The Response of a Compressible, Non-Homogeneous Earth to Internal Loading: Theory

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Convective flow inside the mantle will cause topography at the Core Mantle Boundary (CMB). The amplitude and shape of the topography depend on details of the flow pattern, and on the Earth’s rheology. The topography can be modeled by using seismic tomography results. When lateral variations density are inferred, they are used as internal loads in a viscous Earth. So far, all models have assumed the Earth is incompressible and composed of homogeneous layers.

In this paper, we describe a formalism for computing mantle flow, geoid perturbations, and boundary deformation (at the surface or at the CMB) due to internal loading by lateral density heterogeneities for a spherically-symmetric Earth that is compressible and that has radially dependent viscosity and density. We assume the Earth is neutrally stable below the lithosphere. The solution vector, as well as the density anomalies, are expanded as sums of spherical harmonics with coefficients that depend on radius. Green’s functions, responses to a unit mass load, are computed for each order of the spherical harmonics. These functions can be convolved with tomography results inside the mantle to get the geoid or the CMB topography, for example.

1. Introduction

Mantle convection is one of the most important, but still not fully understood, of all geophysical processes. Our understanding of the convection depends ultimately on the success of fully self-consistent, dynamic models in explaining observable features of the flow. Although the recent, rapid increase in available computer power is allowing geophysicists to construct increasingly complicated and complete models (see, as one example, BERCOVICI et al., 1989), it is far from certain that any completely self-consistent model will ever be able to predict the Earth’s actual convection pattern. Instead, what those models can predict that is testable, are only general characteristics of the flow.

To permit comparison with specific features of the flow, RICHARDS and HAGER (1984) (among others) considered a more modest problem. Suppose you already know the density anomalies caused by the convection. For example, you might be willing to infer the density from someone’s map of three-dimensional seismic velocities in the mantle (DZIEWONSKI, 1984; WOODHOUSE and DZIEWONSKI, 1984; MORELLI and DZIEWONSKI, 1987; etc.). What flow pattern is consistent with those density anomalies? And what are the accompanying deformations of the Earth’s outer surface and of internal boundaries?

RICHARDS and HAGER (1984) (see, also HAGER and O’CONNELL, 1981; RICARD et
al., 1984; Forte and Peltier, 1987) found answers to this problem by solving the linearized momentum equation for flow inside the Earth, using the density anomalies as forcing terms. (The density anomalies are assumed to be superimposed on a much larger, spherically-symmetric density field, which is not directly related to the convection and which is used as a parameter in the differential equation.) The problem is not fully consistent in the sense that there is no attempt to relate the density anomalies back to the flow through an equation of state and a heat equation. Details of the solution depend both on the parameter values used in the momentum equation, and on the assumed density anomalies. For example, by estimating the long-wavelength components of the geoid from his solution and comparing them with the observed geoid, Hager (1984) was able to make inferences about the viscosity contrast across the 670 km discontinuity. In another study, Forte and Peltier (1989) compared their predicted results for the core-mantle boundary (CMB) topography with the seismic CMB results of Morelli and Dziewonski (1987), to infer information about slab penetration depths (in their study, the slabs provided upper mantle density anomalies that drove the flow).

In all of these models, the convection is assumed to be occurring in an Earth that is composed of a few homogeneous, incompressible layers (typically: a lithosphere, an upper and lower mantle, a fluid core, and, possibly, a D" layer). The viscosity and the spherically-symmetric density field are both taken as constant in each layer, and the bulk modulus is assumed to be infinite throughout the Earth. Also, the 670 km discontinuity is routinely assumed to be either a chemical boundary (and so a barrier to flow), or a phase boundary with no density contrast (so that its deformation induces no additional gravitational signal).

These assumptions make it difficult to fully address certain questions. For example, suppose you wish to place bounds on the amplitude of the CMB topography that could be induced by a given density anomaly. The topography will be strongly sensitive to the density contrast across the boundary and to the viscosity of the D" layer. But it will also depend to some extent on any radial gradient that might exist in the density or viscosity close to the boundary. And the effects of arbitrary radial gradients can not be explored with existing models. Or suppose your goal is to estimate the range of possible displacements of the 670 km discontinuity. Besides not being able to consider the possible effects of radial gradients in the viscosity or density (or of compressibility, for that matter), you would also be unable to include a phase boundary at 670 km, which would almost certainly have an important effect on your conclusions.

In this paper, we describe a formalism for computing the induced flow and boundary deformation, for a spherically-symmetric Earth that is compressible and that has radially-dependent viscosity and density. We assume, though, that the spherically-symmetric model is neutrally stable below the lithosphere, so that the Brunt-Väisälä frequency vanishes. The solution vector is expanded as a sum of vector spherical harmonics, with coefficients that depend on radius. The partial differential equations (the momentum equation, Poisson's equation, and the stress-strain relation) are then used to derive a set of ordinary differential equations for these coefficients. These equations, together with the appropriate boundary conditions, can be solved on a computer, once the density, viscosity, and compressibility profiles have been chosen. The effects of a phase transition at 670 km, or at any other depth, can be included, once certain thermodynamic parameters associated with the phase transition are specified (though values for those parameters are highly uncertain, at present). We are presently using this
formalism to consider certain internal loading problems, and we will report our results in a future paper.

2. Formulation: General Idea

We begin by expanding the lateral heterogeneities in the density field as the sum:

$$\delta \rho(\vec{r}_0) = \sum_{l,m} \delta \rho_i^m(r_0) Y_l^m(\theta_0, \lambda_0)$$  \hspace{1cm} (1)

where $Y_l^m(\theta_0, \lambda_0)$ are the spherical harmonic functions, $\vec{r}_0$ is the position of the load, $r_0$ is the distance from the Earth center up to the load position, $\theta_0$ and $\lambda_0$ are the co-latitude and eastward longitude of the load.

The idea is to compute first the Earth's transfer function at a chosen point inside the Earth, $\vec{r}$, for every unit load situated at each point inside the Earth (for a lot of $\vec{r}_0$), for every $l$ in the sum (1) (for a spherically-symmetric Earth, the results are independent of $m$). These functions are thus Green's functions, depending on $\vec{r}_0$ and $\vec{r}$. Afterwards, the Earth's response at $\vec{r}$ to a set of internal loads can be computed by convolution over $\vec{r}_0$ using Green's functions and the assumed amplitudes of the loads (lateral heterogeneities in the density). For this problem (see, for example, RICHARDS and HAGER, 1984), the internal loads are assumed to have been in place much longer than the relevant Maxwell relaxation times of the mantle. (The internal loading changes with time scales typical of mantle convection: tens of millions of years and longer. The Maxwell relaxation times are typically a few tens of thousands of years.) Thus, all visco-elastic disturbances have had time to decay away, and the flow is steady. To represent this steady-state case, the inner core and mantle (below the lithosphere) are assumed here to be Newtonian viscous fluids and all time derivatives of Eulerian variables are set to zero. The lithosphere is assumed to be an elastic solid.

In the following paragraphs, we will describe the differential equations, boundary conditions, and starting solutions (used to numerically solve the equations) for the internal loading problem, without assuming the medium is homogeneous or incompressible, but still considering a neutrally stratified mantle.

2.1 New equations

The equations for the fluid outer core in the static case have already been developed by Dahlen in 1974 in the inviscid fluid case. In that case the Eulerian density $\rho_1^E$ and the Eulerian potential $\phi_1^E$ are related by:

$$\rho_1^E = \frac{1}{\rho_0} \partial_t \rho_0 \phi_1^E$$  \hspace{1cm} (2)

where $\rho_0$ is the initial spherically-symmetric density, and by Poisson's equation:

$$\nabla^2 \phi_1^E = 4\pi G \rho_1^E.$$  \hspace{1cm} (3)

Using the expansion in spherical harmonic functions of $\phi_1^E$

$$\phi_1^E = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \phi_{1l}^{Em}(r) Y_l^m(\theta, \lambda)$$  \hspace{1cm} (4)
the equations (2) and (3) reduce to:

\[
\begin{align*}
\frac{d\phi_i^E}{dr} &= g_i^E, \\
\frac{dg_i^E}{dr} &= 4\pi G \frac{1}{\rho_0} \partial_r \rho_0 \phi_i^E + 2L^2 \frac{1}{r} \phi_i^E - 2 \frac{1}{r} g_i^E
\end{align*}
\] 

where \(\rho_0\) is the initial density as given in PREM \(\text{Dziewonski and Anderson, 1981}\) and where \(\phi_i^E\) and \(g_i^E\) are written for \(\phi_{ii}^{Em}\) and \(g_{ii}^{Em}\) (\(g_{ii}^{Em}\) is defined by the first equation of system (5)).

We have thus to develop new equations for the “solid” part of the Earth. These are equations for the deformations of a Newtonian fluid in the static case. Two possibilities can be considered: one can either let the internal density readjust, or not. The second case would arise when one imposes mass loads everywhere inside the Earth. The first case is pertinent if internal loads are specified only in certain regions of the Earth, and it is of interest to deduce the change in density induced in other regions by those loads. The Eulerian density is then left as a component of the solution in those other regions.

2.2 With mass readjustment

If mass readjustments are considered, the Eulerian density is one of the variables. The vectorial equation of motion for a hydrostatically prestressed Earth is:

\[
-\rho_0 \nabla \phi_i^E - \rho_i^E \nabla \phi_0 + \nabla \cdot \mathbf{T}_E = 0
\] 

where \(\rho_0\) is the initial gravitational potential and \(\mathbf{T}_E\) is the Eulerian stress tensor. The Eulerian potential is related to the Eulerian density \(\rho_i^E\) \((\rho_i^E = -\nabla \cdot (\rho_0 \mathbf{u}_L))\), where \(\mathbf{u}_L\) is the Lagrangian displacement), by Poisson’s equation (3) and the stress tensor is related to the deformations by the stress-strain relationship for an isentropic, Newtonian, compressible fluid. If \(\eta\) is the Newtonian viscosity and \(k\) is the bulk modulus, this relation can be written:

\[
\mathbf{T}_E = \left[ k(\nabla \cdot \mathbf{u}_L) + \mathbf{u}_L \cdot \nabla P_0 \mathbf{I} + \frac{2}{3} \nabla \cdot \mathbf{v}_E \mathbf{I} + \nabla \mathbf{v}_E + \nabla \mathbf{v}_E^T \right]
\] 

where \(\mathbf{v}_E\) is the Eulerian velocity, \(P_0\) is the initial hydrostatic pressure, \(\mathbf{I}\) is the identity tensor, and the subscript T means transpose. For a spherical Earth, these partial differential equations ((7), (3) and (8)) can be transformed into a set of ordinary scalar differential equations of the first order in \(d/dr\) by using spherical harmonic expansions. For example, define together with Eq. (4), the expansion:
where $D_l^m(\theta, \lambda)$ are the Generalised Spherical Harmonics of PHINNEY and BURRIDGE (1973), and the $\hat{e}_n$ are the basis vectors as defined in SMITH (1974).

Then, after some manipulation to eliminate the Lagrangian displacements from Eqs. (7), (3) and (8) transform to:

\[
\frac{dV_R}{dr} = -\frac{1}{\rho_0} \partial_r \rho_0 V_R - \frac{1}{r} (L V_T + 2 V_R),
\]

\[
\frac{dV_T}{dr} = \frac{1}{\eta} Q + \frac{1}{r} (V_T + 2 L V_R),
\]

\[
\frac{dP}{dr} = \frac{12 \eta}{r^2} V_R + \frac{8 \eta}{r \rho_0} \partial_r \rho_0 V_R - \frac{4 \eta}{3 \rho_0^2} (\partial_r \rho_0)^2 V_R
\]

\[
+ \frac{6 L \eta}{r^2} V_R + \frac{2 L \eta}{r \rho_0} \partial_r \rho_0 V_T + \frac{1}{\rho_0} \partial_r \rho_0 P - \frac{L}{r} Q + \rho_0 g_1^E,
\]

\[
\frac{dQ}{dr} = \frac{12 L \eta}{r^2} V_R + \frac{4 L \eta}{r \rho_0} \partial_r \rho_0 V_R + \frac{(6L^2 + 2L^2 \eta)}{r^2} V_T
\]

\[
+ \frac{2L}{r} P - \frac{3}{r} Q - \frac{2L \rho_0}{r} \phi_1^E,
\]

\[
\frac{d\phi_1^E}{dr} = g_1^E,
\]

\[
\frac{dg_1^E}{dr} = -\left( \frac{4 \eta_0}{kr} + \frac{4 \eta}{3k} \partial_r \rho_0 \right) 4\pi G V_R - \frac{8 \pi G L \eta_0}{kr} V_T + \frac{2L^2}{r^2} \phi_1^E - \frac{2}{r} g_1^E,
\]

with $L$ defined in (6) and $L'$ by:

\[
L' = \sqrt{\frac{(l-1)(l+2)}{2}}
\]

where $V_R$, $V_T$ are written for $V_{Rl}^m$, $V_{Tl}^m$, and are the radial and tangential components of the velocity field, where $P$ and $Q$ are written for $P_{ll}^m$, $Q_{ll}^m$, and are, respectively, the radial and tangential components of the stress-tensor. $\phi_1^E$ and $g_1^E$ are written for $\phi_{ll}^E$ and $g_{ll}^E$. $\phi_1^E$ is the Eulerian potential, and $g_1^E$ is defined by the fifth equation of (10). $P$ is linearly related to the Eulerian pressure $P_{ll}^E$ by:

\[
P = -P_{ll}^E + \frac{2 \eta}{3} \frac{1}{\rho_0} \partial_r \rho_0 V_R + 2 \eta \partial_r V_R.
\]
where $\partial_r V_R$ can be replaced by the first equation of (10). In deriving (10), we assumed the spherically-symmetric Earth is neutrally stable, except possibly across internal boundaries, so that:

$$\partial_r \rho_0 = -\frac{\rho_0^2 g_0}{k}$$  \hspace{1cm} (13)

(zero Brunt-Väisälä frequency). This is equivalent to assuming the density obeys the Adams-Williamson condition.

2.3 Without mass readjustment

The second case for the equations of a static Newtonian compressible fluid is for no mass readjustment i.e. for a zero Eulerian density change:

$$\rho E = 0.$$  \hspace{1cm} (14)

For the same choice of variables as in Subsection 2.2, the system of Eqs. (7), (3) and (8) reduces to:

$$\frac{dV_R}{dr} = -\frac{1}{\rho_0} \partial_r \rho_0 V_R - \frac{1}{r} (LV_T + 2V_R),$$

$$\frac{dV_T}{dr} = \frac{1}{\eta} Q + \frac{1}{r} (V_T + 2LV_R),$$

$$\frac{dP}{dr} = \frac{12\eta}{r^2} V_R + \frac{4\eta}{r^2} \partial_r \rho_0 V_R + \frac{6L\eta}{r^2} V_T - \frac{L}{r} Q + \rho_0 \phi^E,$$

$$\frac{dQ}{dr} = \frac{12L\eta}{r^2} V_R + \frac{4L\eta}{r^2} \partial_r \rho_0 V_R + \frac{2(2l^2 + 2l - 1)\eta}{r^2} V_T$$

$$\hspace{1cm} + \frac{2L}{r} P - \frac{3}{r} Q - \frac{2L\rho_0}{r} \phi^E,$$

$$\frac{d\phi^E}{dr} = g^E,$$

$$\frac{dg^E}{dr} = \frac{2L^2}{r^2} \phi^E - \frac{2}{r} g^E,$$

with the same convention as the system of Eq. (10) for the order $l$ and degree $m$. Again, in deriving (15) we assumed the Earth is neutrally stable.

2.4 Equation for the lithosphere

The lithosphere is considered as a solid Hooke body with a shear and a bulk modulus profile, so that in the case where mass readjustment is allowed, the equation are the same as DAHLEN (1974). Nevertheless, their form is different due to our choice of variables:
\[
\frac{dU}{dr} = - \frac{\lambda}{\lambda + 2\mu} \frac{1}{r} (LV + 2U) + \frac{1}{\lambda + 2\mu} P,
\]
\[
\frac{dV}{dr} = \frac{1}{\mu} Q + \frac{1}{r} (V + 2LU),
\]
\[
\frac{dP}{dr} = \frac{4\gamma}{r^2} U - \frac{4\rho_0 g_0}{r} U + \frac{2L\gamma}{r^2} V - \frac{L\rho_0 g_0}{r} V
\]
\[
+ \frac{4\mu}{r(\lambda + 2\mu)} P - \frac{L}{r} Q + \rho_0 g_1^E - 4\pi G \rho_0 U,
\]
\[
\frac{dQ}{dr} = \frac{4L\gamma}{r^2} U - \frac{2L\rho_0 g_0}{r} U + \frac{(2L\gamma + 2L^2 \mu)}{r^2} V
\]
\[
+ \frac{2L\lambda}{(\lambda + 2\mu) r} P - \frac{3}{r} Q - \frac{2L\rho_0}{r} \phi_1^E,
\]
\[
\frac{d\phi_1^E}{dr} = g_1^E - 4\pi G \rho_0 U,
\]
\[
\frac{dg_1^E}{dr} = -4\pi G \frac{L}{r} \rho_0 V + \frac{2L^2}{r^2} \phi_1^E - \frac{2}{r} g_1^E
\]

with
\[
\gamma = \frac{3\lambda + 2\mu}{\lambda + 2\mu} \mu.
\]

In the case where no readjustment is allowed ($\rho_1^E = 0$), the equations become:

\[
\frac{dU}{dr} = -\frac{1}{r} (LV + 2U) - \frac{1}{\rho_0} \partial_r \rho_0 U,
\]
\[
\frac{dV}{dr} = \frac{1}{\mu} Q + \frac{1}{r} (V + 2LU),
\]
\[
\frac{dP}{dr} = 4\frac{\mu}{r \rho_0} \partial_r \rho_0 U + 6\frac{\mu}{r^2} (LV + 2U) - \frac{L}{r} Q + \rho_0 g_1^E,
\]
\[
\frac{dQ}{dr} = 12\frac{L\mu}{r^2} U + \frac{4L\mu}{r \rho_0} \partial_r \rho_0 U + \frac{2(2L^2 + 2L - 1)\mu}{r^2} V
\]
\[
+ \frac{2L}{r} P - \frac{3}{r} Q - \frac{2L\rho_0}{r} \phi_1^E,
\]
\[
\frac{d\phi_1^E}{dr} = g_1^E,
\]
\[
\frac{dg_1^E}{dr} = \frac{2L^2}{r^2} \phi_1^E - \frac{2}{r} g_1^E.
\]
Here $U$ and $V$ are the spherical harmonics coefficients of the Lagrangian displacement in the lithosphere:

$$
\bar{u}_l = \sum_{n=0}^{+} \sum_{l=0}^{\infty} \sum_{m=0}^{l} S_l^{mn}(r) D_l^{mn}(\theta, \lambda), \\
V_l^m = S_l^{m+} + S_l^{m-}, \\
U_l^m = S_l^{m+}.
$$

In deriving (16) and (17), we did not require that the Earth be neutrally stable.

3. The Internal Load and the Outer Surface Boundary Condition

The homogeneous systems (10) or (15), (16) or (17), do not include forcing terms. Suppose the internal load is applied at radius $r = r_0$, and has the form:

$$
\delta p(\bar{r}_{0}) = \delta p_l^m \delta(r - r_0) Y_l^m(\theta_0, \lambda_0)
$$

where $\delta(r - r_0)$ is the dirac-delta function. Then, the equations described in Section 2 can be applied everywhere inside the Earth, except at the loading radius, $r = r_0$ where inhomogeneous boundary conditions are introduced. All the variables are continuous there, except the Eulerian traction $P$ and $g^E_l$. The step in these two variables is due to the density perturbation:

$$
P_+ = P_0 + g_0 \delta p_l^m, \\
g^E_{l+} = g^E_{l-} + 4\pi G \delta p_l^m
$$

where $\delta p_l^m = 1$ for unit forcing, $-$ is written for the side of the boundary nearer the Earth center and $+$ for the opposite side.

As usual, the final solution of the integration inside the Earth is written as a sum of a general solution of the homogeneous system, where $\delta p_l^m$ is set to zero in (20), plus a particular solution which must satisfy (20) at the load position. These solutions are propagated from the center of the Earth up to the surface of the Earth. There homogeneous boundary conditions allow us to find the coefficients of the linear combination of the general and particular solutions. The coefficient of the particular solution in this final linear combination must be equal to one to preserve the unity of the forcing.

For this combination, one uses the superscript 1, 2 or 3 for the three linearly independent vectors of the general solution and the superscript “part” for the particular solution. The surface boundary conditions (no externally-applied surface stresses or gravitational force) lead to the relation:

$$
\begin{pmatrix}
P^1 \\
Q^1 \\
g^E_1 + \frac{l+1}{a} \phi^E_1 + 4\pi G\rho_0 U^1
\end{pmatrix}
\begin{pmatrix}
P^2 \\
Q^2 \\
g^E_2 + \frac{l+1}{a} \phi^E_2 + 4\pi G\rho_0 U^2
\end{pmatrix}
\begin{pmatrix}
P^3 \\
Q^3 \\
g^E_3 + \frac{l+1}{a} \phi^E_3 + 4\pi G\rho_0 U^3
\end{pmatrix}
$$
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\[ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = - \begin{pmatrix} p_{\text{part}} \\ Q_{\text{part}} \\ g_1^{\text{Epart}} + \frac{l + 1}{a} \phi_1^{\text{Epart}} + 4\pi G \rho_0 U_{\text{part}} \end{pmatrix} \]  

(21)

at \( r=a \), where \( a_i \) are the coefficients of the homogeneous solutions in the final linear combination:

\[ \bar{Y} = \sum_{i=1}^{3} a_i \bar{Y}^i + \bar{Y}_{\text{part}}. \]  

(22)

Here

\[ \bar{Y}^i = \begin{pmatrix} U^i \\ V^i \\ P^i \\ Q^i \\ \phi_1^E \\ g_1^E \end{pmatrix} \]

and \( \bar{Y}_{\text{part}} = \begin{pmatrix} U_{\text{part}} \\ V_{\text{part}} \\ P_{\text{part}} \\ Q_{\text{part}} \\ \phi_1^{\text{Epart}} \\ g_1^{\text{Epart}} \end{pmatrix} \)

and the final solution vector \( \bar{Y} \) is:

\[ \bar{Y} = \begin{pmatrix} U \\ V \\ P \\ Q \\ \phi_1^E \\ g_1^E \end{pmatrix} \]

in the lithosphere, and:

\[ \bar{Y} = \begin{pmatrix} V_R \\ V_T \\ P \\ Q \\ \phi_1^E \\ g_1^E \end{pmatrix} \]

everywhere else. The latter solution includes the velocity components but does not directly give the boundary displacements. Those displacements can be computed from the components of \( \bar{Y} \), as described in Section 8 below.

4. Starting Solution

The equations described in Section 2 are solved, once \( \rho_0(r), k(r), \) and \( \eta(r) \) are chosen, by picking starting solutions for \( \bar{Y}^i \) and \( \bar{Y}_{\text{part}} \) at some very small radius (well inside the
inner core), propagating each of those solutions upwards through the Earth using the section 2 equations and some sort of finite difference scheme, propagating across internal boundaries using the boundary conditions described in Section 5 below (plus the boundary condition (20) for the particular solution), and then matching the outer surface boundary condition as described above. Useful starting solutions are the three linearly independent solutions to (10) or (15) for an homogeneous, incompressible sphere, that are regular at the origin. These are:

\[
\begin{pmatrix}
\frac{1}{L} & \frac{1}{L + 1} \\
\frac{2(l - 1)\eta}{L r} & -2(l - 1)(l + 1)\eta \frac{L r}{r}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{1}{L} & \frac{1}{L + 3} \\
\frac{2(l^2 - l - 3)\eta}{L r} & -2l(l + 2)\eta \frac{L r}{r}
\end{pmatrix}
\]

5. New Boundary Conditions

At the “solid”–“solid” and “liquid”–“solid” interfaces and at the surface of the Earth, new boundary conditions adapted to our special static case are to be applied.

5.1 At the inner core-outer core boundary

The inner core-outer core boundary is a “solid”–“liquid” boundary. It is a boundary between a Newtonian fluid and an inviscid fluid. For both the homogeneous solution and the particular solution, three linearly independent starting solutions are found in the inner core and are propagated to the boundary. The radial velocity at each side of the boundary and the tangential stress at each side must be equal to zero by continuity of these variables and thus by their value inside the liquid outer core, a single linear combination of the three solutions is found at the top of the inner core (see Fig. 1(a)). Consequently, the number of degrees of freedom is reduced from 3 to 1. The one remaining solution is then passed to the other side of the boundary so that \( \phi_1^E \) and the following linear combination of \( P \) and \( g_1^E \) is continuous:

\[
g_0 \phi_1^E - 4\pi G P \quad \text{is continuous}
\]

where \( g_0 \) is the initial gravitational acceleration.

5.2 At the outer core-mantle boundary

The outer core-mantle boundary is again a “liquid”–“solid” boundary. It is a boundary between an inviscid fluid and a Newtonian fluid. One solution has been propagated inside the liquid outer core up to this boundary. There, the tangential velocity can be discontinuous. \( P \) and \( g_1^E \) can be discontinuous too, although the continuity equation (24) must be satisfied. The radial velocity and tangential stress must vanish on the mantle side (they are both already set to zero on the fluid core side). Two degrees of
Fig. 1. Description of the internal boundaries.
freedom are again added and three solutions are built at the bottom of the mantle (see Fig. 1(b)).

5.3 **Boundary conditions inside the mantle**

Inside the mantle, the spherical rheological models may have some steps, such as at depths of 400 km or 670 km. At these positions, a "solid"-"solid" boundary is considered. It is a boundary between two Newtonian fluids which have different rheological parameters. Three solutions are propagated up to the boundary inside the mantle. One of the classical "solid"-"solid" boundary conditions for the non-static case (for real solids, rather than viscous fluid) is that the jump in the Eulerian pressure is proportional to the jump in density times the radial Lagrangian displacement. In the static case, the time derivative of this relation leads to the condition that the radial velocity is equal to zero on both sides of the boundary if the density is discontinuous across the boundary. One degree of freedom must then be subtracted and the general solution (just below the boundary) is a linear combination of two new basis vectors (see Fig. 1(c)). Again \( P \) and \( g^E \) can be discontinuous so long as (24) is satisfied, so that one degree of freedom must be added again to get the solution just above the boundary. The solution is then again a linear combination of three basis vectors.

However, this boundary condition (density jump times radial velocity equals zero) is only appropriate for a chemical boundary. It assumes that mantle material in one side of the boundary cannot discontinuously change its material properties by moving to the other side of the boundary. But that is exactly what does happen if the boundary marks a phase transition. Phase transition boundaries are described in Section 6 below.

5.4 **At the upper mantle-lithosphere boundary**

The upper mantle-lithosphere boundary is a "solid"-"solid" boundary. On one side, the upper mantle is a viscous Newtonian fluid for the static case, on the other side, the lithosphere is a Hooke body with defined shear and bulk moduli profiles. On both sides of this boundary, the radial velocity must be equal to zero and this reduces by one the degree of freedom (reduction from 3 to 2), one can consider either the free-slip condition (the tangential stress is equal to zero on both sides of the boundary) or the no-slip condition (the tangential velocity is equal to zero inside the upper mantle and can be anything inside the lithosphere due to the discontinuity). Both cases reduce the degree of freedom from 2 to 1 just below the boundary. There is no continuity condition for the radial or tangential Lagrangian displacements at the bottom of the lithosphere, and so this increases the number of degrees of freedom in the lithosphere back to 3 (see Fig. 1(d)).

6. **Phase Transition Boundary**

Following Christensen's (1985) idea, the density jump across an internal boundary can be either adiabatic or not. Adiabatic density variations imply isentropic compression and decompression with an immediate adjustment of the pressure. Non-adiabatic density variations carried along with the displacement field are able to produce restoring buoyancy forces. Chemical boundaries behave fully non-adiabatically. In the regions between discontinuities, density variations are assumed to be due to self-compression and thus behave fully adiabatically. Phase transitions, if the reaction is fast enough, also
behave adiabatically. Some authors believe that the 670 km discontinuity is a chemical discontinuity with also some phase transition for spinel to post-spinel. It is commonly believed that the 400 km discontinuity is a phase transition zone for olivine to spinel. A phase transition at the 670 km discontinuity would be associated with whole mantle convection and a chemical discontinuity at 670 km would favour two layer convection.

Let us consider, as in SCHUBERT and TURCOTTE (1978), a down-going global displacement. The material going down is cooler than the stagnant mantle at the same depth. It will tend to change phase to the denser form above the normal position of the boundary. Because it becomes denser (due to the zero order temperature gradient which is positive) its downward motion will then tend to increase the global velocity. However if the phase change is exothermic, heat is released. The material is warmed upon changing. This implies a decrease in the element density by thermal expansion, and thus an upward stabilizing body force. The heat evolved due to thermo-dynamic equilibrium on the Clapeyron curve, implies a displacement of the boundary at greater depth and thus the particule (lighter) will experience an upward motion. This results in two upward motions and one downward motion which together restore the boundary close to its original position. In other words, following the explanation of SCHUBERT et al. (1970) or TURCOTTE and SCHUBERT (1971), if relatively cold material from above the interface comes downward near the phase boundary, since the interface must satisfy the Clapeyron relation, the lower temperature forces the interface to a region of lower hydrostatic pressure, and so upward. With the interface displaced upward, the heavier material below the interface gives an hydrostatic pressure variation tending to drive the flow downward, which leads to instability. The balance between the opposing factors is critically dependent upon the values of the heat of phase change and the gradient of the phase change curve (the Clapeyron curve). The net effect is much smaller that the extremes, and either assists or inhibits the displacement. For an exothermic phase change as considered in the example above, and which is usually assumed to hold for the upper-mantle, the net result would tend to be a downward displacement of the boundary. On the other hand, if the phase change boundary is endothermic, which is usually assumed to be the case in the lower mantle, the boundary will be warped upward at the sites of downgoing currents and downward at rising currents.

The amplitude of the net effect for the interface between the upper mantle and the lower mantle will depend on the number of layers in the mantle.

All of these conclusions assume that the particule proceeds through the phase transition zone. But this is not always the case. It depends on the material velocity at the boundary. If the velocity is high enough (above a certain critical value: the VERHOOGEN (1965) limit), the particule reaches the Clapeyron curve earlier and a small amount of material will transform and change structure liberating or using heat to continue to stay on the Clapeyron curve. Below this velocity limit, the material never passes through the boundary. OLSON and YUEN (1982) emphasize that the transition would offer resistance and would hinder convection unless the particule started from a position far above or below the transition zone, and they computed the associated critical velocity. This critical velocity has also been computed by CHRISTENSEN and YUEN (1985) including the effects of an endothermic phase transition and extended Boussinesq approximation (excepting for the driving buoyancy forces, the fluid is treated as being incompressible everywhere including across the phase change region). They also consider the effects of an adiabatic gradient, latent heat, and frictional heating in the energy equation. Their results depend
on the mean depth of the layer, the thickness of the layer, the Rayleigh number (effect of the stratification over effect of thermal and viscous diffusion), the latent heat involved and the viscoelastic rheology. But the major effects are due to lateral temperature differences causing deflection of the boundary.

Schubert et al. (1975) argue that although the "solid"-"solid" transitions in the Earth are polyvariant in nature, the univariant system is a relevant approximation because of the relatively small widths of transition zones compared with their depths. Another reason that the univariant case is considered here is that, if the transition zone is not too large, the results for a divariant case would be similar to the results for an univariant case, where the mean Clapeyron curve is between the two Clapeyron curves. However, the mean Clapeyron curve would have a different slope than either of the two Clapeyron curves.

Following Navrotsky's definition (1980), a phase transition to a first approximation is a rearrangement of the packing of isolated units with a decrease in the empty space between them. Under these conditions, the entropy of a phase correlates well with its molar volume change and the associated variation of the entropy. In the case of low pressure, the variation of molar volume and the variation of the entropy have the same sign. At high pressure however, a denser and more symmetrical packing is associated with an increase of bound distances so that the Clapeyron curve slope is now negative. Negative $P$-$T$ slopes (larger entropies for denser phases) may be the rule rather than the exception under pressure-temperature condition of the lowest part of the upper mantle and in the lower mantle. If one consider a phase transition at the 400 and 670 km, the Clapeyron curve would be positive in the first transition and negative in the second one.

Figure 2 shows the transition for an univariant phase transition for the particular case of a positive Clapeyron curve. This figure is due to the summary of the work of different authors: Jeanloz and Thompson (1983), Turcotte and Schubert (1971) and Verhoogen (1865). Also shown on this figure is the effect of downward and upward motions of a particle, following Verhoogen (1965).

6.1 The 400 km-discontinuity

Olson and Yuen (1982) evaluate the 400 km discontinuity critical velocity at 0.2 cm/yr. Turcotte and Schubert (1971) show that the critical velocity is given by:

$$V_{\text{crit}} = \frac{k_g}{\beta} T_0 \Delta S$$

where $\beta$ is the Clapeyron curve slope, $\Delta S$, the entropy difference (heat of reaction $= T \Delta S$), $k$, the thermal conductivity and $T_0$, the temperature when the phase change is completed.

Richter (1973) assumes a supercritical vertical stratification of the convective velocity for the 400 km olivine-spinel discontinuity. He computes the change in the amplitude of motion due to buoyancy changes and latent heat. In estimating the importance of these effects, one must keep in mind the uncertainties which exist regarding other dynamically important quantities in the upper mantle. Nevertheless Richter could conclude that the changes in the motion due to the phase transition were not significant for his purpose, buoyancy changes at the phase boundary being offset by
corresponding sources and sinks of latent heat. There is only a numerical factor of 3 or 4 between the amplitudes; and this, for convection, following his idea, has no influence on the structure of the motion itself. But it might be important for our purpose.

6.2 The 670 km discontinuity

Examining the situation after the convection current has made its half turn, Vening Meinesz (1957) finds the transition-layer shifted upwards where material is subsiding, and downwards where material is rising. The topography thus caused at the Earth's surface is not isostatically compensated, as the crust's topography normally is, but the mass-compensation is situated deep down in the mantle in the transition-layer.

Fig. 2. Pressure-Temperature relation at a phase transition along the Clapeyron curve Cl for an upwelling, and Cl, for a downwelling, (a) when the motion is emphasised, (b) when the motion is inhibited.
OLSON and YUEN (1982) discuss the stability of the two-layer and one-layer convection, and its dependence on the origin of the 670 km discontinuity. Their conclusions follow immediately from the previous paragraphs. If all the observed density increase is due to change in composition, then mantle can stably convect in two layers. If the density jump is due to combined effects of phase transition and chemical differences, than the conclusion is modified depending on the percent of the effects. OLSON and YUEN (1982) add also that if the chemical component of the density jump is lower than 4 percent, it is possible that whole-mantle convection occurs, being driven thermally and mixing the upper and lower mantle. In that case, the phase transition does not exert a controlling influence on the mantle circulation pattern but it may affect the amplitude of the velocity field.

PELTIER (1980) argues that the 670 km phase transition can contribute constructively to instability and provide modest enhancement of convection rather than inhibition. He also states that the 670 km phase change could have a greater stability effect if there were some increase in viscosity associated with it.

JEANLOZ and RICHTER (1979a, b) examine the thermal state of the lower mantle and the ensuing implication for convection and chemical composition throughout the mantle. They conclude that there is a necessity of two thermal boundary layers, one at the CMB (the D") and one elsewhere within the lower mantle, either at the bottom or at a phase transition zone. This further non-adiabatic region within the lower mantle strongly suggests the existence of a chemical transition which is a barrier to convection.

RICHTER and JOHNSON (1974) consider an increase of the Fe/Mg ratio in addition to the phase transition. They thus consider a chemically layered upper mantle and examine the dynamical consequences. A displacement of the boundary exists also in their computation, but convection can exist only above and below the region of the rapid chemical change that they consider. If there is a phase transition associated with that discontinuity, the two effects can cancel each other, the phase boundary and the chemical boundary responding differently to thermal and velocity perturbation.

RICHTER and MCKENZIE (1981) reconsider this idea of combined effects of chemistry and phase transition. They favour a two-layer mantle. The reservoirs must be isolated from each other on a large scale but they introduce a new idea: the isolation is occasionally and locally violated since the continental float basalts reflect the geochemical properties of a reservoir different from that supplying the mid-ocean ridge basalts. The lower layer should occasionally "leak" into and through the upper 670 km thick layer.

MCKENZIE and WEISS (1975) put forward various arguments in favour of two horizontal scales of convective flow in the mantle at depths less than 670 km. One can immediately think about this in terms of a critical velocity at this boundary.

JEANLOZ and THOMPSON (1983) concluded that a discontinuous reaction occurs in the olivine component of the mantle at the conditions of the 400 km seismological discontinuity; however, following their idea no discontinuous reactions corresponding to the 670 km discontinuity have yet been identified. Thus it may be necessary to involve either an univariant reaction that has not yet been observed experimentally or a chemical discontinuity at this depth.

FJELDSKAAR and CATHLES (1984) also reconsider a chemically stratified mantle and discuss the possibility of distinguishing between adiabatic or partially non-adiabatic mantle from the observations.

CHRISTENSEN and YUEN (1985) find that a possible super- plastic rheology within
the transition zone (due to a strong reduction of the grain size) can favour a layered convection.

In summary, it is evident that no law can be ruled out at the 670 km discontinuity and that whether there is a two- or a one-cell convection in the mantle is still an open question. For this reason, we consider both cases (the radial velocity=0 or not at this boundary) in our numerical computations.

6.3 New boundary conditions at a phase transition

In our case we want to compute the values of our variables on one side of the boundary (+), given their values on the other side (-), i.e. we want to find \( V_R^+, V_T^+, P^+, Q^+, \phi_i^{E+}, g_i^{E+} \) from \( V_R^-, V_T^-, P^-, Q^-, \phi_i^{E-}, g_i^{E-} \) inside the mantle. This "solid" boundary is the same as in Subsection 5.3 except that first, the condition \( V_R^=V_R^- = 0 \) must be replaced by \( V_R^=x\%V_R^- \) where \( x \) is determined from experiments (see below), and second, there is a jump in the pressure at a phase transition boundary. The variation in density \( \delta \rho(r, \theta, \lambda) \) at any point \((r, \theta, \lambda)\) in the Earth is related to the increase or decrease of temperature and pressure:

\[
\delta \rho = - \alpha \delta T(r, \theta, \lambda) + \gamma k_s \delta P(r, \theta, \lambda) \tag{26}
\]

where \( \alpha \) is the thermal expansion, \( \gamma \) is the Gruneisen parameter and \( k_s \) is the bulk modulus. This relation (26) allows us to relate the change in temperature to the change in pressure and density, given values for the parameters \( \alpha, \gamma, \) and \( k_s \). \( \delta P \) is described by the pressure scalar \( P_r^E \) related to \( P \) in Subsection 2.1 by Eq. (12) and \( \delta \rho \) has contributions from the applied internal load, and from \( \rho_i^E \), when mass readjustment is included.

At a phase transition boundary, it is possible to compute the motion of the boundary due to the loading effect. The boundary deformation can be used to find the jump in pressure across the boundary. Suppose that the phase transition is described by the temperature-pressure relation \( P=F(T) \), where \( F \) is some known function. For the non-convecting spherical case, assume that the phase transition occurs at \( r_0 \) for all \((\theta, \lambda)\). Then, the pressure and temperature at the transition, \( P_0(r_0) \) and \( T_0(r_0) \) satisfy:

\[
P_0(r_0) = F(T_0(r_0)). \tag{27}
\]

For the convecting case, suppose the pressure \( (P) \) and the temperature \( (T) \) are:

\[
P(r, \theta, \lambda) = P_0(r) + \delta P(r, \theta, \lambda), \tag{28}
\]

\[
T(r, \theta, \lambda) = T_0(r) + \delta T(r, \theta, \lambda). \tag{29}
\]

The phase transition now occurs at points \((r, \theta, \lambda)\), where

\[
P(r, \theta, \lambda) = F(T(r, \theta, \lambda)). \tag{30}
\]

Suppose the boundary radius at \((\theta, \lambda)\) is

\[
r = \bar{r}(\theta, \lambda) = r_0 + \bar{dr}(r_0, \theta, \lambda). \tag{31}
\]
After Taylor expansions and using (28), (29) and (31), Eq. (30) can be written:

\[
\delta r(r_0, \theta, \lambda) \left[ \frac{dP_0}{dr}(r_0) - \frac{dF}{dT}(T_0) \frac{dT_0}{dr}(r_0) \right] = \frac{dF}{dT}(T_0) \delta T(r_0, \theta, \lambda) - \delta P(r_0, \theta, \lambda). \tag{32}
\]

Here, \(\delta T(r_0, \theta, \lambda)\) and \(\delta P(r_0, \theta, \lambda)\) are the perturbations in temperature and pressure either just inside or just outside the boundary. If values outside the boundary are used, then the values for \(dP_0(r_0)/dr\) and \(dT_0(r_0)/dr\) should also be those just outside the boundary. Otherwise, \(dP_0(r_0)/dr\) and \(dT_0(r_0)/dr\) should be estimated just inside the boundary.

The jump in pressure across the boundary is equal to \(g_0[\rho+(r_0) - \rho-(r_0)]\delta r\). So, using (26) in (32), we find the jump in the pressure scalar \(P\) across a phase boundary:

\[
P_+ - P_- = g_0[\rho+(r_0) - \rho-(r_0)] \left( \frac{gks}{\alpha \frac{dF(T_0)}{dT} - 1} \right) P - \frac{dF(T_0)}{dT} \frac{1}{\alpha} (\delta \rho_l^m + \rho_{li}^{Em}) \frac{dP_0(r_0)}{dr} - \frac{dF(T_0)}{dT} \frac{dT_0(r_0)}{dr} \tag{33}
\]

Here, \(\delta \rho_l^m\) and \(\rho_{li}^{Em}\) are the spherical harmonic coefficients of the applied load, and the perturbed Eulerian density, respectively (if mass adjustments are not allowed, then \(\rho_{li}^{Em}\) vanishes).

Equation (33) gives the pressure jump across the boundary in terms of the pressure and density scalars, and so it, together with the continuity equation for the vertical velocity, provides boundary conditions at phase boundaries. The values for both of these scalars and for the other variables in (33) should either all be taken just outside the boundary, or all just inside the boundary.

For the computation of the effects of the phase transition zones inside the mantle, for supercritical velocities, we need values of the different parameters. One important parameter is the Clapeyron curve slope, \((dF/dT)(T_0)\).

Based on recent experiments (KATSURA and ITO, 1989; CHAPELAS, 1990b) and thermodynamic calculations (AKAOGI et al., 1989), the slope of the phase boundary between olivine and modified spinel is well constrained to be:

\[
\frac{dP}{dT} = 26 \pm 1 \text{ bars}/\text{°K}.
\]

CHRISTENSEN and YUEN (1985) estimate the Clapeyron slope at \(-60\text{ bar}/\text{°K}\) for the \(\gamma\)-spinel-perovskite phase transition, presumably at the 670 km discontinuity. ITO and TAKAHASHI (1989) experimentally determine a Clapeyron slope of \(-25\text{ bar}/\text{°K}\) from phase equilibria studies in a multi-anvil press.

Concerning the values of the change in the velocity fields at phase transitions at mantle conditions, a summary of data is provided in DUDDY and ANDERSON (1989) by adiabatically compressing the mantle. Using new results for the volume dependence of
the thermal expansion coefficient of \( \langle \partial \ln \alpha / \partial \ln V \rangle = 5.5 \pm 0.5 \) (CHOPELAS and BOEHLER, 1989), CHOPELAS (1990a, b) computes a 7.5% increase in the compressional velocity \( (\omega_k) \) for the olivine to \( \beta \)-spinel transition at mantle pressure and temperature at 400 km depth. This corresponds to a mantle with 55 to 60% olivine since the average seismic velocity increase at 400 km is about 4.5%.

7. Summary and Strategy

The computation of the internal loading Green's function can be done in the following way:

I. Computation of the general solution
   1. Computation of the starting solution
   2. Integration inside the inner core
   3. Application of the inner core-outer core boundary conditions
   4. Integration inside the outer core
   5. Application of the outer core-mantle boundary conditions
   6. Integration inside the mantle
   7. Application of the mantle-mantle boundary conditions at density steps
   8. Integration inside the mantle
   9. Application of the upper mantle-lithosphere boundary conditions
   10. Integration inside the lithosphere up to the surface of the Earth.

II. Computation of the particular solution.

This computation follows the same chart as the general solution up to the load position. There, particular boundary conditions are applied.

At the next integration step, the propagation up to the surface of the Earth can be done as for the general solution, following the same chart.

III. Combination of the general and the particular solution.

At the surface of the Earth, a linear combination of the general solution and the particular solution (with an unit coefficient) can be found according to the homogeneous surface boundary conditions.

For each degree \( l \) and order \( m \) and for each loading position \( r_0 \), a solution for unit \( \delta \rho_l^m \) is found everywhere inside the Earth. Appropriate combinations of variables will give Green's functions for things like the surface geoid, internal boundary deformation, etc., for fixed \( (l, m, r_0) \).

8. Other Green's Function

Inside any region of the viscous fluid where no loading has been considered (i.e. where \( \rho_i^E \) is not zero), one can easily get \( \rho_i^E \) from the values of the variables:

\[
\rho_i^E = -\frac{\rho_0 P}{k} - \frac{4}{3} \frac{\eta}{k} \delta \rho_i V_r - \frac{4\eta \rho_0}{kr} V_r - \frac{2\eta L \rho_0}{kr} V_t. \tag{34}
\]

From \( \rho_i^E \), one can get the figure of the equi-density surfaces:
\[ \varepsilon^m_i(r) = -\frac{\rho^E_{m}(r)}{r \varrho_{0}(r)} \]  

where \( \varepsilon^m_i \) are defined so that those surfaces are described by:

\[ r' = r \left( 1 + \sum_{l,m} \varepsilon^m_i(r) Y^m_l(\theta, \lambda) \right) \]  

Boundary displacements, \( U \), can be computed from the discontinuity of \( g^E_i \), at boundaries where there is a jump in the density:

\[ U = \frac{g^E_{i+} - g^E_{i-}}{4\pi G(\varrho_+ - \varrho_-)} \]  

where + and − are again written for the two sides of the boundary, −, nearest the Earth center, +, nearest the surface.

9. Conclusion

We have described a method for computing the response of the Earth to the internal loading associated with convection. The Earth need not be incompressible or composed of homogeneous layers, though we do require it to be neutrally stable below the lithosphere. Phase transition boundaries can be included, if the relevant thermodynamic parameters associated with the phase transition are specified.

The generality of the approach makes it possible to more fully address certain questions related to boundary deformation and parameter estimation. We will discuss applications in a forthcoming paper (DEHANT and WAHR, 1991).

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The Response of a Compressible, Non-Homogeneous Earth to Internal Loading


