Magnetic Field Due to Polygonal Pyramidal Bodies

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Three-dimensional topographic bodies are approximated more precisely by polygonal pyramids than either by polygonal laminas or by polygonal prisms. Exact forms of indefinite integrals for magnetic anomalies are obtained for a polygonal pyramid whose sides of top and bottom planes are parallel. Magnetic anomalies calculated for such a polygonal pyramid by both polygonal laminas and polygonal prisms show that the large differences from the exact solution occurred in particular at the flank of the polygonal pyramid. It implies that the large number of laminas or prisms are needed to approximate the polygonal pyramid especially when the magnetic observations are made near the magnetic sources. Thus, if the source body can be approximated by a few polygonal pyramids computation time is much less than in other methods.

1. Introduction

In the analysis of magnetic and gravity anomalies, horizontal polygonal lamina approximation for a source body (TALWANI, 1965) has been widely adopted. PLOUFF (1976) proposed a polygonal prism approximation by integrating the lamina formulas in the direction of depth. BLAKELY and CHRISTIANSEN (1978) applied Plouff's method to a practical analysis of magnetic anomalies of Mount Shasta. The disadvantage of the polygonal prism approximation is that the vertical slope critically affects the magnetic field near the source. This disadvantage can be overcome by integrating the lamina formula not in the direction of depth but in the direction of slope. The integral in the direction of slope can be done in the case of polygonal pyramid if the sides of the top and bottom surfaces are parallel. Although Barnett (1976) and Okabe (1979) presented the solutions of integration for polyhedral bodies by using coordinate transformations, the solution in this paper is the direct interpretation of integrations in the particular case of polygonal pyramid which is simple to understand and easy to apply to practical use. The computation time of the integral for one side of the polygonal pyramid at a observation point is almost the same as the one in the case of the polygonal prism. Demagnetization effect due to the shape of the source is not taken into account in the calculation throughout this paper.
2. Solution of Indefinite Integration

In a right-handed Cartesian coordinate system the three orthogonal components of the magnetic anomalies due to uniformly magnetized three dimensional body are expressed as

\[ \Delta X = J_x V_1 + J_y V_2 + J_z V_3, \]
\[ \Delta Y = J_x V_2 + J_y V_4 + J_z V_5, \]
\[ \Delta Z = J_x V_3 + J_y V_5 + J_z V_6, \]

(TALWANI, 1965). For a polygonal pyramid, the equations of the \(i\)-th ridge line can be expressed as follows,

\[ y = \alpha_i Z + \beta_i, \]
\[ x = \gamma_i Z + \delta_i, \]

(see Fig. 1). The equation of the \(i\)-th edge of the horizontal cross section of the polygonal pyramid at the depth \(Z (Z_1 \leq Z \leq Z_2)\) is

\[ x = a_i y + b_i, \]

Fig. 1. A polygonal pyramidal model.
where \( a_i \) is assumed to be constant for \( Z(Z_1 \leq Z \leq Z_2) \) in the present case. From (2) and (3), \( b_i \) is reduced to be

\[
 b_i = p_i Z + q_i, \quad (4)
\]

where \( p_i = \gamma_i - a_i \alpha_i \) and \( q_i = \delta_i - a_i \beta_i \). By substituting (2) and (4) into the lamina formula \( S_k \) (TALWANI, 1965, Eq. (4)) we get \( V_k \) \((k = 1, 2, \ldots 6)\) in the Eq. (1) as follows,

\[
 V_k = \sum_{i} \int_{Z_1}^{Z_2} S_{k,i}(Z) dZ. \quad (5)
\]

\( \int S_{k,i}(Z) dZ \) is expressed as linear combinations of terms of six forms, \( T_{1,i}, T_{2,i}, T_{3,i}, T_{1,i+1}, T_{2,i+1} \) and \( T_{3,i+1} \) which are

\[
 T_{1,i} = \int \frac{dZ}{\psi_i(Z) \sqrt{\phi_i(Z)}}, \quad T_{1,i+1} = \int \frac{dZ}{\psi_i(Z) \sqrt{\phi_{i+1}(Z)}},
\]

\[
 T_{2,i} = \int \frac{Z \, dZ}{\psi_i(Z) \sqrt{\phi_i(Z)}}, \quad T_{2,i+1} = \int \frac{Z \, dZ}{\psi_i(Z) \sqrt{\phi_{i+1}(Z)}},
\]

\[
 T_{3,i} = \int \frac{Z^2 \, dZ}{\psi_i(Z) \sqrt{\phi_i(Z)}}, \quad T_{3,i+1} = \int \frac{Z^2 \, dZ}{\psi_i(Z) \sqrt{\phi_{i+1}(Z)}},
\]

where \( \psi_i(Z) = A_{1,i} Z^2 + B_{1,i} Z + C_{1,i}, \phi_i(Z) = A_{2,i} Z^2 + B_{2,i} Z + C_{2,i} \) and \( A_{1,i}, B_{1,i}, C_{1,i}, A_{2,i}, B_{2,i}, C_{2,i}, A_{2,i+1}, B_{2,i+1} \) and \( C_{2,i+1} \) are constants which are the functions of \( \alpha_i, \beta_i, \gamma_i, \delta_i \) and \( a_i \) (See Appendix 1). Using the relation

\[
 T_{3,i} = \int \frac{1}{A_{1,i} (A_{2,i} Z^2 + B_{2,i} Z + C_{2,i})^{1/2}} \left( \frac{C_{1,i}}{A_{1,i}} \right) T_{1,i} - \left( \frac{B_{1,i}}{A_{1,i}} \right) T_{2,i}, \quad (7)
\]

the problem is to obtain the two indefinite integrations of \( T_{1,i} \) and \( T_{2,i} \) because the first term of the right hand side of the relation (7) can be easily integrated.

After the change of variable \( Z \) we have two forms which are easily integrated, and \( T_{1,i} \) and \( T_{2,i} \) are obtained as the linear combinations of these two forms, that are,

\[
 \int \frac{dt}{(t^2 + P_i)(Q_{1,i} t^2 + Q_{2,i})^{1/2}}, \quad 8(a)
\]

and

\[
 \int \frac{tdt}{(t^2 + P_i)(Q_{1,i} t^2 + Q_{2,i})^{1/2}}, \quad 8(b)
\]

where \( t \) is the new variable and \( Q_{1,i}, Q_{2,i} \) and \( P_i \) are constants (See Appendix
2). These two forms are the similar ones obtained by PLOUFF (1976, forms 7(a), 7(b)) of which definite integral can be made in the whole region of Z except for the singular points at the edges of the polygonal prisms, while there are singular points in the definite integrals of 8(a) and 8(b) introduced by the variable transformation. Details are in Appendix 2. \( T_{1,i+1}, T_{2,i+1} \) and \( T_{3,i+1} \) are obtained by the same procedure mentioned above.

Under the assumption of the constant \( a_i \) for \( Z \), we obtain the relations among \( S_{2,i}, S_{3,i}, S_{4,i} \) and \( S_{5,i} \) directly from the lamina formula of TALWANI (1965, Eqs. (12) to (15)), that are,

\[
S_{4,i} = -a_i S_{2,i}, \tag{9(a)}
\]

\[
S_{5,i} = -a_i S_{3,i}, \tag{9(b)}
\]

which hold as a matter of course in the case of the polygonal prism (PLOUFF, 1976, Eq. (9)) and 9(a), 9(b) and the relation \( V_6 = -V_1 - V_4 \) save the computation time considerably.

3. An Example of Computation

Figure 2 shows the total intensity anomalies due to the circular cone (RIKITAKE, 1951). Figure 3 shows the polygonal pyramidal model used for approximation of the circular cone. Figure 3 shows also the total intensity anomalies computed by the method mentioned above, which is almost identical with those in Fig. 2. Figure 4 shows the total intensity anomalies due to the same polygonal pyramid computed by the Taylor’s method (TALWANI, 1965). Figures 4 and 5 are in the case of 5 and 11 laminas approximation, respectively. As shown in Table 1, 5 laminas do not approximate the circular cone compared to the polygonal pyramidal model because the goodness of fit ratio (RICHARDS et al., 1967) is much worse. More than 11 laminas approximate well and 41 laminas give the same goodness of fit ratio as the polygonal pyramidal method.

If the computation time for the polygonal pyramidal model in our method is taken as a unit, the computation times, in the Taiwan’s method are 0.6, 1.2, 2.1 and 4.0 for 5, 11, 21 and 41 laminas approximations, respectively. This implies that our method save more computation time than the TALWANI’s method for the polygonal pyramidal body.

4. Conclusion

In general, it is difficult to know the exact shape of the source of magnetic anomalies. Sometimes the nonmagnetic material overlies the magnetic source like a coral island. Then we should first approximate the source by such a simple shape as a circular cone (RIKITAKE, 1951) or a polygonal pyramid mentioned in this paper. Generally, it is clear that the polygonal pyramid approximates the actual shape of the source better than the circular cone.
Fig. 2. Total intensity anomalies due to a circular cone represented by the thick solid circles. The ratio of the radius of the upper surface, that of the lower surface, the thickness, the depth of the upper surface and the one division of the scale in the figure is 10:20:2:0.5:5. The unit of the values of anomaly contours is $10^{-7}$. The inclination and declination of the ambient geomagnetic field are 50° and 0°, respectively. Those of magnetization are 50° and 45°, respectively, whose intensity is a unit.

The indefinite integration is obtained by transformation of the variable and the definite integral is conducted along the slope of the polygonal pyramid. The computation time is less than by other methods proposed by TALWANI (1965), PLOUFF (1976) and BARNETT (1976).

Appendix 1

For $k = 1$, $\int S_{1,i}(Z) \, dZ$ is expressed explicitly as follows,

$$
\int S_{1,i}(Z) \, dZ = \int \frac{a_i Z^2 - b_i y_{i+1}}{[(a_i^2+1)Z^2+b_i^2]} \left[ (x_{i+1}^2+y_{i+1}^2+Z^2)^{1/2} \right] \, dZ
$$

$$
- \int \frac{a_i Z^2 - b_i y_i}{[(a_i^2+1)Z^2+b_i^2]} \left[ (x_i^2+y_i^2+Z^2)^{1/2} \right] \, dZ \quad \text{(TALWANI, 1965, Eq.(11))}
$$

$$
= \int \frac{(a_i - \alpha_{i+1} p_i)Z^2 - (\alpha_{i+1} q_i + \beta_{i+1} q_i)Z - \beta_{i+1} q_i}{[(a_i^2+p_i^2+1)Z^2+2p_i q_i Z + q_i^2]} \left[ (\alpha_{i+1}^2+y_{i+1}^2+1)Z^2 + 2(\alpha_{i+1} \beta_{i+1} + y_{i+1} \delta_{i+1})Z + \beta_{i+1}^2 + \delta_{i+1}^2 \right]^{1/2} \, dZ
$$
Fig. 3. Total intensity anomalies calculated by the method mentioned in this paper by the thick solid lines. The ratio of the distance from the center to the ridge of the upper surface, that of the lower surface, the thickness, the depth of the upper surface and the one division of the scale in the figure is 10:13:20:26:2:0.5:5. Other parameters are the same as in Fig. 2.

\[
- \int \frac{(a_i - \alpha_i p_i)Z^2 - (\alpha_i q_i + \beta_i p_i)Z - \beta_i q_i}{((a_i^2 + p_i^2 + 1)Z^2 + 2p_i q_i Z + q_i^2)((\alpha_i^2 + \gamma_i^2 + 1)Z^2 + 2(\alpha_i \beta_i + \gamma_i \delta_i)Z + \beta_i^2 + \delta_i^2)^{1/2}} dZ
\]

\[= (\zeta_{3,i+1} T_{3,i+1} + \zeta_{2,i+1} T_{2,i+1} + \zeta_{1,i+1} T_{1,i+1}) - (\zeta_{3,i} T_{3,i} + \zeta_{2,i} T_{2,i} + \zeta_{1,i} T_{1,i}),\]

where

\[\zeta_{3,i+1} = a_i - \alpha_i + 1 p_i,\]
\[\zeta_{2,i+1} = (\alpha_i + 1 q_i + \beta_i + 1 p_i),\]
\[\zeta_{1,i+1} = -\beta_i + 1 q_i,\]
\[\zeta_{3,i} = a_i - \alpha_i p_i,\]
\[\zeta_{2,i} = - (\alpha_i q_i + \beta_i p_i),\]
\[\zeta_{1,i} = -\beta_i q_i.\]

and

\[T_{3,i+1}, \ldots, T_{1,i}\] are expressed as the Eq. (6) in the text,

where

\[A_{1,i} = a_i^2 + p_i^2 + 1,\]
Fig. 4. Total intensity anomalies calculated by the Talwani's method. The polygonal pyramid, the same one as in Fig. 2, is approximated by 5 laminas. All parameters are the same as in Fig. 3.

For $k=2$ and 3, we can get the similar forms, that are,

\[ B_{1,i} = 2p_i q_i, \]
\[ C_{1,i} = q_i^2, \]
\[ A_{2,i+1} = \alpha_i^2 + \gamma_i^2 + 1, \]
\[ B_{2,i+1} = 2(\alpha_i + \beta_i + \gamma_i + \delta_i), \]
\[ C_{2,i+1} = \beta_i^2 + \delta_i^2, \]
\[ A_{2,i} = \alpha_i^2 + \gamma_i^2 + 1, \]
\[ B_{2,i} = 2(\alpha_i \beta_i + \gamma_i \delta_i), \]
\[ C_{2,i} = \beta_i^2 + \delta_i^2. \]

For $k=2$ and 3, we can get the similar forms, that are,

\[ S_{2,i}(Z) \ dz = (\eta_{3,i+1} T_{3,i+1} + \eta_{2,i+1} T_{2,i+1} + \eta_{1,i+1} T_{1,i+1}) \]
\[ - (\eta_{3,i} T_{3,i} + \eta_{2,i} T_{2,i} + \eta_{1,i} T_{1,i}), \]

where \[ \eta_{3,i+1} = \alpha_i + 1a_i p_i + q_i + 1, \]
\[ \eta_{2,i+1} = \alpha_i + 1a_i q_i + \beta_i + 1a_i p_i + 2p_i q_i, \]
Fig. 5. Total intensity anomalies calculated by the Tawkani’s method. The polygonal pyramid, the same one as in Fig. 2, is approximated by 11 laminas. All parameters are the same as in Fig. 3.

\[
\begin{align*}
\eta_{1,i+1} &= \beta_i + \beta_i a_i q_i + q_i^2, \\
\eta_{3,i} &= \alpha_i a_i p_i + p_i^2 + 1, \\
\eta_{2,i} &= \alpha_i a_i q_i + \beta_i a_i p_i + 2 p_i q_i, \\
\eta_{1,i} &= \beta_i a_i q_i + q_i^2.
\end{align*}
\]

and

\[
\int S_{3,i}(Z) \, dz = (\varepsilon_{2,i+1} T_{2,i+1} + \varepsilon_{3,i+1} + T_{3,i+1}) - (\varepsilon_{2,i} T_{2,i} + \varepsilon_{3,i} T_{3,i}),
\]

where

\[
\begin{align*}
\varepsilon_{3,i+1} &= -\alpha_i + 1(a_i^2 + 1) - a_i p_i, \\
\varepsilon_{2,i+1} &= -\beta_i + 1(a_i^2 + 1) - a_i q_i, \\
\varepsilon_{3,i} &= -\alpha_i(a_i^2 + 1) - a_i p_i, \\
\varepsilon_{2,i} &= -\beta_i(a_i^2 + 1) - a_i q_i.
\end{align*}
\]

\[\int S_{4,i}(Z) \, dZ, \int S_{5,i}(Z) \, dZ \text{ and } \int S_{6,i}(Z) \, dZ\] are obtained by the use of the relations 9(a), 9(b) and \( V_6 = - V_1 - V_4 \).

Appendix 2

To obtain the indefinite integrals of \( T_{1,i} \) and \( T_{2,i} \), the variable \( Z \) is used.
Table 1. Comparison between the polygonal pyramidal method and the Talwani's method.

<table>
<thead>
<tr>
<th></th>
<th>Polygonal pyramidal method</th>
<th>Talwani's method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goodness of fit ratio</td>
<td>46.7</td>
<td>6.3  38.8  45.6  46.7</td>
</tr>
<tr>
<td>Relative computation time</td>
<td>1.0</td>
<td>0.6  1.2  2.1  4.0</td>
</tr>
</tbody>
</table>

41 laminas in the Talwani's method give the same goodness of fit ratio (RICHARDS et al., 1967) as the polygonal pyramidal model, although the computation time is 4 times long. The goodness of fit ratio is defined by $\Sigma |\Delta T_i|/\Sigma |R_i|$, where $\Delta T_i$ is the calculated total intensity anomaly due to the circular cone (see Fig. 2) and $R_i$ is the difference of the total intensity anomaly between the one due to the circular cone and the one due to each model.

to be changed to the new variable $t$ which is expressed as

$$t = \frac{R_2 - Z}{Z - R_1},$$

where $R_1$ and $R_2$ are the two different roots of the quadratic equation below,

$$(A_{1,i}B_{2,i} - B_{1,i}A_{2,i})\xi^2 + 2(A_{1,i}C_{2,i} - C_{1,i}A_{2,i})\xi + B_{1,i}C_{2,i} - C_{1,i}B_{2,i} = 0$$

(MORIGUCHI et al., 1956).

The discriminant of the equation is always positive. After the change of the variable $Z$ to $t$, $T_{1,i}$ and $T_{2,i}$ are expressed as follows,

$$T_{1,i} = D_i \int \frac{(t+1) \text{ sgn} (t+1) \, dt}{(t^2 + P_i)(Q_{1,i}t^2 + Q_{2,i})^{1/2}},$$

and

$$T_{2,i} = D_i \int \frac{(R_i t + R_2) \text{ sgn} (t+1) \, dt}{(t^2 + P_i)(Q_{1,i}t^2 + Q_{2,i})^{1/2}},$$

where

$$D_i = (R_1 - R_2)/(A_{1,i}R_{1,i}^2 + B_{1,i}R_1 + C_{1,i}),$$

$$P_i = (A_{1,i}R_{2,i}^2 + B_{1,i}R_2 + C_{1,i})/(A_{1,i}R_{1,i}^2 + B_{1,i}R_1 + C_{1,i}),$$

$$Q_{1,i} = A_{2,i}R_{1,i}^2 + B_{2,i}R_1 + C_{2,i},$$

and

$$Q_{2,i} = A_{2,i}R_{2,i}^2 + B_{2,i}R_2 + C_{2,i}.$$
REFERENCES


