A New Class of Exact Solutions for 3D Compressible Reconnection Problem

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For the steady-state three-dimensional reconnection problem in a compressible plasma, we have found a new class of exact solutions, which consist of ideal-MHD regions, MHD singularities (slow shocks and rotational discontinuities), and a neutral line. This solution predicts the existence of plasma jetting parallel to the neutral line, in addition to the usual plasma jetting in the direction perpendicular to the neutral line. However, contrary to the suggestion by LAU and FINN (1990), and PRIEST and FORBES (1992) that three-dimensional reconnection regions should have velocity singularities, our solution has well-behaved velocity fields without any singularity.

1. Introduction

At the dayside magnetopause, magnetic reconnection controls the energy transfer process from the solar wind to the magnetosphere, when the interplanetary magnetic field has a southward component. Since the interplanetary magnetic field generally has other components, sun-earth and dawn-to-dusk components, three dimensionality is essential for the reconnection problem at the magnetopause. In the study of the magnetotail, where magnetic reconnection plays an essential role during substorm energy conversion processes, theoretical consideration has been focused to the reconnection of anti-parallel field lines. Observations show, however, occasional existence of the dawn-dusk component of the magnetic field, originating in the dawn-dusk component of the interplanetary magnetic field. This component makes the field structure of the magnetotail reconnection region three dimensional also.

Recently, it is suggested that the dynamical properties in three-dimensional reconnection regions are significantly different from those in two-dimensional reconnection regions (SCHINDLER *et al.*, 1988; HESSE and SCHINDLER, 1988; GREENE, 1988; WRIGHT and BERGER, 1989; LAU and FINN, 1990; PRIEST and FORBES, 1992). Since exact solutions for the three-dimensional reconnection problem have not been known so far, the arguments presented by the above authors are either qualitative or based on approximate solutions. In this letter, we present an exact solution for the steady-state three-dimensional reconnection problem in compressible plasmas, in which the reconnection region consists of ideal-MHD regions, MHD singularities and a neutral line. (Since we still drop the spatial dependence in one direction, $\partial/\partial z=0$, our solution might be better called a $2\frac{1}{2}$ -dimensional solution.)

2. Description of the New Solution

We shall consider a reconnection process between plasmas of the same pressure P_1 and mass density ρ_1 . The original magnetic fields (namely the magnetic fields before reconnection) have the same magnitude ($|B_1| = |B_5|$) but make an *oblique* angle $\pi - 2\theta_{B_1}$. The coordinate system is

Fig. 1. Illustration of the solution for $M_1 = 0.5$ and $\beta_1 = 1$. (a) An xy cross section of the reconnection region (namely a view looking into the $-z$ direction) (b) A quarter view of the panel (a). (c) An yz cross section of the reconnection region (namely a view looking into the $-x$ direction) (d) A three-dimensional view of the reconnection region. Dashed lines in panels (a), (c), and (d), $[a'_2 \rightarrow N \rightarrow a_2]$ and $[b_2 \rightarrow N \rightarrow b'_2]$, are separatrices.

taken so as to represent these fields as,

$$\begin{aligned} \vec{B}_1 &= (0, B_1 \cos \theta_{B_1}, B_1 \sin \theta_{B_1}), & (x > 0) \\ \vec{B}_5 &= (0, -B_1 \cos \theta_{B_1}, B_1 \sin \theta_{B_1}). & (x < 0) \end{aligned} \quad (1)$$

We require that during the reconnection process the field lines as well as the stream lines are straight except bends at shocks and discontinuities. We note that the requirement for the straightness is rather technical but not physical. Physically admissible solutions could have curved magnetic field lines and/or curved stream lines in the regions among shocks/discontinuities. To make the problem numerically tractable, however, we shall not consider these general solutions

in this letter. Under the above requirement, we have found that four rotational discontinuities (R_a , R_b , R'_a , and R'_b) and four slow shocks (S_a , S_b , S'_a , and S'_b) are needed for the field lines to reconnect. (Without rotational discontinuities, the field lines cannot have necessary three-dimensional bends. Without slow shocks the necessary energy conversion from the magnetic field to the particles cannot occur.) The geometry of our solution is illustrated in Fig. 1, where the intersection line of four discontinuities is taken as the z axis. Since the y component of the magnetic field is zero along this line, this is the “neutral” line ($\mathbf{N}$) as is called traditionally.

In the regions between the rotational discontinuities and slow shocks, the magnetic fields are,

$$\begin{aligned}\vec{B}_2 &= (B_1 \cos \theta_{B_2} \cos \varphi_{B_2}, B_1 \cos \theta_{B_2} \sin \varphi_{B_2}, B_1 \sin \theta_{B_2}) \text{ between } R_a \text{ and } S_a, \\ \vec{B}'_2 &= (-B_1 \cos \theta_{B_2} \cos \varphi_{B_2}, B_1 \cos \theta_{B_2} \sin \varphi_{B_2}, B_1 \sin \theta_{B_2}) \text{ between } R'_a \text{ and } S'_a,\end{aligned}\quad (2)$$

$$\begin{aligned}\vec{B}_4 &= (B_1 \cos \theta_{B_2} \cos \varphi_{B_2}, -B_1 \cos \theta_{B_2} \sin \varphi_{B_2}, B_1 \sin \theta_{B_2}) \text{ between } R_b \text{ and } S_b, \\ \vec{B}'_4 &= (-B_1 \cos \theta_{B_2} \cos \varphi_{B_2}, -B_1 \cos \theta_{B_2} \sin \varphi_{B_2}, B_1 \sin \theta_{B_2}) \text{ between } R'_b \text{ and } S'_b.\end{aligned}\quad (3)$$

The magnetic fields downstream of pairs of slow shocks are,

$$\begin{aligned}\vec{B}_3 &= (B_3 \cos \theta_{B_3}, 0, B_3 \sin \theta_{B_3}) \text{ downstream of } S_a \text{ and } S_b, \\ \vec{B}'_3 &= (-B_3 \cos \theta_{B_3}, 0, B_3 \sin \theta_{B_3}) \text{ downstream of } S'_a \text{ and } S'_b.\end{aligned}\quad (4)$$

For simplicity, we have required in the above that the magnetic field $\vec{B} = (B_x, B_y, B_z)$ and flow velocity $\vec{V} = (V_x, V_y, V_z)$ satisfy the following symmetry property: With respect to the yz ($x = 0$) plane, B_x , B_z , and V_y are symmetric, and B_y , V_x , and V_z are anti-symmetric. With respect to the xz ($y = 0$) plane, B_y , B_z , and V_x are symmetric, and B_x , V_y , and V_z are anti-symmetric. If we do not pose these requirements, a variety of asymmetric solutions results. We shall describe these asymmetric solutions in a future publication.

Figure 1a shows the xy cross section of the reconnection region. A quarter view of Fig. 1a ($x \geq 0$ and $y \geq 0$) is shown in Fig. 1b, where we define the attack angles of R_a and S_a , φ_1 and φ_3 . Dashed lines show the stream lines in the three regions, where flow Alfvén Mach numbers are $M_1 (=|V_{1x}|/V_{A1})$ upstream of R_a , $M_2 (=|\vec{V}_2|/V_{A1})$ between R_a and S_a , and $M_3 (=V_{3y}/V_{A1})$ downstream of S_a . V_{A1} is the Alfvén velocity $B_1/\sqrt{4\pi\rho_1}$ in the upstream region of R_a . While the structure shown in Fig. 1a and 1b looks similar to the Sonnerup's solution (SONNERUP, 1970), it differs in the three-dimensional sense. In particular, the coplanarity planes at four slow shocks all differ and do not coincide with the xy plane. The yz cross section and a three-dimensional view of the same reconnection region are shown in Fig. 1c and 1d. A field line, $[a'_1 \rightarrow R'_a \rightarrow R_a \rightarrow a_1]$, is convected toward $\mathbf{N}$ in the $-x$ direction. This field line becomes a separatrix $[a'_2 \rightarrow \mathbf{N} \rightarrow a_2]$, when it arrives at the neutral line $\mathbf{N}$. At the neutral line, this field line is to be reconnected with another separatrix, $[b_2 \rightarrow \mathbf{N} \rightarrow b'_2]$, which has been convected toward $\mathbf{N}$ in the $+x$ direction ($[b_1 \rightarrow R_b \rightarrow R'_b \rightarrow b'_1] \Rightarrow [b_2 \rightarrow \mathbf{N} \rightarrow b'_2]$). After the reconnection, these two field lines become a pair of field lines, $[b_3 \rightarrow S_b \rightarrow S_a \rightarrow a_3]$ and $[a'_3 \rightarrow S'_a \rightarrow S'_b \rightarrow b'_3]$, ejected respectively in the $+y$ and $-y$ directions. (Note that the parts $[S_b \rightarrow S_a]$ and $[S'_a \rightarrow S'_b]$ in Fig. 1c, whose projections are parallel to the z axis in this panel, are not well resolved graphically.)

The numerical method by which solutions are obtained is similar to that used by HAMEIRI (1979). We search a solution for a given angle φ_3 of the slow shock S_a ($0 < \varphi_3 < 90^\circ$). First, we set trial values of φ_{B_2} and θ_{B_2} . For these angles, we determine $\vec{V}_2$ and $\vec{V}_3$, the upstream and downstream velocities around S_a , using the shock Rankin-Hugoniot condition and the symmetry condition, $V_{3x} = V_{3z} = 0$. Second, we determine the angle φ_1 so that the stationary condition for

Fig. 2a. Dependence of reconnection parameters on the inflow Alfvén Mach number M_1 for the case of upstream $\beta_1 = 0$. For the angles, θ 's and ϕ 's, the unit of degree is used.

the rotational discontinuity R_a , $\vec{V}_2 \cdot \vec{n}_{\text{rot}} = \vec{B}_2 \cdot \vec{n}_{\text{rot}} / \sqrt{4\pi\rho_2}$, is satisfied, where $\rho_2 (= \rho_1)$ is the mass density, and $\vec{n}_{\text{rot}}$ the unit normal vector of the rotational discontinuity, $(-\sin\varphi_1, \cos\varphi_1, 0)$. Across R_a , the magnetic field rotates around $\vec{n}_{\text{rot}}$ as the axis, keeping its magnitude ($|\vec{B}_2| = |\vec{B}_1|$). Utilizing this freedom of rotation, we set $B_{1x}=0$. Using the Walén relation for rotational discontinuities, $\vec{V}_2 - \vec{V}_1 = -(\vec{B}_2 - \vec{B}_1)/\sqrt{4\pi\rho_2}$, we determine $\vec{V}_1$, the velocity in the upstream region of R_a . By changing φ_{B_2} and θ_{B_2} we make $\vec{V}_1$ point to the $-x$ direction. (The actual algorithm consists of a double bisection method searching of φ_{B_2} and θ_{B_2} for $V_{1y} = V_{1z} = 0$). Note that the theta angle of the original magnetic field, θ_{B_1} , is not given a priori, but determined when the relation at R_a is found.

The parameters (M_2 , M_3 , B_3/B_1 , φ_1 , φ_{B_2} , φ_3 , θ_{B_1} , θ_{B_2} , θ_{B_3} , φ_{V_2} and θ_{V_2}) are plotted against the inflow Alfvén Mach number M_1 in Fig. 2a and 2b for the cases of $\beta_1 = 0$ and $\beta_1 = 1$, respectively. The azimuthal angle φ_{V_2} and the elevation angle θ_{V_2} are flow deflection angles at the

Fig. 2b. Dependence of reconnection parameters on the inflow Alfvén Mach number M_1 for the case of upstream $\beta_1 = 1$.

rotational discontinuity R_a . Since $\theta_{V_2} < 0$, V_z is negative in the region between R_a and S_a (and also between R'_b and S'_b). In the regions between R_b and S_b , and R'_a and S'_a , on the other hand, V_z is positive. Therefore, the distribution of V_z around the neutral line N has a quadrupole-type structure. Downstream the slow shocks (S_a , S_b , S'_a , and S'_b) the plasma flow is deflected in the $\pm y$ directions. These plasma jets in the $\pm y$ direction downstream of the slow shocks are typical of the reconnection geometry.

Above certain values of M_1 (>0.8 for $\beta_1=0$ and >1.0 for $\beta_1=1$), the solution becomes non-unique: For the same M_1 there are two solutions with higher and lower M_3 . This finding is similar to what Hameiri found in his model. However, as seen in Fig. 2a and 2b, the attack angle of the rotational discontinuity against the upstream flow (M_1), φ_1 , becomes $\lesssim 30^\circ$ for $M_1 \gtrsim 1$. For the realistic reconnection geometry, we may require that $\varphi_1 > 45^\circ$, so that we should limit the discussion to the solutions with $M_1 \lesssim 0.5$. For low to moderate M_1 (0–0.5), the value range of the

angle of the original magnetic fields, θ_{B_1} , is quite limited: For $\beta = 0$ (Fig. 2a), $55^\circ < \theta_{B_1} < 58^\circ$, and for $\beta = 1$ (Fig. 2b), $56^\circ < \theta_{B_1} < 60^\circ$. Outside the above range of θ_{B_1} , solutions of the new class do not exist.

3. Summary and Comment

We have described a new class of exact solutions for the three-dimensional reconnection process, which occurs between *oblique* original magnetic fields. These solutions are not compressible counterparts of the Sonnerup's incompressible solutions, since the new solutions do not connect to the anti-parallel geometry ($\theta_{B_1} \rightarrow 0$). For the pure anti-parallel geometry, HAMEIRI (1979) found a class of exact solutions for compressible reconnection problem. Although our solutions are similar to his in that both solutions have three-dimensional geometry and consist of slow shocks and rotational discontinuities, our solutions have different symmetry property and form an independent class of exact solutions.

The new solution for the three-dimensional reconnection predicts the existence of plasma jets along the neutral line ($\pm V_z$) in the regions between the rotational discontinuities and the slow shocks, in addition to the usual plasma jets in the perpendicular direction to the neutral line ($\pm V_y$) in the downstream regions of the slow shocks. Since our solution has well-behaved velocity fields without any singularity, it provides a counter example against the suggestion by LAU and FINN (1990), and PRIEST and FORBES (1992), that three-dimensional reconnection regions should have velocity singularities. Note that in these previous studies the magnetic field component parallel to the neutral line, B_z , is *assumed* to be uniform in the entire reconnection region. We suggest that this assumption is too restrictive and leads these authors to singular (possibly unphysical) solution. In our solution, B_z does change at shocks and discontinuities. However, since what we have obtained is mathematically a *particular* solution but not a *general* solution, we need further studies to prove or disprove the existence of the velocity singularities in the three-dimensional reconnection region.

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