Fluid Motions in a Sphere II

Thermal Instability of a Conducting Fluid Sphere Heated within under a Uniform Magnetic Field

By Tomikazu NAMIKAWA

Institute of Politechnics, Osaka City University

(Read Oct. 9, 1957 ; Received July 20, 1958)

Abstract

In this paper the problem of thermal instability of non-viscous fluid sphere heated within under a uniform magnetic field is discussed. Differing from the former case of thermal instability of a rotating sphere, there exist axially symmetric steady state solutions and only $(U_n^m, V_{n+1}^m, \dots)$ motions can arise. Calculation suggests a relation between perturbations in magnetic field and vorticity. Under normal terrestrial conditions, instability arises as ordinary cellular convections, but under astrophysical conditions, it can arise depending on intensity of the uniform magnetic field, as cellular convections or as overstability, oscillations of increasing amplitude.

1. Introduction

In paper [I], we have examined thermal instability of a rotating fluid sphere and found that there is no steady state solutions; instability first sets in as overstability. We shall examine in this paper the influence of magnetic field on thermal convection in a sphere.

2. Equation of Motion

We shall consider a homogeneous conducting fluid sphere of radius a under a uniform magnetic field, the direction of which is taken to be z -axis. Assumption about the source of heat generation is the same as that of paper [I].

The equation of motion, of Maxwell, and of heat conduction appropriate to the problem on hand are

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla P + \rho \nu \nabla^2 \mathbf{u} + \rho \nabla V + \mu \mathbf{J} \times \mathbf{H}, \quad (1)$$

$$\text{curl } \mathbf{H} = 4\pi \mathbf{J}, \quad (2)$$

$$\text{curl } \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \quad (3)$$

$$\text{div } \mathbf{H} = 0, \quad \text{div } \mathbf{E} = 4\pi c^2 q, \quad (4)$$

$$\mathbf{J} = \sigma(\mathbf{E} + \mu \mathbf{u} \times \mathbf{H}) \quad (5)$$

and

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \kappa \nabla^2 T, \quad (6)$$

where $\mathbf{E}$, $\mathbf{J}$, $\mathbf{H}$, σ and μ denote the intensity of electric field, current density, intensity of magnetic field, electrical conductivity and the magnetic permeability. The meaning of other notations is equal to those in paper [I].

The equations governing small departures from the stationary state are (Appendix of [1]; [2])

$$\frac{\partial \theta}{\partial t} = \kappa \nabla^2 \theta + 2\beta(r u_r), \quad (7)$$

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_0} \nabla P + \gamma \theta \mathbf{r} + \frac{\mu H}{4\pi \rho_0} \frac{\partial \mathbf{h}}{\partial z} + \nu \nabla^2 \mathbf{u}, \quad (8)$$

$$\frac{\partial \mathbf{h}}{\partial t} = H \frac{\partial \mathbf{u}}{\partial z} + \eta \nabla^2 \mathbf{h}, \quad (9)$$

and

$$\text{div } \mathbf{u} = \text{div } \mathbf{h} = 0, \quad (10)$$

where H and $\mathbf{h}$ denote magnitude of uniform magnetic field and perturbation in magnetic field and

$$\eta = \frac{1}{4\pi \mu \sigma}. \quad (11)$$

Taking the curl of eqns. (8) and (9), we have

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \gamma \nabla \theta \times \mathbf{r} + \frac{\mu H}{4\pi \rho_0} \frac{\partial}{\partial z} \text{curl } \mathbf{h} + \nu \nabla^2 \boldsymbol{\omega}, \quad (12)$$

and

$$\frac{\partial}{\partial t} \text{curl } \mathbf{h} = H \frac{\partial}{\partial z} \boldsymbol{\omega} + \eta \nabla^2 (\text{curl } \mathbf{h}). \quad (13)$$

Again, taking the curl of eqn. (12), we get

$$-\frac{\partial}{\partial t} \nabla^2 \mathbf{u} = \gamma \text{curl}(\nabla \theta \times \mathbf{r}) - \frac{\mu H}{4\pi \rho_0} \frac{\partial}{\partial z} \nabla^2 \mathbf{h} - \nu \nabla^4 \mathbf{u}. \quad (14)$$

We can express the velocity field and the magnetic field as a superposition of a poloidal and a toroidal vector in terms of four scalars.

$$\mathbf{u} = \nabla V \times \mathbf{i}_r + \text{curl}(\nabla U \times \mathbf{i}_r), \quad (15)$$

and

$$\mathbf{h} = \nabla T \times \mathbf{i}_r + \text{curl}(\nabla S \times \mathbf{i}_r). \quad (16)$$

Five scalars U , V , T , S and θ are expressed as follows

$$\begin{aligned} U &= \sum_{n,m} U_n^m(r, t) Y_n^m, & V &= \sum_{n,m} V_n^m(r, t) Y_n^m, \\ S &= \sum_{n,m} S_n^m(r, t) Y_n^m, & T &= \sum_{n,m} T_n^m(r, t) Y_n^m, \end{aligned} \quad (17)$$

and

$$\theta = \sum_{n,m} \Theta_n^m(r, t) Y_n^m,$$

where Y_n^m is a surface spherical harmonic.

Multiplying eqns. (9), (12), (13) and (14) by r , we get

$$\frac{\partial(rhr)}{\partial t} = H\mathbf{r} \frac{\partial \mathbf{u}}{\partial z} + \eta \nabla^2(rhr), \quad (18)$$

$$\frac{\partial}{\partial t}(r\omega r) = \frac{\mu H}{4\pi\rho_0} \mathbf{r} \frac{\partial}{\partial z}(\text{curl } \mathbf{h}) + \nu \nabla^2(r\omega_r), \quad (19)$$

$$\frac{\partial}{\partial t} r(\text{curl } \mathbf{h})_r = H\mathbf{r} \frac{\partial}{\partial z} \boldsymbol{\omega} + \eta \nabla^2 r(\text{curl } \mathbf{h})_r, \quad (20)$$

and

$$-\frac{\partial}{\partial t} \nabla^2(ru_r) = rL^2\theta - \frac{\mu H}{4\pi\rho_0} \mathbf{r} \frac{\partial}{\partial z} \nabla^2 \mathbf{h} - \nu \nabla^4(ru_r), \quad (21)$$

Operating Y_n^m to eqns. (7), (18), (19), (20) and (21) and executing the integration with respect to ϑ and φ , we get

$$a^2 \frac{\partial}{\partial t} \Theta_n^m - \kappa \frac{1}{r^2} \left\{ \frac{d}{dr} \left(r^2 \frac{d}{dr} - n(n+1) \right) \right\} \Theta_n^m - 2\beta a n(n+1) \frac{U_n^m}{r} = 0, \quad (22)$$

$$\begin{aligned} & a^2 \frac{\partial}{\partial t} n(n+1) \frac{S_n^m}{r} - \eta n(n+1) \frac{1}{r^2} \left\{ \frac{d}{dr} \left(r^2 \frac{d}{dr} - n(n+1) \right) \right\} \frac{S_n^m}{r} \\ & - Ha \left[\frac{(n-1)(n+1)(n-m)}{2n-1} r^{n-1} \frac{d}{dr} (r^{-n} U_n^m) + \frac{n(n+2)(n+m+1)}{2n+3} r^{-(n+2)} \right. \\ & \quad \left. \times \frac{d}{dr} (r^{n+1} U_{n+1}^m) + im \frac{Y_n^m}{r^2} \right] = 0, \quad (23) \end{aligned}$$

$$\begin{aligned} & a^2 \frac{\partial}{\partial t} n(n+1) \frac{V_n^m}{r^2} - \nu n(n+1) \frac{1}{r^2} \left\{ \frac{d}{dr} \left(r^2 \frac{d}{dr} - n(n+1) \right) \right\} \frac{V_n^m}{r^2} \\ & - \frac{Ha}{4\pi\rho_0} \left[\frac{(n-1)(n+1)(n-m)}{2n-1} r^{n-1} \frac{d}{dr} (r^{-n} T_{n-1}^m) + \frac{n(n+2)(n+m+1)}{2n+3} r^{-(n+2)} \right. \\ & \quad \left. \times \frac{d}{dr} \left(r^{n+1} \frac{T_{n+1}^m}{r} \right) - \frac{im}{r} \left\{ \frac{d^2}{dr^2} - \frac{n(n+1)}{r^2} \right\} S_n^m \right] = 0, \quad (24) \end{aligned}$$

$$\begin{aligned} & -n(n+1) a^2 \frac{\partial}{\partial t} \frac{1}{r^2} \left\{ \frac{d}{dr} \left(r^2 \frac{d}{dr} - n(n+1) \right) \right\} \frac{U_n^m}{r} + \nu n(n+1) \left[\frac{1}{r^2} \left\{ \frac{d}{dr} \left(r^2 \frac{d}{dr} - n(n+1) \right) \right\} \right. \\ & \quad \left. \times \frac{U_n^m}{r} + \frac{\mu Ha}{4\pi\rho_0} \frac{1}{r^2} \left\{ \frac{d}{dr} \left(r^2 \frac{d}{dr} - n(n+1) \right) \right\} \left[\frac{(n-1)(n+1)(n-m)}{2n-1} \right. \right. \\ & \quad \left. \times r^{n-1} \frac{d}{dr} (r^{-n} S_{n-1}^m) + \frac{n(n+2)(n+m+1)}{2n+3} r^{-(n+2)} \frac{d}{dr} (r^{n+1} S_{n+1}^m) + im \frac{T_n^m}{r^2} \right] \\ & \quad \left. - r n(n+1) a^5 \Theta_n^m = 0 \quad (25) \end{aligned}$$

and

$$\begin{aligned} & a^2 \frac{\partial}{\partial t} n(n+1) \frac{T_n^m}{r^2} - \eta n(n+1) \frac{1}{r^2} \left\{ \frac{d}{dr} \left(r^2 \frac{d}{dr} - n(n+1) \right) \right\} \frac{T_n^m}{r^2} \\ & - Ha \left[\frac{(n-1)(n+1)(n-m)}{2n-1} r^{n-1} \frac{d}{dr} (r^{-n} V_{n-1}^m) + \frac{n(n+2)(n+m+1)}{2n+3} r^{-(n+2)} \right. \\ & \quad \left. \times \frac{d}{dr} (r^{n+1} V_{n+1}^m) - im \frac{1}{r} \left\{ \frac{d^2}{dr^2} - \frac{n(n+1)}{r^2} \right\} U_n^m \right] = 0. \quad (26) \end{aligned}$$

Eqns. (22), (23), (24), (25) and (26) are the basic equations of this problem.

3. Boundary Conditions

As in [1], we shall discuss eqns. (24) and (25) neglecting the viscous term.

We must require that

$$\Theta_n^m(1) = U_n^m(1) = 0, \quad (27)$$

Additional boundary conditions are the continuity of magnetic field and tangential components of electric field on the boundary surface.

4. Solutions of Basic Equations

From eqns, (22), (23), (24), (25) and (26), we find that there exist groups of fluid motions and magnetic field as follows

$$\left. \begin{array}{l} U_n^m \longleftrightarrow V_{n+1}^m \longleftrightarrow U_{n+2}^m \longleftrightarrow \dots \\ \updownarrow \quad \updownarrow \quad \updownarrow \\ T_n^m \longleftrightarrow S_{n+1}^m \longleftrightarrow T_{n+2}^m \longleftrightarrow \dots \\ V_n^m \longleftrightarrow U_{n+1}^m \longleftrightarrow V_{n+2}^m \longleftrightarrow \dots \\ \updownarrow \quad \updownarrow \quad \updownarrow \\ S_n^m \longleftrightarrow T_{n+1}^m \longleftrightarrow S_{n+2}^m \longleftrightarrow \dots \end{array} \right\} \quad (28)$$

We shall examine the $(U_n^m, T_n^m, V_{n+1}^m, S_{n+1}^m, \dots)$ motions taking only the first four terms in considerations. Simple calculation shows that there is no $(V_n^m, U_{n+1}^m, S_n^m, T_{n+1}^m, \dots)$ motion because continuity of the magnetic field cannot be satisfied on the boundary surface. We expand $\frac{U_n^m}{r}$, $\frac{V_{n+1}^m}{r^2}$, $\frac{T_n^m}{r^2}$, $\frac{S_{n+1}^m}{r}$, and Θ_n^m as follows:

$$\left. \begin{array}{l} \frac{U_n^m}{r} = \sum_j U_{n,j}^m \frac{J_{n+\frac{1}{2}}(a_j r)}{\sqrt{r}}, \quad \frac{V_{n+1}^m}{r^2} = \sum_j Y_{n+1,j}^m \frac{J_{n+\frac{3}{2}}(a_j r)}{\sqrt{r}} \\ \frac{T_n^m}{r^2} = \sum_j T_{n,j}^m \frac{J_{n+\frac{1}{2}}(a_j r)}{\sqrt{r}}, \quad \frac{S_{n+1}^m}{r} = \sum_j S_{n+1,j}^m \frac{J_{n+\frac{3}{2}}(a_j r)}{\sqrt{r}} \\ \Theta_n^m = \sum_j \Theta_{n,j}^m \frac{J_{n+\frac{1}{2}}(a_j r)}{\sqrt{r}}, \end{array} \right\} \quad (29)$$

and

$$\Theta_n^m = \sum_j \Theta_{n,j}^m \frac{J_{n+\frac{1}{2}}(a_j r)}{\sqrt{r}},$$

where a_j 's is the roots of $J_{n+\frac{1}{2}}(a_j) = 0$.

With above expressions, the boundary conditions (27) are satisfied.

The external magnetic field must satisfy eqns.

$$\operatorname{div} \mathbf{H}^e = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{H}^e = 0 \quad (30)$$

and expressed as follows

$$\mathbf{H}^e = \nabla \psi, \quad (31)$$

where

$$\psi = \sum_{n,m} A_n^m r^{-(n+1)} Y_n^m(\vartheta, \varphi). \quad (32)$$

From eqns. (29), (31) and (32), it is found, after simple calculations, that the continuity of the magnetic field is satisfied on the boundary surface.

The internal electric field is given, from eqns, (2) and (5),

$$\mathbf{E} = \eta \operatorname{curl} \mathbf{H} - \mu \mathbf{u} \times \mathbf{H}. \quad (33)$$

Making use of eqn. (29), the internal electric field is calculated from eqn. (33).

The external electric field must satisfy eqns.

$$\operatorname{div} \mathbf{E}^e = 0 \quad \text{and} \quad -\mu \frac{\partial \mathbf{H}^e}{\partial t} = \operatorname{curl} \mathbf{E}^e \quad (34)$$

and expressed as follows:

$$\mathbf{E}^e = \sum \{ B_n^m \nabla(r^{-(n+1)} Y_n^m) + C_n^m \nabla(r^{-(n+1)} Y_n^m) \}. \quad (35)$$

From eqns. (33), (34) and (35), it is found that the continuity of tangential components of electric field is satisfied.

If all quantities vary as $e^{\lambda t}$, substituting relations (29), eqns. (22), (23), (24), (25) and (26) are reduced to

$$(\lambda a^2 + \kappa a_j^2) \Theta_{n,j}^m - 2\beta a n(n+1) U_{n,j}^m = 0, \quad (36)$$

$$(n+1)(n+2) \lambda a^2 V_{n+1,j}^m - \frac{Ha}{4\pi\rho_0} \left[-\frac{n(n+2)(n+1-m)}{2n+1} a_j T_{n,j}^m + i m a_j^2 S_{n+1,j}^m \right] = 0, \quad (37)$$

$$n(n+1) \lambda a^2 a_j^2 U_{n,j}^m - \frac{Ha}{4\pi\rho_0} a_j^2 \left[\frac{n(n+2)(n+m+1)}{2n+3} a_j S_{n+1,j}^m + i m T_{n,j}^m \right] - r n(n+1) a^5 \Theta_{n,j}^m = 0, \quad (38)$$

$$(n+1)(n+2) (\lambda a^2 + \eta a_j^2) S_{n+1,j}^m - Ha \left[-\frac{n(n+2)(n+1-m)}{2n+1} a_j U_{n,j}^m + i m V_{n+1,j}^m \right] = 0 \quad (39)$$

and

$$n(n+1) (\lambda a^2 + a_j^2) T_{n,j}^m - Ha \left[\frac{n(n+2)(n+m+1)}{2n+2} a_j V_{n+1,j}^m + i m a_j^2 U_{n,j}^m \right] = 0. \quad (40)$$

From above eqns., we get

$$\left\{ \omega^2 + \eta a_j^2 \omega + \frac{f_1}{(n+1)^2(n+2)} a_j^2 \eta^2 Q \right\} \left[\omega^3 + (\kappa + \eta) a_j^2 \omega^2 + \left\{ \kappa \eta a_j^4 + \frac{f_2}{n(n+1)^2} a_j^2 \eta^2 Q \right. \right. \\ \left. \left. - \frac{n(n+1)}{a_j^2} \kappa \eta R \right\} \omega + \frac{f_2}{n(n+1)^2} \kappa \eta^2 a_j^4 Q - n(n+1) \kappa \eta^2 R \right] - 4 m^2 \\ \times \frac{(n+1-m)(n+m+1)}{n(n+1)^2(n+2)(2n+1)(2n+3)} (a_j^2 \eta^2 Q)^2 (\omega + \kappa a_j^2) = 0, \quad (41)$$

where

$$\left. \begin{aligned} \omega &= \lambda a^2, \quad Q = \frac{\mu^2 H^2 \sigma a^2}{\rho_0 \eta}, \quad f_1 = \frac{n(n+2)^2(n+m+1)(n+1-m)}{(2n+1)(2n+3)} + \frac{m^2}{n+2}, \\ f_2 &= \frac{n(n+2)(n+m+1)(n+1-m)}{(2n+1)(2n+3)} + \frac{m^2}{n} \quad \text{and} \quad R = \frac{2\beta r a^6}{\kappa \eta}. \end{aligned} \right\} \quad (42)$$

5. Solutions of Marginal Stability

At first, assuming the validity of the principle of exchange of stabilities, we get the value of non dimensional number R of the marginal stabilities.

$$R_c^{\text{con}} = \left\{ \frac{f_2}{n+1} - 4 m^2 \frac{(n+m+1)(n+1-m)(n+1)}{(2n+1)(2n+3) f_1} \right\} \frac{a_j^4 Q}{n^2(n+1)^2} \quad (43)$$

As, for all value of n , the right hand side of (43) is negative, we have only axially

symmetric solutions of marginal stability, $[U_1^0, V_2^0, T_1^0, S_2^0, \dots]$ motion, letting $m \rightarrow 0$ in (43).

Thus

$$\begin{aligned} R_c^{\text{con}} &= \frac{(n+2)a_j^4 Q}{n(2n+1)(2n+3)(n+1)} \\ &= \frac{a_j^4 Q}{10}. \end{aligned} \quad (44)$$

6. An Examination of the Principle of the Exchange of Stabilities

Here, we examine axially symmetric case of eqn. (41).

In this case eqn. (41) is reduced to

$$\omega^2 + \eta a_j^2 \omega + \frac{n(n+2)}{(2n+3)(2n+1)} a_j^2 \eta^2 Q = 0, \quad (45)$$

or

$$\begin{aligned} \omega^3 + (\kappa + \eta) a_j^2 \omega^2 + \kappa \eta \left\{ a_j^4 + \frac{(n+2)}{(2n+1)(2n+3)} a_j^2 Q \cdot \frac{\eta}{\kappa} - \frac{n(n+1)}{a_j^2} R \right\} \omega \\ + \kappa \eta^2 a_j^2 \left\{ \frac{(n+2)a_j^2}{(2n+1)(2n+3)} - \frac{n(n+1)}{a_j^2} R \right\} = 0. \end{aligned} \quad (46)$$

Eqn. (45) has no real positive root, but eqn. (46) has real positive root if

$$\frac{n+2}{(2n+1)(2n+3)} a_j^2 Q < \frac{n(n+1)}{a_j^2} R, \quad (47)$$

and no real positive root if

$$\frac{n+2}{(2n+1)(2n+3)} a_j^2 Q > \frac{n(n+1)}{a_j^2} R \quad (48)$$

and

$$a_j^4 + \frac{(n+2)}{(2n+1)(2n+3)} a_j^2 Q \cdot \frac{\eta}{\kappa} - \frac{n(n+1)}{a_j^2} R > 0. \quad (49)$$

As eqn. (45) has no complex root with positive real part, we examine whether eqn. (46) has a complex root with positive real part under the condition (48) and (49).

We write eqn. (46) as follows:

$$\omega^3 + p\omega^2 + q\omega + r = 0, \quad (50)$$

where

$$\left. \begin{aligned} p &= (\kappa + \eta) a_j^2, \quad q = \kappa \eta \left\{ a_j^4 + \frac{n+2}{(2n+1)(2n+3)} a_j^2 Q \cdot \frac{\eta}{\kappa} - \frac{n(n+1)}{a_j^2} R \right\} \\ \text{and} \quad r &= \kappa \eta^2 a_j^2 \left\{ \frac{(n+2)}{(2n+1)(2n+3)} a_j^2 Q - \frac{n(n+1)}{a_j^2} R \right\}. \end{aligned} \right\} \quad (51)$$

If we put $\omega = x + iy$, where x and y are real numbers, eqn. (50) is reduced to

$$3x^2 + 2px + q = y^2 \quad (52)$$

and

$$x^3 + px^2 + qx + r - (3x + p)y^2 = 0. \quad (53)$$

From above two eqns. we have

$$8x^3 + 8px^2 + 2(p^2 + r)x + pq - r = 0. \quad (54)$$

If

$$pq - r < 0, \quad (55)$$

eqn. (54) admits a positive real root.

Thus if

$$q > 0, \quad r > 0 \quad \text{and} \quad pq - r < 0, \quad (56)$$

the principle of the exchange of stabilities can not be applied and we have a case of overstability.

According to eqn. (51), the condition (56) is equivalent to

$$\left(1 + \frac{\eta}{\kappa}\right) \alpha_j^4 + \frac{(n+2)}{(2n+1)(2n+3)} \alpha_j^2 Q \left(\frac{\eta}{\kappa}\right)^2 < \frac{n(n+1)}{\alpha_j^2} R < \alpha_j^4 + \frac{(n+2)}{(2n+1)(2n+3)} \alpha_j^2 Q \cdot \frac{\eta}{\kappa}, \quad (57)$$

or

$$\left(1 + \frac{\eta}{\kappa}\right) \alpha_j^4 + \frac{(n+2)}{(2n+1)(2n+3)} \alpha_j^2 Q \left(\frac{\eta}{\kappa}\right)^2 < \frac{n(n+1)}{\alpha_j^2} R < \frac{n+2}{(2n+1)(2n+3)} \alpha_j^2 Q, \quad (58)$$

according to whether

$$\alpha_j^4 + \frac{n+2}{(2n+1)(2n+3)} \alpha_j^2 Q \cdot \frac{\eta}{\kappa} < \frac{n+2}{(2n+1)(2n+3)} \alpha_j^2 Q, \quad (59)$$

or

$$\alpha_j^4 + \frac{n+2}{(2n+1)(2n+3)} \alpha_j^2 Q \cdot \frac{\eta}{\kappa} > \frac{n+2}{(2n+1)(2n+3)} \alpha_j^2 Q. \quad (60)$$

Rearranging the outer inequalities of (57) or (58), we get

$$0 < \alpha_j^4 < \frac{n+2}{(2n+1)(2n+3)} \alpha_j^2 Q \left(1 - \frac{\eta}{\kappa}\right), \quad (61)$$

or

$$0 < \left(1 + \frac{\eta}{\kappa}\right) \alpha_j^2 < \frac{n+2}{(2n+1)(2n+3)} \alpha_j^2 Q \left(1 + \frac{\eta}{\kappa}\right) \left(1 - \frac{\eta}{\kappa}\right). \quad (62)$$

These inequalities are the necessary conditions for the principle of exchange of stabilities not to be valid.

According to eqn. (61) or (62), a sufficient condition for the principle of exchange of stabilities to be valid is

$$\eta > \kappa. \quad (63)$$

This inequality is satisfied for the fluid under terrestrial conditions (for example, mercury at room temperature) [2].

The frequency of oscillation at marginal stability is given by

$$\lambda = \frac{\sqrt{q}}{a^2} \quad (64)$$

and the critical value of R for the onset of convection is given by

$$R_c^0 = \frac{(n+2)\alpha_j^2 Q}{n(n+1)(2n+1)(2n+3)} \left\{ \left(\frac{\eta}{\kappa}\right)^2 + \left(1 + \frac{\eta}{\kappa}\right) \frac{\alpha_j^2(2n+1)(2n+3)}{Q(n+2)} \right\}. \quad (65)$$

7. Onset of Instability in a Magnetic Field under Astrophysical Conditions

The inequality

$$\kappa \gg \eta \quad (66)$$

is characteristic of astrophysical conditions [2]. When (66) holds, the inequality (61) or (62) can be simplified to

$$Q > \frac{(2n+1)(2n+3)}{n+2} \alpha_j^2 = Q_1. \quad (67)$$

From eqn. (65), we can write for a considerable range of Q satisfying (67),

$$R_c^0 = \frac{\alpha_j^6}{n(n+1)} \left(Q < \frac{\kappa^2}{\eta^2} Q_1 \right). \quad (68)$$

This is independent of the strength of the prevailing magnetic field. In axially symmetric case $n=1$ and minimum value of α_j is α_1 : thus when (66) holds, instability in the form of oscillation of increasing amplitude can arise for

$$Q > 5\alpha_1^2 \quad (69)$$

and the critical value R_c^0 remains practically constant and has the value

$$R_c^0 = \frac{\alpha_1^6}{2} \left(Q < \frac{\kappa^2}{\eta^2} 5\alpha_1^2 \right) \quad (70)$$

For $Q > \frac{\kappa^2}{\eta^2} 5\alpha_1^2$, R_c^0 can be written

$$R_c^0 = R_c^{\text{con}} \left(\frac{\eta}{\kappa} \right)^2, \quad (71)$$

where

$$R_c^{\text{con}} = \frac{\alpha_1^4}{10} Q. \quad (72)$$

On the other hand, if

$$Q < 5\alpha_1^2, \quad (73)$$

the principle of the exchange of stabilities will be applicable and instability sets in as the ordinary cellular convection. The critical value of R_c^{con} is given by eqn. (72) and it is less than that given by (70) so long as (73) holds. therefore we may say that for $Q < Q_1$ instability will arise by cellular convection while for $Q > Q_1$ it will arise through overstability.

According to eqns. (51) and (64), for $Q > Q_1$

$$\lambda = \frac{\alpha_1 \eta}{a^2} \sqrt{\frac{Q}{5}} \quad (74)$$

Magnetic field does not influence the critical value of R_c^0 (eqn. (70)), but does the frequency of oscillation with which overstability sets in. The meaning of this frequency becomes clearer when we express Q in terms of the velocity of the magneto-hydrodynamic wave [2].

$$V = \left(\frac{\mu H^2}{4\pi\rho_0} \right)^{\frac{1}{2}} \quad (75)$$

From eqns. (11), (42) and (75)

$$Q = V^2 \frac{a^2}{\eta^2}. \quad (76)$$

Thus we find

$$\lambda = \frac{a_1 V}{\sqrt{5} a}, \quad (77)$$

where $\frac{2\pi}{\lambda}$ is the time required for a magneto-hydrodynamic wave to travel a distance equal to $\frac{2\pi\sqrt{5}}{a_1} a$.

Letting

$$Q_1 = 5a_1^2 = V_1^2 \frac{a^2}{\eta^2}, \quad (78)$$

we get

$$V_1 = \sqrt{5} a_1 \frac{\eta}{a}. \quad (79)$$

Thus we may summarize that so long as the velocity of magneto-hydrodynamic wave is less than V_1 , instability will arise through cellular convection, but for

$V_1 < V < \sqrt{5} a_1 \frac{\kappa}{a}$, we will have overstability when R^0 reaches the value given by (70).

8. Examinations of Onset of Instability in the General Case

Using eqns. (65) and (72) we can draw in the (R, Q) — plane the critical value of R . The convection curve starts on the R -axis at the point

$$R_e^{\text{con}} = 0 \quad (Q=0) \quad (80)$$

and for $Q \rightarrow \infty$

$$R_e^{\text{con}} = \frac{a_1^4 Q}{10} \quad (Q \rightarrow \infty). \quad (81)$$

On the other hand, the overstability curve starts on the R axis at

$$R_e^0 = \frac{\left(1 + \frac{\eta}{\kappa}\right)}{2} a_1^6 \quad (Q=0) \quad (82)$$

and for $Q \rightarrow \infty$ becomes asymptotic to the line

$$R_e^0 = \frac{a_1^4 Q}{10} \frac{\eta^2}{\kappa^2} \quad (Q \rightarrow \infty). \quad (83)$$

It follows that the overstability curve always starts above the convection curve.

i) From eqns. (81) and (83), it follows that the overstability curve lies entirely above the convection curve when

$$\eta \geq \kappa. \quad (84)$$

In this case instability always arises as a cellular convection.

ii) According to Eqns. (81) and (83), when

$$\kappa > \eta, \quad (85)$$

the overstability curve intersects the convection curve and for $Q \rightarrow \infty$ lies below it. The

value of Q at the point where the two curves intersect is given by

$$Q_1' = \frac{5\alpha_1^2}{1 - \frac{\eta}{\kappa}} = \frac{Q_1}{1 - \frac{\eta}{\kappa}}. \quad (86)$$

Thus for $Q \leq Q_1'$, instability arises as a cellular convection, but for $Q > Q_1'$, we shall have overstability when the critical value of R^0 is reached.

9. Concluding Remarks

It has been found that only $(U_n^m, V_{n+1}^m, T_n^m, S_{n+1}^m, \dots)$ motion can arise in a non-viscous sphere under a uniform magnetic field. The remarkable feature is that the order and degree of the spherical surface harmonics of perturbations in magnetic field and vorticity are equal. This may suggest a relation or resemblance between the magnetic field and vorticity, which is first proposed by Batchelor [3].

Under the normal terrestrial conditions ($\eta > \kappa$), the inhibition of convection by a magnetic field is a very pronounced effect, but under astrophysical conditions ($\kappa > \eta$), this is not. The results are quantitatively equal to those of Chandrasekhar's investigation of instability of a layer of fluid heated below under a uniform magnetic field. We shall treat the problem of instability of a fluid sphere under simultaneous action of rotation and magnetic field in the next paper.

10. Acknowledgements

The author wishes to express his sincere thanks to Prof. M. Hasegawa of Fukui University, Prof. Y. Tamura of Kyoto University Prof. Y. Saito of Osaka City University and Dr. H. Miki for their continuous interest and encouragement in the course of this study and to Pro. K. Maeda of Kyoto University, Prof. T. Nagata, Dr. T. Rikitake of Tokyo University, Dr. M. Hirono of Radio Research Laboratories and Dr. K. Nagashima of Nagoya University for their valuable discussions and advice on this problem.

References

- [1] Namikawa, T., J.G.G., **9**, 182 (1958)
- [2] Chandrasekhar, S., Phil. Mag., **7**, **43**, 501 (1952)
- [3] Batchelor, G.K., Proc. Roy. Soc. A **201**, 405 (1950)