of each degree to the M.F. and S.V., it is useful to use LOWES' (1966 and 1974) expression for the mean-square value over the Earth's surface of the field produced by harmonics of a given degree n . For the M.F. this is given by

$$[R_n]_{\text{M.F.}} = (n+1) \sum_{m=0}^n [(g_n^m)^2 + (h_n^m)^2]. \quad (6)$$

For the S.V. field it will be the same expression with dots over coefficients to designate the time derivatives.

When (2) and (3) are substituted in (6),

$$[R_n]_{\text{M.F.}} = (n+1)n^2 R^{2(n-1)} \sum_{m=0}^n (P_n^m(\theta_0))^2 \quad (7)$$

is obtained for the M.F. When (4) and (5) are substituted in (6), with dots over the SHC, after using expressions for $dP_n^m(\theta_0)/d\theta_0$ as given by CHAPMAN and BARTELS (1940, p. 622) it can be shown that

$$[R_n]_{\text{S.V.}} = (n+1)n^2 R^{2(n-1)} \left\{ a_3^2 (P_n^1(\theta_0))^2 (\dot{\theta}_0)^2 + \sum_{m=1}^n [(Q_n^m(\theta_0))^2 (\dot{\theta}_0)^2 + m^2 (P_n^m(\theta_0))^2 (\dot{\lambda}_0)^2] \right\}, \quad (8)$$

$$\text{or } [R_n]_{\text{S.V.}} = [R_n]_{\dot{\theta}_0} + [R_n]_{\dot{\lambda}_0}, \quad (9)$$

where $Q_n^m(\theta_0) = a_1 P_n^{m-1}(\theta_0) - a_2 P_n^{m+1}(\theta_0)$,

$$a_1 = \frac{1}{2} \{ \delta_{m-1} (n+m)(n-m+1) \}^{1/2}$$

$$a_2 = \frac{1}{2} \{ (n+m+1)(n-m) \}^{1/2},$$

$$a_3 = \left\{ \frac{1}{2} n(n+1) \right\}^{1/2}$$

$$\delta_0 = 2, \delta_1 = \delta_2 = \dots = 1,$$

$[R_n]_{\dot{\theta}_0}$ = all the terms in (8) depending on $\dot{\theta}_0$,

and $[R_n]_{\dot{\lambda}_0}$ = all the terms in (8) depending on $\dot{\lambda}_0$.

$$\text{Because } \sum_{m=0}^n (P_n^m(\theta_0))^2 = 1 \quad (10)$$

(SCHMIDT, 1935), (7) can be written as

$$[R_n]_{\text{M.F.}} = (n+1)n^2 R^{2(n-1)}, \quad (11)$$

which is independent of θ_0 . The $\log_{10}$ of this function is plotted versus n for $R = 0.2222$ as curve (a) in Fig. 1 and for $R = 0.45$ as curve (d) in the same figure. The $R = 0.2222$ value is used because it is the value found by ALLDREDGE and STEARNS (1969) for all but the centered dipole when 35 dipoles were used. The value $R = 0.45$ is used to provide an example of a shallower source.

Simplification of (8) is not easy because (10) cannot be applied directly. Nevertheless, calculations for a few different values of θ_0 show that $[R_n]_{\dot{\theta}_0}$ does not depend on θ_0 as would be expected because a simple transformation of coordinates

Fig. 1. $\log [R_n]$ values vs n . (a) $[R_n]_{M.F.}$ for $R = 0.2222$. (b) $[R_n]_{\theta_0}$ for $R = 0.2222$. (c) is (b) moved up to match (a) at $n = 20$ so as to more clearly visualize the difference in slope between $[R_n]_{M.F.}$ and $[R_n]_{\theta_0}$. (d) $[R_n]_{M.F.}$ for $R = 0.45$. (e) $[R_n]_{\theta_0}$ for $R = 0.45$. (f) is (e) moved up to match (d) at $n = 20$ so as to more clearly visualize the difference in slope between $[R_n]_{M.F.}$ and $[R_n]_{\theta_0}$. (g) $[R_n]_{\lambda_0}$ for $R = 0.45$ and $\theta_0 = 30^\circ$. For all cases $\theta_0 = \lambda_0 = 0.2^\circ/\text{year}$.

can change the pole to any position and the motion is still along a great circle. The value of $\log_{10} [R_n]_{\dot{\theta}_0}$ is plotted as curve (b) in Fig. 1 for $R = 0.2222$ and $\dot{\theta}_0 = 0.2^\circ/\text{year}$ and as curve (e) for $R = 0.45$ and $\dot{\theta}_0 = 0.2^\circ/\text{year}$ in the same figure. In Fig. 1, the dashed curves (c) and (f) represent curves (b) and (e), respectively, moved up to match curves (a) and (d), respectively, at $n = 20$ so as to more clearly visualize the difference in slope between $\log_{10} [R_n]_{\text{M.F.}}$ and $\log_{10} [R_n]_{\dot{\theta}_0}$.

Calculations show that, for $\theta_0 = 90^\circ$, $[R_n]_{\dot{\lambda}_0}$ is exactly the same as $[R_n]_{\dot{\theta}_0}$ when $\dot{\lambda}_0 = \dot{\theta}_0$ as would be expected for the reasons given above. It is quite obvious from curves (a) and (c) and from curves (d) and (f) in Fig. 1 that $[R_n]_{\text{M.F.}}$ falls off more rapidly with n than does $[R_n]_{\dot{\theta}_0}$ and hence $[R_n]_{\dot{\lambda}_0}$ (for $\theta_0 = 90^\circ$). This means that, at least for these dipole models, a higher harmonic degree model is needed for the S.V. than for the M.F. for equally good representations of the two fields.

In general $[R_n]_{\dot{\lambda}_0}$ is a function of θ_0 because the implied motion is along parallels instead of great circles. Curve (g) in Fig. 1 shows $\log_{10} [R_n]_{\dot{\lambda}_0}$ for $R = 0.45$, $\theta_0 = 30^\circ$ and $\dot{\lambda}_0 = -0.2^\circ/\text{year}$. This curve appears to be roughly parallel to curve (e) so that it is plausible to assume that the above conclusion holds for any rate-of-change in the angular position of the radial dipole source.

A more pictorial demonstration of the need for higher harmonics to properly describe the S.V. field than the M.F. is shown in Fig. 2 and Table 1. The upper four panels in Fig. 2 ((a)–(d)) relate to the vertical component (Z) from a dipole at lat 40°N. and long 90°E. at a distance of 3381 km from the center of the Earth (100 km below the core-mantle boundary) with a magnetic moment that gives a Z component of 1,000 nT at the surface of the Earth directly over the dipole. Each panel shows only a portion of the field either to the west or east of long 90°E. That is all that is needed because of the symmetry or asymmetry about that meridian. The lower four panels ((e)–(h)) relate to the rate-of-change of the vertical component ($\dot{Z}$) when the dipole is given a westward drift of $0.2^\circ/\text{year}$.

Panel (a) in Fig. 2 shows the complete Z computed from the closed form starting with the potential given by HURWITZ (1960). Panel (b) shows a spherical harmonic approximation to Z when the expansion in (1) is carried to $n = N_{\text{max}} = 5$, and Z is taken from the gradient of V in the usual way. Panels (c) and (d) show the Z component for $N_{\text{max}} = 8$ and 10, respectively. Panel (d) is barely distinguishable from a mirror image of panel (a) showing a rapid convergence of the spherical harmonic expression for the Z component.

Panel (e) in Fig. 2 shows the complete $\dot{Z}$ computed from the closed form starting with the time derivative of the potential given by HURWITZ (1960).

Panels (f), (g) and (h) give spherical harmonic approximations of $\dot{Z}$ for $N_{\text{max}} = 5, 8$, and 10, respectively. When the panels for Z and $\dot{Z}$ are overlaid, it is obvious that more harmonics are required for $\dot{Z}$ than for Z to closely approximate the closed form calculations.

The above conclusion is presented in still another way in Table 1 in which the root-mean-square (RMS) residuals of the spherical harmonic approximations from the closed form values are given as a percentage of the closed form RMS field for a 2° grid of 1681 points centered over the dipole. Note that the residuals decrease much

Fig. 2. Z (nT) and $\dot{Z}$ (nT/yr) from a dipole at lat 40° N. long 90° E. at a distance of 3381 km from the Earth's center (100 km below the core-mantle boundary). The dipole has a magnetic moment that gives a Z component of 1,000 nT at the surface of the Earth directly above the dipole. The dipole has a westward drift of $0.2^\circ/\text{yr}$ (a) Complete Z component, to the west of the N-S line of symmetry, computed from a closed form. (b) Spherical harmonic approximation to Z , to the east of the N-S line of symmetry for $N_{\max} = 5$. (c) Same as (b) except $N_{\max} = 8$. (d) Same as (b) except $N_{\max} = 10$. (e), (f), (g), and (h) correspond to (a), (b), (c), and (d), but they refer to $\dot{Z}$ instead of Z .

more rapidly with $N_{\max}$ for Z than for $\dot{Z}$, clearly demonstrating that higher harmonics are required for $\dot{Z}$ than for Z for equally good representations of the two fields.

The above trends were also evident for the other components not shown here.

Table 1. RMS residuals of the spherical harmonic approximations from the closed form values expressed as a percentage of the closed form RMS field, for various values of $N_{\max}^*$

$N_{\max}$	Z	$\dot{Z}$
1	82.58	97.74
2	61.93	88.54
3	45.17	73.25
4	31.55	56.58
5	20.84	41.85
6	13.27	30.16
7	8.32	21.30
8	5.14	14.65
9	3.12	9.75
10	1.88	6.32
11	1.12	4.04
12	0.67	2.57
13	0.39	1.63
14	0.23	1.02
15	0.13	0.64

*A 2° grid of 1681 points centered over the dipole was used for these calculations.

4. Varying Dipole Sources Representing the Main Field and the Secular Variation Field.

ALLDREDGE and STEARNS (1969) used 35 varying radial dipoles to successfully model the field and its secular variation. The dipole parameters of that study were used in Eqs. (2) and (3) for each dipole, the parameters were then summed to get the final resultant g_n^m and h_n^m , and were then used in (6) to get $[R_n]_{\text{M.F.}}$. A similar procedure was used to get $[R_n]_{\text{S.V.}}$ for the 35 dipoles. The $\log_{10}$ of these values, not shown here, exhibited the same characteristics as described in Fig. 1, showing that in this real case the same conclusion applies that was found for moving single dipole sources. The same conclusion was reached by BARRACLOUGH and MALIN (1979) by an entirely different method.

5. Discussion and Conclusions

When $[R_n]_{\text{M.F.}}$ is compared with $[R_n]_{\text{S.V.}}$ when both are computed from SHA of real data it is seen that $[R_n]_{\text{M.F.}}$ falls off faster with the degree n , than does $[R_n]_{\text{S.V.}}$ to the largest harmonic available for the S.V. Both BULLARD (1967) and LOWES (1974) showed examples of this. Lowes stated that "It is clear that $[R_n]_{\text{S.V.}}$ decreases much less rapidly with n than does $[R_n]_{\text{M.F.}}$. There is in fact reason to expect that the S.V. spectrum will be less steep than the field spectrum, because a substantial part of the S.V. is due to the westward drift of the non-dipole field pattern with respect to the Earth's surface. . . . the S.V. produced is proportional to the longitudinal gradient of

the appropriate field, so that shorter wavelengths are emphasized compared with longer wavelengths."

It is concluded from all available evidence that S.V. models need higher degree spherical harmonics than M.F. models for equally good representations of the two fields. This is indeed unfortunate because the nature of the S.V. field makes the S.V. data more difficult to obtain than data for the M.F. Secular variation data always requires measurements made at different epochs. This is true whether S.V. models are made in separate analyses (PEDDIE and FABIANO, 1982) or in one solution together with the M.F. model (CAIN *et al.*, 1965).

Satellite data, if obtainable on a regular basis, might help solve this problem, but varying fields from external sources, which are not wanted in the S.V. field, will always be a real problem in the use of satellite data.

If S.V. data is limited to that provided by observatories and repeat stations, it will be a long time before adequate S.V. data are available.

Secular variation models have several uses. When a S.V. model is used as a part of a predictive model for charting purposes it should use the same number of spherical harmonics as the corresponding M.F. model if possible. Even higher degree harmonics than those usually employed in M.F. models might be useful when studying deep mantle conductivity and dynamo characteristics.

LOWES and RUNCORN (1951) used 12 vertical dipoles below the surface of the core to explain the secular variation for epoch 1922.5. Their graphical method was no doubt useful before high speed computers became available, but as can be seen by a comparison of the lower panels in Fig. 2 compared with the upper panels, many more vertical dipoles would be required to approximate the S.V. field than would be required to approximate the M.F. A more natural procedure is to use radial dipoles with varying parameters to represent both the M.F. and the S.V. as was done by ALLDREDGE and STEARNS (1969).

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