Fluid Motions in a Sphere I

Thermal Instability of a Rotating Fluid Sphere Heated within

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Abstract

Thermal instability of an incompressible rotating fluid sphere with negligible viscosity heated within is examined. It is found that we are not able to have stationary convections; Instability first sets in as overstability.

I. Introduction

In recent years the onset of thermal instability in horizontal layers of viscous fluid, heated from below under the influence of Coriolis acceleration and magnetic field has been investigated [1]–[4]. These studies have disclosed a number of novel features. Extent of these investigation to a fluid sphere was carried out by Chandrasekhar [5] and to a rotating sphere by Takeuchi and Shimazu [6] and Chandrasekhar [7] in the case of axial symmetry under modified boundary conditions. We are interested in the convection in the earth's core or in the stars under the influence of Coriolis force and magnetic field. The present paper is devoted, as the first step, to problems of a rotating fluid sphere with internal heat sources.

2. The Equation of the Problem

We shall consider a homogeneous sphere of radius a rotating with an angular velocity Ω about the z -axis. A heat generating source is supposed. The rate of heat generation is assumed such that in the absence of conduction and convection the temperature would rise at a uniform rate ϵ . The deduction of the equation governing small departures was done by Chandrasekhar [5], [7]. The treatment is given in appendix.

The temperature distributions in the state of no convection is given by

$$T = \beta(a^2 - r^2), \quad \beta = \epsilon/6\kappa, \quad (1)$$

where κ is the coefficient of thermometric conductivity. The equations governing small departures from the stationary state are [A, (14), (15)]

$$\frac{\partial \theta}{\partial t} = \kappa \nabla^2 \theta + 2\beta(r u_r), \quad (2)$$

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla \frac{\partial p}{\partial \rho} + \gamma \theta \mathbf{r} + \nu \nabla^2 \mathbf{u} + 2\mathbf{u} \times \Omega, \quad (3)$$

and $\text{div } \mathbf{u} = 0, \quad (4)$

where θ is the perturbations in temperature.

We can eliminate δp from eqn. (3). by taking its curl. Making use of eqn. (4). we have

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \gamma \nabla \theta \times \mathbf{r} + \nu \nabla^2 \boldsymbol{\omega} + 2\Omega \frac{\partial \mathbf{u}}{\partial z}, \quad (5)$$

where $\boldsymbol{\omega}$ denotes the vorticity. Next, by taking the curl of eqn. (5), we obtain

$$-\frac{\partial}{\partial t} \nabla^2 \mathbf{u} = \gamma \text{curl}(\nabla \theta \times \mathbf{r}) - \nu \nabla^4 \mathbf{u} + 2\Omega \frac{\partial \boldsymbol{\omega}}{\partial z}. \quad (6)$$

We can express the velocity field as a superposition of poloidal and toroidal vector in terms of two scalars.

$$\mathbf{u} = \nabla T \times \mathbf{i}_r + \text{curl}(\nabla S \times \mathbf{i}_r), \quad (7)$$

where $\mathbf{i}_r$ is the unit vector of radius direction. Two scalars T and S are expressed as follows

$$T = \sum_{m,n} T_n^m(r, t) Y_n^m \text{ and } S = \sum_{m,n} S_n^m(r, t) Y_n^m, \quad (8)$$

where Y_n^m is a surface spherical harmonic.

The following procedure is equivalent to take the poloidal and the toroidal components of eqn. (5).

Multiplying eqn. (5) by $\mathbf{r}$, we get

$$\frac{\partial(r\omega_r)}{\partial t} = \nu \nabla^2(r\omega_r) + 2\Omega \mathbf{r} \frac{\partial \mathbf{u}}{\partial z}, \quad (9)$$

and

$$-\frac{\partial}{\partial t} \nabla^2(rur) = \gamma L^2 \theta - \nu \nabla^4(rur) + 2\Omega \mathbf{r} \frac{\partial \boldsymbol{\omega}}{\partial z}, \quad (10)$$

where

$$L^2 = r^2 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \nabla^2 \right).$$

We express $\theta = \sum_{m,n} \Theta_n^m(r, t) Y_n^m$. Operating Y_n^m to eqns. (2), (9) and (10) and executing the integration with respect to ϑ and φ , we get

$$\frac{\partial}{\partial t} \Theta_n^m - \kappa \frac{1}{r^2} \left[\frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - n(n+1) \right] \Theta_n^m - 2\beta n(n+1) \frac{S_n^m}{r} = 0, \quad (11)$$

$$\begin{aligned} & \frac{\partial}{\partial t} n(n+1) \frac{T_n^m}{r} - \nu n(n+1) \frac{1}{r^2} \left[\frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - n(n+1) \right] \frac{T_n^m}{r} - 2\Omega \left[\frac{(n-1)(n+1)(n-m)}{2n-1} \right. \\ & \times r^{n-1} \frac{d}{dr} (r^{-n} S_{n-1}^m) + \frac{n(n+2)(n+m+1)}{2n+3} r^{-(n+2)} \frac{d}{dr} (r^{n+1} S_{n+1}^m) + im \frac{T_n^m}{r} \left. \right] = 0, \end{aligned} \quad (12)$$

and

$$\begin{aligned} & -n(n+1) \frac{\partial}{\partial t} \frac{1}{r^2} \left[\frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - n(n+1) \right] \left\{ \frac{S_n^m}{r} + \nu n(n+1) \left[\frac{1}{r^2} \left[\frac{d}{dr} \left(r^2 \frac{d}{dr} \right) \right. \right. \right. \right. \\ & \left. \left. \left. - n(n+1) \right] \right] \frac{S_n^m}{r} - 2\Omega \left[\frac{(n-1)(n+1)(n-m)}{2n-1} r^{n-1} \frac{d}{dr} (r^{-n} T_{n-1}^m) \right. \right. \\ & \left. \left. + \frac{n(n+2)(n+m+1)}{2n+3} r^{-(n+2)} \frac{d}{dr} (r^{n+1} T_{n+1}^m) - im \frac{1}{r} \left[\frac{d^2}{dr^2} - \frac{n(n+1)}{r^2} \right] S_n^m \right] \right\} \\ & = \gamma n(n+1) \Theta_n^m. \end{aligned} \quad (13)$$

The following transformation of notations are made

$$r' = \frac{r}{a} \quad \text{and} \quad T_n^m(r, t) = \frac{1}{r} T_n^m(r, t), \quad (14)$$

but for simplicity, we write r' and $T_n^m(r, t)$ as r and $T_n^m(r, t)$. Thus eqns. (11), (12) and (13) becomes as follows.

$$\frac{\partial}{\partial t} a^2 \Theta_n^m - \kappa \frac{1}{r^2} \left\{ \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - n(n+1) \right\} \Theta_n^m - 2\beta a n(n+1) \frac{S_n^m}{r} = 0, \quad (15)$$

$$\begin{aligned} n(n+1)a^2 \frac{\partial}{\partial t} \frac{T_n^m}{r^2} - \nu n(n+1) \frac{1}{r^2} \left\{ \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - n(n+1) \right\} \frac{T_n^m}{r^2} \\ - 2\Omega a^2 \left\{ \frac{(n-1)(n+1)(n-m)}{2n-1} r^{n-1} \frac{d}{dr} (r^{-n} S_{n-1}^m) + \frac{n(n+2)(n+m+1)}{2n+3} \right. \\ \left. \times r^{-(n+2)} \frac{d}{dr} (r^{n+1} S_{n+1}^m) + im \frac{T_n^m}{r^2} \right\} = 0, \end{aligned} \quad (16)$$

and

$$\begin{aligned} -n(n+1)a^2 \frac{\partial}{\partial t} \frac{1}{r^2} \left\{ \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - n(n+1) \right\} \frac{S_n^m}{r} + \nu n(n+1) \left[\frac{1}{r^2} \left\{ \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) \right. \right. \\ \left. \left. - n(n+1) \right\} \right] \frac{S_n^m}{r} - 2\Omega a^2 \left[\frac{(n-1)(n+1)(n-m)}{2n-1} r^{n-1} \frac{d}{dr} (r^{-n-1} T_{n-1}^m) \right. \\ \left. + \frac{n(n+2)(n+m+1)}{2n+3} r^{-(n+2)} \frac{d}{dr} (r^n T_{n+1}^m) - im \frac{1}{r} \left\{ \frac{d^2}{dr^2} - \frac{n(n+1)}{r^2} \right\} S_n^m \right] \\ = \gamma n(n+1) a^5 \Theta_n^m \end{aligned} \quad (17)$$

Equations (15), (16) and (17) are the basic equations of this problem.

3. Boundary Conditions

We treat eqns. (15), (16) and (17) neglecting the viscous term. This simplification is allowed when we consider some geophysical or astrophysical problems. For example, in the Earth's Core, kinematic viscosity is estimated to be $10^{-3} \sim 10^9$. The order of magnitude of viscous force, $\rho_0 \nu U/a^2$, is $10^{-20} \sim 10^{-8}$, which is much smaller than 10^{-4} , order of magnitude of Coriolis force [9].

Thus boundary conditions are not dependent on whether the bounding surface considered is rigid or free. In all cases, we require that

$$\theta = 0 \quad \text{and} \quad u_r = 0 \quad (\text{on a bounding spherical surface}) \quad (18)$$

4. The Solutions of Basic Equations

From eqns. (15), (16) and (17), we can find that there exist groups of fluid motions which contain S_n^m or T_n^m with $n=m$ as their first number. Axially symmetric case is an exceptional one from this rule.

The way of coupling of fluid motions is shown schematically as follows

$$\begin{aligned} S_1^0 &\longleftrightarrow T_2^0 \longleftrightarrow S_3^0 && \dots\dots\dots \\ T_1^0 &\longleftrightarrow S_2^0 \longleftrightarrow T_3^0 && \dots\dots\dots \\ S_1^1 &\longleftrightarrow T_2^1 \longleftrightarrow S_3^1 && \dots\dots\dots \\ T_1^1 &\longleftrightarrow S_2^1 \longleftrightarrow T_3^1 && \dots\dots\dots \\ S_2^2 &\longleftrightarrow T_3^2 \longleftrightarrow S_4^2 && \dots\dots\dots \end{aligned}$$

$$T_{\frac{3}{2}}^2 \longleftrightarrow S_{\frac{3}{2}}^2 \longleftrightarrow T_{\frac{5}{2}}^2 \dots \dots \dots \quad (18)$$

etc.

We can satisfy the boundary conditions on $\Theta_n^m(r, t)$ and $S_n^m(r, t)$ by expanding them in terms of the various modes which form a complete set of orthogonal functions.

Thus

$$\Theta_n^m(r, t) = \sum_j \Theta_{n,j}^m \frac{J_{n+\frac{1}{2}}(\alpha_j r)}{\sqrt{r}} \quad \text{and} \quad \frac{S_n^m(r, t)}{r} = \sum_j S_{n,j}^m \frac{J_{n+\frac{1}{2}}(\alpha_j r)}{\sqrt{r}}, \quad (18)$$

where $J_{n+\frac{1}{2}}$ denote the Bessel functions of order $n+1/2$. The α_j 's ($j=1, 2, \dots$) are its zeros.

5. An Examination of the Principle of the Exchange of Stabilities

a) [$S_n^m, T_{n+1}^m, \dots$] motion

We shall first examine stability of [$S_n^m, T_{n+1}^m, \dots$] motion taking only the first two motions S_n^m and T_{n+1}^m . It seems that this simplification does not alter the results to any great extent [6], [7]. We expand $S_n^m(r, t)$, $T_{n+1}^m(r, t)$ and $\Theta_n^m(r, t)$ as follows,

$$\frac{S_n^m(r, t)}{r} = \sum_j S_{n,j}^m \frac{J_{n+\frac{1}{2}}(\alpha_j r)}{\sqrt{r}}, \quad \frac{T_{n+1}^m}{r^2} = \sum_j T_{n+1,j}^m \frac{J_{n+\frac{3}{2}}(\alpha_j r)}{\sqrt{r}}$$

and

$$\Theta_n^m(r, t) = \sum_j \Theta_{n,j}^m \frac{J_{n+\frac{1}{2}}(\alpha_j r)}{\sqrt{r}}, \quad (20)$$

where α_j 's are the j th roots of $J_{n+\frac{1}{2}}(\alpha_j) = 0$.

If all quantities very like $e^{\lambda t}$, eqns. (15), (16) and (17) are reduced to

$$(\lambda a^2 + \kappa \alpha_j^2) \Theta_{n,j}^m - 2\beta a n(n+1) S_{n,j}^m = 0, \quad (21)$$

$$(n+1)(n+2) \lambda a^2 T_{n+1,j}^m - 2\Omega a^2 \left\{ -\frac{n(n+2)(n+1-m)}{2n+1} \alpha_j S_{n,j}^m + im T_{n+1,j}^m \right\} = 0 \quad (22)$$

and

$$n(n+1) \lambda a^2 \alpha_j^2 S_{n,j}^m - 2\Omega a^2 \left\{ \frac{n(n+2)(n+m+1)}{2n+3} \alpha_j T_{n+1,j}^m + im \alpha_j^2 S_{n,j}^m \right\} - \gamma n(n+1) a^5 \Theta_{n,j}^m = 0. \quad (23)$$

From above equations we get

$$\begin{aligned} \{ (n+1)(n+2) \omega - im \kappa T \} \left\{ n(n+1) \alpha_j^2 \omega - im \alpha_j^2 \kappa T - \kappa^2 \frac{n^2(n+1)^2}{\omega + \kappa \alpha_j^2} R \right\} \\ + \frac{n^2(n+2)^2(n+m+1)(n+1-m)}{(2n+1)(2n+3)} \alpha_j^2 \kappa^2 T^2 = 0, \end{aligned} \quad (24)$$

where

$$\omega = \lambda a^2, \quad T = \frac{2\Omega}{\kappa} a^2 \quad \text{and} \quad R = \frac{2\beta r}{\kappa^2} a^6. \quad (25)$$

Eqn. (24) is reduced to

$$\omega^3 + (p_1 + ip_2) \omega^2 + (q_1 - R_1 + ip_1 p_2) \omega + p_1 q_1 + i \frac{m \kappa T}{(n+1)(n+2)} R_1 = 0, \quad (26)$$

where

$$\left. \begin{aligned} p_1 &= \kappa \alpha_j^2, & p_2 &= -2m\kappa T, \\ q_1 &= \left\{ \frac{n(n+2)(n+m+1)(n+1-m)}{(n+1)^2(2n+1)(2n+3)} - \frac{m^2}{n(n+1)^2(n+2)} \right\} \kappa^2 T^2 \text{ and } R_1 = \frac{n(n+1)}{\alpha_j^2} \kappa^2 R. \end{aligned} \right\} \quad (27)$$

Substituting $\omega = x + iy$ in eqn. (26), we get

$$x^3 + p_1 x^2 - (3_1 y^2 + 2p_2 y - q_1 + R_1)x - p_1 y^2 - p_1 p_2 y + p_1 p_1 = 0 \quad (28)$$

and

$$(3y + p_2)x^2 + p_1(2y + p_2)x - y^3 - p_2 y^2 + (q_1 - R_1)y + \frac{m\kappa T}{(n+1)(n+2)} R_1 = 0. \quad (29)$$

As p_1 , q_1 and $\frac{m\kappa T}{(n+1)(n+2)}$ are positive, we have no steady state solutions.

i) axially symmetric solutions

In this case, eqn. (26) is reduced to

$$\omega^3 + p_1 \omega^2 + (q_1 - R_1)\omega + p_1 q_1 = 0 \quad (30)$$

When $q_1 - R_1 < 0$ this eqn. is able to have positive real roots, but when $q_1 - R_1 > 0$ it is not possible to have positive real roots. Eqn. (30) allows a negative real root and the corresponding solutions lead to an exponential damping of an initial disturbance.

We have to examine whether under these conditions the complex roots of (30) can have a positive real part. If x and y denote the real and the imaginary parts of complex roots of eqn. (30), they must satisfy the equations

$$3x^2 + 2p_1 x + q_1 - R_1 = y^2 \quad (31)$$

and

$$x^3 + p_1 x^2 + (q_1 - R_1)x + p_1 q_1 - (3x + p_1)y^2 = 0. \quad (32)$$

From the above two eqns. we have

$$8x^3 + 8p_1 x^2 + 2(p_1^2 + q_1 - R_1)x - p_1 R_1 = 0. \quad (33)$$

This eqn. admits a positive real root and letting $R_1 \rightarrow 0$, we shall have the critical frequency

$$\lambda = \frac{\kappa T}{\sqrt{5} a^2} = \frac{2\Omega}{\sqrt{5}}. \quad (34)$$

Thus we know the critical frequency of axially symmetric oscillation of a rotating non viscous fluid sphere is nearly equal to the angular velocity of the sphere.

ii) asymmetric solutions

The real roots of eqn. (26) must satisfy the equations

$$x^3 + p_1 x^2 + (q_1 - R_1)x + p_1 q_1 = 0 \quad (35)$$

and

$$p_2 x^2 + p_1 p_2 x + \frac{m\kappa T}{(n+1)(n+2)} R_1 = 0 \quad (36)$$

As $p_2 < 0$ and the third term of eqn. (36) is positive, eqn. (36) has a positive and a negative real roots. When $q_1 - R_1 < 0$, eqn. (35) is able to have a positive root, but when $q_1 - R_1 > 0$ this has a negative root.

From eqns. (35) and (36) we get

$$x = \frac{-p_1 q_1}{q_1 - \frac{2(n+1)(n+2)-1}{2(n+1)(n+2)} R_1}. \quad (37)$$

When

$$R_1 > \frac{2(n+1)(n+2)}{2(n+1)(n+2)-1} q_1 > q_1, \quad (38)$$

we have positive value of x .

When eqn. (26) has no positive roots, its complex roots must satisfy

$$x^3 + p_1 x^2 - \{(y^2 + p_2 y - q_1) + y(2y + p_2) + R_1\} x - p_1(y^2 + p_2 y - q_1) = 0 \quad (39)$$

and

$$(3y + p_2)x^2 + p_1(2y + p_2)x - y(y^2 + p_2 y - q_1) + R_1 \left\{ \frac{m\kappa T}{(n+1)(n+2)} - y \right\} = 0. \quad (40)$$

Eqn. (39) admits positive roots when

$$y^2 + p_2 y - q_1 > 0 \quad (41)$$

Eqn. (40) admits a positive root when

$$3y + p_2 > 0, \quad (42)$$

and

$$-y(y^2 + p_2 y - q_1) + R_1 \left\{ \frac{m\kappa T}{(n+1)(n+2)} - y \right\} < 0. \quad (43)$$

General examination of the inequality of (41), (42) and (43) is troublesome, but these inequality can be clearly satisfied by the condition

$$y > \frac{m\kappa T}{(n+1)(n+2)} \quad (44)$$

Thus we have a case of overstability when period of oscillation is smaller than a critical value. The critical number of vibrations is

$$\lambda = \frac{2m}{(n+1)(n+2)} \Omega \quad (45)$$

which is smaller than the angular velocity of sphere for all value of n tends to zero when n tends to infinity.

b) $[T_n^m, S_{n+1}^m, \dots]$ motion

We express $T_n^m(r, t)$, $S_{n+1}^m(r, t)$ and $\Theta_{n+1}^m(r, t)$ as follows

$$\frac{T_n^m(r, t)}{r^2} = \sum_j T_{n,j}^m \frac{J_{n+\frac{1}{2}}(\alpha_j r)}{\sqrt{r}}, \quad \frac{S_{n+1}^m(r, t)}{r} = \sum_j S_{n+1,j}^m \frac{J_{n+\frac{1}{2}}(\alpha_j r)}{\sqrt{r}} \quad (46)$$

and

$$\Theta_{n+1}^m(r, t) = \sum_j \Theta_{n+1,j}^m \frac{J_{n+\frac{1}{2}}(\alpha_j r)}{\sqrt{r}},$$

where α_j 's are the j th roots of $J_{n+\frac{1}{2}}(\alpha_j) = 0$.

Corresponding to eqns. (21), (22) and (23), we have

$$(\lambda a^2 + \kappa \alpha_j^2) \Theta_{n+1,j}^m - 2\beta a(n+1)(n+2) S_{n+1,j}^m = 0, \quad (47)$$

$$n(n+1) \lambda a^2 T_{n,j}^m - 2\Omega a^2 \left\{ \frac{n(n+2)(n+m+1)}{2n+3} \alpha_j S_{n+1,j}^m + im T_{n,j}^m \right\} = 0 \quad (48)$$

and

$$\begin{aligned} (n+1)(n+2) \alpha_j^2 \lambda a^2 S_{n+1,j}^m - 2\Omega a^2 \left\{ -\frac{n(n+2)(n+1-m)}{2n+1} \alpha_j T_{n,j}^m + im \alpha_j^2 S_{n+1,j}^m \right\} \\ = r(n+1)(n+2) a^5 \Theta_{n+1,j}^m. \end{aligned} \quad (49)$$

From above three eqn. we get

$$\{n(n+1)\omega - im\kappa T\} \left\{ (n+1)(n+2)\alpha_j^2 - im\alpha_j^2\kappa T - \kappa^2 \frac{(n+1)^2(n+2)^2}{\omega + \kappa\alpha_j^2} R \right\} \\ + \frac{n^2(n+2)^2(n+n+1)(n+1-m)}{(2n+1)(2n+3)} \alpha_j^2 \kappa^2 T^2 = 0. \quad (50)$$

Eqn. (50) is reduced to

$$\omega^3 + (p_1 + ip_2)\omega^2 + (q_1 - R_1' + ip_1p_2)\omega + p_1q_1 + i \frac{m\kappa T}{n(n+1)} R_1' = 0, \quad (51)$$

where p_1 , p_2 and q_1 are the same notations in (27); and $R_1' = \kappa^2 \frac{(n+1)(n+2)}{\alpha_j^2} R$.

As eqn. (51) has the same form as eqn. (26), we can apply the examinations of eqn. (26) to eqn. (51). We have no steady state solutions.

i) axially symmetric solutions

In this case the discussion of $[S_n, T_{n+1} \dots]$ motion is entirely applicable; When $R_1 > q_1$ we have an increasing amplitude instability, but when $R_1 < q_1$, we get increasing amplitude oscillations and the critical frequency of oscillations is equal to that of $[S_n, T_{n+1} \dots]$ motion.

ii) asymmetric solutions

The real roots of eqn. (51) must satisfy the equations

$$x^3 + p_1x^2 + (q_1 - R_1')x + p_1q_1 = 0, \quad (52)$$

and

$$p_2x^2 + p_1p_2x + \frac{m\kappa T}{n(n+1)} R_1' = 0. \quad (53)$$

The discussions of $[S_n^m, T_{n+1}^m \dots]$ motion is entirely applicable to $[T_n^m, S_{n+1}^m \dots]$ motions, letting $n \rightarrow n-1$ and $R_1 \rightarrow R_1'$. When

$$R_1' < q_1 \quad \text{and} \quad y > \frac{m\kappa T}{n(n+1)} \quad (54)$$

we have a case of overstability. The critical frequency of vibration is

$$\lambda = \frac{2}{n+1} \Omega. \quad (55)$$

Thus

$$\lambda \leq \Omega \quad (56)$$

and tends to zero when n tends to infinity.

6. Concluding Remarks

We have found that thermal instability of rotating fluid sphere with negligible viscosity sets in as oscillations of increasing amplitude when R is smaller than a critical value. The critical frequencies of oscillations are smaller than the angular velocity of sphere. It is our results of this investigation that in a rotating fluid sphere with negligible viscosity the principle of the exchange of stabilities can not be applied and we have a case of overstability.

We are interested in the convection in the Earth's Core and in this case the effect of magnetic field can not be neglected. We shall discuss this problem further in the following paper.

7. Appendix

Equation of Motion [5], [7].

As we are seeking the effect of Coriolis force, we shall ignore the rotational flattening of the sphere in treating this problem. In the state of no convection, the temperature distribution, $T(r)$, is governed by

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T + \varepsilon = 0, \quad (\text{A1})$$

where κ is the coefficient of thermometric conductivity. The solution of eqn. (A.1) appropriate to the problem on hand is

$$T_0 = \frac{\varepsilon}{6\kappa} (a^2 - r^2) = \beta (a^2 - r^2). \quad (\text{A2})$$

We have assumed that $T_0 = 0$ at $r = a$; This entails no loss of generality.

If we assume the temperature distribution in the perturbed state

$$T = T_0 + \theta, \quad (\text{A3})$$

the equation governing θ is

$$\frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla) T = \kappa \nabla^2 \theta, \quad (\text{A4})$$

where $\mathbf{u}$ denotes the velocity.

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla P + \rho \nabla V + \rho \nu \nabla^2 \mathbf{u} + \rho \mathbf{u} \times \boldsymbol{\Omega}, \quad (\text{A5})$$

where ρ , P , V and ν denote density, pressure, gravitational potential and kinematic viscosity respectively.

Following Rayleigh [8], we shall allow in eqn. (A5) for the variation of density only is so far as it modifies the external field. Thus in eqn. (A.5) we shall replace ρ which occurs in front of ∇V by

$$\rho = \rho_0 (1 - \alpha T), \quad (\text{A6})$$

where α denotes the coefficient of volume expansion and ρ_0 is the density at $r = a$ (where $T = 0$) and regard ρ occurring elsewhere in eqn. (A.6) as ρ_0 .

Now making use of relation (A.3), eqn. (A.5) becomes

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \left(\frac{P}{\rho_0} - V \right) - \alpha [\beta (a^2 - r^2) + \theta] \nabla V + \nu \nabla^2 \mathbf{u} + 2\mathbf{v} \times \boldsymbol{\Omega}. \quad (\text{A7})$$

With variation of density due to thermal expansion allowed for in this manner, we shall treat $\mathbf{u}$ as a solenoidal vector

$$\text{div } \mathbf{u} = 0. \quad (\text{A8})$$

Substituting the relation

$$\nabla V = -\frac{4}{3} \pi \bar{\rho} G \mathbf{r}, \quad (\text{A9})$$

where G and $\bar{\rho}$ denote the constant of gravitation and mean density of the sphere, eqn. (A.7) becomes

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla W + \gamma \theta \mathbf{r} + \nu \nabla^2 \mathbf{u} + 2\mathbf{u} \times \boldsymbol{\Omega}, \quad (\text{A10})$$

where

$$\gamma = \frac{4}{3} \pi \bar{\rho} G \alpha \quad (\text{A11})$$

and

$$W = \frac{\rho}{\rho_0} - V - \frac{1}{4} \beta \gamma (2a^2 r^2 - r^4). \quad (\text{A12})$$

We shall suppose that $\mathbf{u}$ and θ are small quantities of the first order and that we can ignore products and squares of them.

Then, the equations in the perturbed state are

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla \frac{\delta p}{\rho} + \gamma \theta \mathbf{r} + \nu \nabla^2 \mathbf{u} + 2\mathbf{u} \times \boldsymbol{\Omega} \quad (\text{A13})$$

and

$$\frac{\partial \theta}{\partial t} = \kappa \nabla^2 \theta + 2\beta(r U_r), \quad (\text{A14})$$

where δp is the perturbation in pressure.

Boundary Conditions

Boundary conditions which the solutions of eqns. (A.13) and (A.14) must satisfy depend on whether the bounding surface is rigid or free, but in all cases we require that

$$\theta = 0 \quad \text{and} \quad u_r = 0 \quad (\text{on a bounding spherical surface}). \quad (\text{A.15})$$

Additional boundary conditions follow from the equation of continuity and depend on the nature of the bounding surface.

Equation of continuity in spherical polar coordinates (r, ϑ, φ) is

$$\frac{\partial u_r}{\partial r} + 2 \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\vartheta}{\partial \vartheta} + \frac{u_\vartheta \cot \vartheta}{r} + \frac{1}{r \sin \vartheta} \frac{\partial u_\varphi}{\partial \varphi} = 0, \quad (\text{A16})$$

where u_r , u_ϑ and u_φ are the components of velocity along the principal elements of arc dr , $r d\vartheta$ and $r \sin \vartheta d\varphi$ respectively.

(i) on a rigid bounding surface

According to eqn.(15) and (17) the conditions (A.15) require

$$S_n^m = 0,$$

and

$$\left. \begin{aligned} & -n(n+1)a^2 \frac{\partial}{\partial t} \frac{1}{r^2} \left\{ \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - n(n+1) \right\} \frac{S_n^m}{r} - \nu n(n+1) \left[\frac{1}{r^2} \left\{ \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) \right. \right. \\ & \left. \left. - n(n+1) \right\} \right]^2 \frac{S_n^m}{r} - 2\Omega a^2 \left[\frac{(n-1)(n+1)(n-m)}{2n-1} r^{n-1} \frac{d}{dr} (r^{n-1} T_{n-1}^m) \right. \\ & \left. + \frac{n(n+2)(n+m+1)}{2n+3} r^{-(n+2)} \frac{d}{dr} (r^n T_{n+1}^m) - im \frac{1}{r} \left\{ \frac{d^2}{dr^2} - \frac{n(n+1)}{r^2} \right\} S_n^m \right] = 0, \end{aligned} \right\} \quad (\text{A17})$$

on $r=1$.

As no slip occurs on the bounding surface, u_r , u_ϑ and u_φ must vanish. From eqns. (7), (16) and (A.16), it follows that

$$\left. \begin{aligned}
 & n(n+1)a^2 \frac{\partial}{\partial t} \frac{T_n^m}{r^2} - \nu n(n+1) \frac{1}{r^2} \left\{ \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - n(n+1) \right\} \frac{T_n^m}{r^2} \\
 & - 2\Omega a^2 \left[\frac{(n-1)(n+1)(n-m)}{2n-1} r^{n-1} \frac{d}{dr} (r^{-n} S_{n-1}^m) + \frac{n(n+2)(n+m+1)}{2n+3} r^{-(n+2)} \right. \\
 & \quad \left. \times \frac{d}{dr} (r^{n+1} S_{n+1}^m) \right] = 0, \\
 & \frac{dS_n^m}{dr} = 0, \text{ and } T_n^m = 0 \text{ on } r=1.
 \end{aligned} \right\} \quad (\text{A18})$$

(ii) on a free bounding surface

In this case we must have in addition to the condition (A.17), the viscous stresses $P_{r\theta}$ and $P_{r\varphi}$ vanishes.

The expression for these stresses are

$$\left. \begin{aligned}
 P_{r\theta} &= \rho \nu \left(\frac{\partial u_r}{r \partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} \right), \\
 P_{r\varphi} &= \rho \nu \left(\frac{\partial u_r}{r \sin \theta \partial \varphi} - \frac{u_\varphi}{r} + \frac{\partial u_\varphi}{\partial r} \right)
 \end{aligned} \right\} \quad (\text{A19})$$

Since u_r vanishes on a bounding surface, the vanishing of $P_{r\theta}$ and $P_{r\varphi}$ requires

$$\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} = \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r} = 0. \quad (\text{A20})$$

According to eqn. (7) these conditions are equivalent to

$$\frac{d}{dr} \left(\frac{1}{r} \frac{dS_n^m}{dr} \right) - \frac{1}{r^2} \frac{dS_n^m}{dr} = \frac{1}{r} \left(\frac{d^2 S_n^m}{dr^2} - \frac{2}{r} \frac{dS_n^m}{dr} \right) = 0$$

and

$$\frac{dT_n^m}{dr} = 0 \text{ on } r=1. \quad (\text{A21})$$

Since $S_n^m = 0$ on $r=1$, the first of (A.21) can be expressed as

$$\frac{\partial^2}{\partial r^2} \left(\frac{S_n^m}{r} \right) = 0 \text{ on } r=1. \quad (\text{A22})$$

	Symbols
a	radius of sphere
G	Constant of gravitation
$\mathbf{i}_r$	unit vector of radius direction
p	pressure
T	temperature
$\mathbf{u}$	Velocity
V	Gravitational potential
α	Coefficient of volume expansion
$2\beta a$	temperature gradient at the boundary
γ	$\frac{4}{3} \pi \bar{\rho} G \alpha$
ϵ	the rate at which temperature would rise in the absence of conduction and convection
θ	perturbation in temperature
κ	Coefficient of thermometric conductivity
ν	kinematic viscosity
ρ	density
Ω	angular velocity of sphere
ω	vorticity

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