Fluid Motions in a Sphere III

Thermal Instability of a Rotating Fluid Sphere Heated Within under a Uniform Magnetic Field

By Tomikazu NAMIKAWA

Institute of Politecnics, Osaka City University

(Read Oct. 9, 1957; Received July 20, 1958)

Abstract

The influence of simultaneous action of Coriolis force and magnetic field on convection in a non-viscous fluid sphere heated within is examined. It is found that there is only axially symmetric solutions of the marginal stability, and inhibition of convection by a magnetic field is pronounced when the intensity of magnetic field is larger than a critical value.

1. Introduction

In paper [1] and [2], we examined the influence of rotation and magnetic field on the thermal instability of fluid sphere heated within. We are interested in the origin of the earth's magnetic field and stellar magnetic field. According to the Elsasser-Bullard's dynamo theory of cosmic magnetic field, the maintenance of a magnetic field is induction effect in the earth's core or in the stars in which a thermal convection may exist. As the earth and the stars are rotating, we must solve the equation of fluid motion, of Maxwell and of heat conduction, taking into consideration Coriolis force.

As this is a very difficult problem, we treat in this paper, these equations in linearized form and examine the influence of a simultaneous action of magnetic field and rotation on the thermal instability of a fluid sphere heated within.

2. Equation of Motion

Consider a homogeneous conducting fluid sphere of radius a rotating with an angular velocity about z -axis under a uniform magnetic field, the direction of which is taken to be z -axis. Assumption about the source of heat generation is the same as that of paper [1] and [2]. The equation of motion, of Maxwell, and of heat conduction appropriate to the problem on hand are

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + 2\rho \mathbf{u} \times \boldsymbol{\Omega} + \rho \nu \nabla^2 \mathbf{u} + \rho \nabla V + \mu \mathbf{J} \times \mathbf{H}, \quad (1)$$

$$\text{curl } \mathbf{H} = 4\pi \mathbf{J}, \quad (2)$$

$$\text{curl } \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \quad (3)$$

$$\text{div } \mathbf{H} = 0, \quad (4)$$

$$\text{div } \mathbf{E} = 4\pi c^2 q, \quad (5)$$

$$\mathbf{J} = c(\mathbf{E} + \mu \mathbf{u} \times \mathbf{H}), \quad (6)$$

and

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \kappa \nabla^2 T. \quad (7)$$

The meaning of notations is equal to those in paper [1] and [2].

The equations governing small departures from the stationary state are ([1], [2], [3])

$$\frac{\partial \theta}{\partial t} = \kappa \nabla^2 \theta + 2\beta(r u_r), \quad (8)$$

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_0} \nabla p + 2\mathbf{u} \times \boldsymbol{\Omega} + \gamma \theta \mathbf{r} + \frac{\mu H}{4\pi \rho_0} \frac{\partial \mathbf{h}}{\partial z} + \nu \nabla^2 \mathbf{u}, \quad (9)$$

$$\frac{\partial \mathbf{h}}{\partial t} = H \frac{\partial \mathbf{u}}{\partial z} + \eta \nabla^2 \mathbf{h} \quad (10)$$

and

$$\text{div } \mathbf{u} = \text{div } \mathbf{h} = 0. \quad (11)$$

Taking the curl of eqns. (9) and (10), we get

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \gamma \nabla \theta \times \mathbf{r} + 2\boldsymbol{\Omega} \frac{\partial \mathbf{u}}{\partial z} + \frac{\mu H}{4\pi \rho_0} \frac{\partial}{\partial z} \text{curl } \mathbf{h} + \nu \nabla^2 \boldsymbol{\omega} \quad (12)$$

and

$$\frac{\partial}{\partial t} \text{curl } \mathbf{h} = H \frac{\partial}{\partial z} \boldsymbol{\omega} + \eta \nabla^2 (\text{curl } \mathbf{h}). \quad (13)$$

Again, taking the curl of eqn. (12), we have

$$-\frac{\partial}{\partial t} \nabla^2 \mathbf{u} = \gamma \text{curl}(\nabla \theta \times \mathbf{r}) - \frac{\mu H}{4\pi \rho_0} \frac{\partial}{\partial z} \nabla^2 \mathbf{h} - \nu \nabla^4 \mathbf{u}. \quad (14)$$

We express the velocity field and the magnetic field as a superposition of a poloidal and a toroidal vector in terms of four scalars as in eqn. (15) and (16) of [2], and the four scalars are given in eqn. (17) of [2].

Multiplying eqns. (10), (12), (13) and (14) by $\mathbf{r}$, we get

$$\frac{\partial}{\partial t} (r h_r) = H \mathbf{r} \frac{\partial \mathbf{h}}{\partial z} + \eta \nabla^2 (r h_r), \quad (15)$$

$$\frac{\partial}{\partial t} (r \omega_r) = 2\boldsymbol{\Omega} \mathbf{r} \frac{\partial \mathbf{u}}{\partial z} + \frac{\mu H}{4\pi \rho_0} \mathbf{r} \frac{\partial}{\partial z} (\text{curl } \mathbf{h}) + \nu \nabla^2 (r \omega_r), \quad (16)$$

$$\frac{\partial}{\partial t} (\text{curl } \mathbf{h})_r = H \mathbf{r} \frac{\partial}{\partial z} \boldsymbol{\omega} + \eta \nabla^2 r (\text{curl } \mathbf{h})_r \quad (17)$$

and

$$-\frac{\partial}{\partial t} \nabla^2 (r u_r) = \gamma L^2 \theta + 2\boldsymbol{\Omega} \mathbf{r} \frac{\partial \boldsymbol{\omega}}{\partial z} - \frac{\mu H}{4\pi \rho_0} \mathbf{r} \frac{\partial}{\partial z} \nabla^2 \mathbf{h} - \nu \nabla^4 (r U_r). \quad (18)$$

Operating Y_n^m to eqns. (8), (15), (16), (17) and (18) and executing the integration with respect to ϑ and φ , we have

$$a^2 \frac{\partial}{\partial t} \Theta_n^m - \kappa \frac{1}{r^2} \left\{ \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - n(n+1) \right\} \Theta_n^m - 2\beta a n(n+1) \frac{U_n^m}{r} = 0, \quad (19)$$

$$\begin{aligned} n(n+1)a^2 \frac{\partial}{\partial t} \frac{S_n^m}{r} - \eta n(n+1) \frac{1}{r^2} \left\{ \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - n(n+1) \right\} \frac{S_n^m}{r} \\ - Ha \left[\frac{(n-1)(n+1)(n-m)}{2n-1} r^{n-1} \frac{d}{dr} (r^{-n} U_n^m) + \frac{n(n+2)(n+m+1)}{2n+3} r^{-(n+2)} \right. \\ \left. \times \frac{d}{dr} (r^{n+1} U_{n+1}^m) + im \frac{V_n^m}{r^2} \right] = 0, \end{aligned} \quad (20)$$

$$\begin{aligned} a^2 \frac{\partial}{\partial t} n(n+1) \frac{V_n^m}{r^2} - \nu n(n+1) \frac{1}{r^2} \left\{ \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - n(n+1) \right\} \frac{V_n^m}{r^2} \\ - 2\Omega a^2 \left[\frac{(n-1)(n+1)(n-m)}{2n-1} r^{n-1} \frac{d}{dr} (r^{-n} U_{n-1}^m) + \frac{n(n+2)(n+m+1)}{2n+3} r^{-(n+2)} \right. \\ \left. \frac{d}{dr} (r^{n+1} U_{n+1}^m) + im \frac{V_n^m}{r^2} \right] - \frac{\mu Ha}{4\pi\rho_0} \left[\frac{(n-1)(n+1)(n-m)}{2n-1} r^{n-1} \frac{d}{dr} (r^{-n} T_{n-1}^m) \right. \\ \left. + \frac{n(n+2)(n+m+1)}{2n+3} r^{-(n+2)} \frac{d}{dr} (r^{n+1} T_{n+1}^m) - \frac{im}{r} \left\{ \frac{d^2}{dr^2} - \frac{n(n+1)}{r^2} \right\} S_n^m \right] = 0, \end{aligned} \quad (21)$$

$$\begin{aligned} -n(n+1)a^2 \frac{\partial}{\partial t} \frac{1}{r^2} \left\{ \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - n(n+1) \right\} \frac{U_n^m}{r} + \nu n(n+1) \left[\frac{1}{r^2} \left\{ \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) \right. \right. \\ \left. \left. - n(n+1) \right\} \right]^2 \frac{U_n^m}{r} - 2\Omega a^2 \left[\frac{(n-1)(n+1)(n-m)}{2n-1} r^{n-1} \frac{d}{dr} (r^{-n} V_{n-1}^m) \right. \\ \left. + \frac{n(n+2)(n+m+1)}{2n+3} r^{-(n+2)} \frac{d}{dr} (r^{n+1} V_{n+1}^m) - im \frac{1}{r} \left\{ \frac{d^2}{dr^2} - \frac{n(n+1)}{r^2} \right\} U_n^m \right] \\ + \frac{\mu Ha}{4\pi\rho_0} \frac{1}{r^2} \left\{ \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - n(n+1) \right\} \left[\frac{(n-1)(n+1)(n-m)}{2n-1} r^{n-1} \frac{d}{dr} (r^{-n} S_{n-1}^m) \right. \\ \left. + \frac{n(n+2)(n+m+1)}{2n+3} r^{-(n+2)} \frac{d}{dr} (r^{n+1} S_{n+1}^m) + im \frac{T_n^m}{r^2} \right] = r n(n+1) a^5 \Theta_n^m \end{aligned} \quad (22)$$

and

$$\begin{aligned} a^2 n(n+1) \frac{\partial}{\partial t} \frac{T_n^m}{r^2} - \eta n(n+1) \frac{1}{r^2} \left\{ \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - n(n+1) \right\} \frac{T_n^m}{r^2} \\ - Ha \left[\frac{(n-1)(n+1)(n-m)}{2n-1} r^{n-1} \frac{d}{dr} (r^n V_{n-1}^m) + \frac{n(n+2)(n+m+1)}{2n+3} r^{-(n+2)} \right. \\ \left. \frac{d}{dr} (r^{n+1} V_{n+1}^m) - \frac{im}{r} \left\{ \frac{d^2}{dr^2} - \frac{n(n+1)}{r^2} \right\} U_n^m \right] = 0. \end{aligned} \quad (23)$$

Eqns. (19), (20), (21), (22) and (23) are the basic equations for this problem.

3. Boundary Conditions

As in [1] and [2], we shall treat eqns. (21) and (22) neglecting the viscous term.

The boundary conditions are the same as that of [2]. The vanishing of perturbation in temperature and radius component of velocity, and continuity of magnetic field and tangential components of electric field on the boundary surface are required.

4. Solutions of Basic Equations

As in [2], we shall examine the $(U_n^m, T_n^m, V_n^m, S_{n+1}^m \dots)$ motions taking only the first four terms in consideration. There is no $(V_n^m, S_n^m, U_{n+1}^m, T_{n+1}^m \dots)$ motion because the continuity of the magnetic field cannot be satisfied on boundary surface.

Expansions of velocity and magnetic field inside sphere or outside sphere, are equal to eqns. (29)–(35) of [2].

If all quantities may like $e^{\lambda t}$, basic equations are reduced to

$$(\lambda a^2 + \kappa a_j^2) \Theta_{n,j}^m - 2\beta a n(n+1) U_{n,j}^m = 0, \quad (24)$$

$$(n+1)(n+2) \lambda a^2 V_{n+1,j}^m - 2\Omega a^2 \left[-\frac{n(n+2)(n+1-m)}{2n+1} a_j U_{n,j}^m + im V_{n+1,j}^m \right] - \frac{Ha}{4\pi\rho_0} \left[-\frac{n(n+2)(n+1-m)}{2n+1} a_j T_{n,j}^m + im a_j^2 S_{n+1,j}^m \right] = 0, \quad (25)$$

$$n(n+1) \lambda a^2 a_j^2 U_{n,j}^m - 2\Omega a^2 \left[\frac{n(n+2)(n+m+1)}{2n+3} a_j V_{n+1,j}^m + im a_j^2 U_{n,j}^m \right] - \frac{Ha}{4\pi\rho_0} a_j^2 \left[\frac{n(n+2)(n+m+1)}{2n+3} a_j S_{n+1,j}^m + im T_{n,j}^m \right] = \gamma n(n+1) a^5 \Theta_{n,j}^m, \quad (26)$$

$$(n+1)(n+2) (\lambda a^2 + \eta a_j^2) S_{n+1,j}^m - Ha \left[-\frac{n(n+2)(n+1-m)}{2n+1} a_j U_{n,j}^m + im V_{n+1,j}^m \right] = 0, \quad (27)$$

$$n(n+1) (\lambda a^2 + \eta a_j^2) T_{n,j}^m - Ha \left[\frac{n(n+2)(n+m+1)}{2n+3} a_j V_{n+1,j}^m + im a_j^2 U_{n,j}^m \right] = 0. \quad (28)$$

From eqns. (24), (27) and (28), we obtain

$$\Theta_{n,j}^m = \frac{2\beta a n(n+1)}{\lambda a^2 + \kappa a_j^2} U_{n,j}^m, \quad (29)$$

$$S_{n+1,j}^m = \frac{Ha \left[-\frac{n(n+2)(n+1-m)}{2n+1} a_j U_{n,j}^m + im V_{n+1,j}^m \right]}{(n+1)(n+2) (\lambda a^2 + \eta a_j^2)} \quad (30)$$

and

$$T_{n,j}^m = \frac{Ha \left[\frac{n(n+2)(n+m+1)}{2n+3} a_j V_{n+1,j}^m + im a_j^2 U_{n,j}^m \right]}{n(n+1) (\lambda a^2 + \eta a_j^2)} \quad (31)$$

Substituting eqns. (29), (30) and (31) in eqns. (25) and (26), we get

$$\left[(n+1)(n+2) \omega - im \eta T + \frac{\alpha_j^2 \eta^2 Q}{(n+1)(\omega + \eta a_j^2)} \left\{ \frac{n(n+2)^2(n+1-m)(n+m+1)}{(2n+1)(2n+3)} + \frac{m^2}{n+2} \right\} \right] V_{n+1,j}^m + \left[\frac{n(n+2)(n+1-m)}{2n+1} a_j \eta T + \frac{2im a_j^3 \eta^2 Q (n+1-m)}{(2n+1)(\omega + \eta a_j^2)} \right] U_{n,j}^m = 0 \quad (32)$$

and

$$\left[\frac{n(n+2)(n+m+1)}{2n+3} a_j \eta T + \frac{2i m a_j^3 \eta^2 Q (n+m+1)}{(2n+3)(\omega + \eta a_j^2)} \right] V_{n+b,j}^m - \left[n(n+1) a_j^2 \omega - i m a_j^2 \eta T + \frac{a_j^4 \eta^2 Q}{(n+1)(\omega + \eta a_j^2)} \left\{ \frac{n(n+2)(n+m+1)(n+1-m)}{(2n+1)(2n+3)} + \frac{m^2}{n} \right\} - \frac{n^2(n+1)^2}{(\omega + \kappa a_j^2)} \kappa \eta R \right] U_{n,j}^m = 0, \quad (33)$$

where

$$\left. \begin{aligned} \omega &= \lambda a^2, & T &= \frac{2\Omega}{\eta} a^2, & Q &= \frac{\mu^2 H^2 \sigma a^2}{\rho_0 \eta} \\ R &= \frac{2\beta \gamma}{\kappa \eta} a^6, \end{aligned} \right\} \quad (34)$$

and

From eqns. (32) and (33), we get

$$\left[(n+1)(n+2)\omega - i m \eta T + \frac{f_1 a_j^2 \eta^2 Q}{(n+1)(\omega + \eta a_j^2)} \right] \left[n(n+1)\omega - i m \eta T + \frac{f_2 a_j^2 \eta^2 Q}{(n+1)(\omega + \eta a_j^2)} - \frac{n^2(n+1)^2 \kappa \eta R}{a_j^2(\omega + \kappa a_j^2)} \right] + \frac{(n+1-m)(n+m+1)}{(2n+1)(2n+3)} \left[n(n+2)\eta T + \frac{2i m a_j^2 \eta^2 Q}{\omega + \eta a_j^2} \right]^2 = 0, \quad (35)$$

where

$$f_1 = \frac{n(n+2)^2(n+1-m)(n+m+1)}{(2n+1)(2n+3)} + \frac{m^2}{n+2},$$

and

$$f_2 = \frac{n(n+2)(n+m+1)(n+1-m)}{(2n+1)(2n+3)} + \frac{m^2}{n}. \quad (36)$$

Eqn. (35) is reduced to

$$\begin{aligned} & \left[(n+1)(n+2)\omega^2 + \eta \{ (n+1)(n+2)a_j^2 - i m T \} \omega + \eta^2 a_j^2 \left\{ \frac{f_1}{n+1} \cdot Q - i m T \right\} \right] \\ & \times \left[n(n+1)\omega^3 + \kappa \left\{ n(n+1) \left(1 + \frac{\eta}{\kappa} \right) - i m T \frac{\eta}{\kappa} \right\} \omega^2 + \kappa \eta a_j^2 \left\{ n(n+1)a_j^2 \right. \right. \\ & + \frac{f_2}{n+1} Q \frac{\eta}{\kappa} - \frac{n^2(n+1)^2}{a_j^4} R - i m T \left(1 + \frac{\eta}{\kappa} \right) \left. \left. \right\} \omega + \kappa \eta^2 a_j^4 \left\{ \frac{f_2}{n+1} Q \right. \right. \\ & \left. \left. - \frac{n^2(n+1)^2}{a_j^4} R - i m T \right\} \right] + \frac{(n+1-m)(n+m+1)}{(2n+1)(2n+3)} (\omega + \kappa a_j^2) \eta^2 \\ & \times [n(n+2)T\omega + n(n+2)a_j^2 \eta T + 2i m a_j^2 \eta Q]^2 = 0. \quad (37) \end{aligned}$$

5. Solutions of Marginal Stability

Firstly we assume the validity of the principle of exchange of stabilities. Letting $\omega \rightarrow 0$ in eqn. (37), we get then

$$\left\{ \frac{f_1}{n+1} Q - i m T \right\} \left\{ \frac{f_2}{n+1} Q - \frac{n^2(n+1)^2}{a_j^4} R - i m T \right\} + \frac{(n+1-m)(n+m+1)}{(2n+1)(2n+3)} \times \{ n(n+2)T + 2i m Q \}^2 = 0. \quad (38)$$

This is reduced to

$$\left\{ \frac{f_1 f_2}{(n+1)^2} - 4 m^2 \frac{(n+1-m)(n+m+1)}{(2n+1)(2n+3)} \right\} Q^2 + \left\{ \frac{n^2(n+2)^2(n+1-m)(n+m+1)}{(2n+1)(2n+3)} - m^2 \right\} T^2 - \frac{f_1}{n+1} Q \cdot \frac{n^2(n+1)^2}{a_j^4} R = 0 \quad (39)$$

and

$$R/Q = \left\{ \frac{f_1 + f_2}{n+1} - \frac{4n(n+2)(n+m+1)(n+1-m)}{(2n+1)(2n+3)} \right\} \frac{\alpha_j^4}{n^2(n+1)^2}. \quad (40)$$

As the right hand side of eqn. (40) is negative for all value of $n(=m)$, we have no steady state solutions in general case.

Letting $m \rightarrow 0$ in (39), we get

$$R = \frac{\alpha_j^4}{n^2(n+1)f_1} \left[\frac{f_1 f_2}{(n+1)^2} Q + \frac{n^2(n+2)^2(n+1)^2}{(2n+1)(2n+3)} \frac{T^2}{Q} \right]. \quad (41)$$

We have only axially symmetric solutions of marginal stability, the $(U_1^0, V_2^0, T_1^0, S_2^0, \dots)$ motion.

The minimum value of α_j is $\alpha_1 = 4.5$; critical value of R is given by

$$R_c^{\text{con}} = \alpha_1^4 \left\{ \frac{Q}{10} + \frac{5T^2}{Q} \right\}. \quad (42)$$

R_c^{con} has its minimum value when

$$Q = 5\sqrt{2}T. \quad (43)$$

6. Concluding Remarks

It is found, from eqn. (42) that when angular velocity and other constant of fluid, κ, η, ρ , etc., are given, the inhibition of convection by a magnetic field is pronounced only when intensity of magnetic field is larger than a critical value. The behaviour of R resembles to that of Chandrasekhar's investigation of instability of a layer of conducting fluid under simultaneous action of Coriolis force and uniform magnetic field.

We shall report in the near-future the results of the examination of non-steady state. It is supposed from the results of this paper that it is very interesting to investigate the convection in the earth's core, solving equations of motion, of Maxwell and of heat conduction in more complete way.

7. Acknowledgements

The author wishes to express his sincere thanks to Prof. M. Hasegawa of Fukui University, Prof. Y. Tamura of Kyoto University, Prof. Y. Saito of Osaka City University and Dr. H. Miki for their continuous interest and encouragements in the course of this study and to Prof. K. Maeda of Kyoto University, Prof. T. Nagata, Dr. T. Rikitake

Dependence of the characteristic number R for the onset of convection on the number T and Q .

of Tokyo University, Dr. M. Hirono of Radio Research Laboratories and Dr. K. Nagashima of Nagoya University for their valuable discussions and advices on this problem.

References

- [1] Namikawa, T., J.G.G. **9**, 182 (1958)
- [2] Namikawa, T., J.G.G. **9**, 193 (1958)
- [3] Chandrasekhar, S., Proc. Roy. Soc., **A 225**, 173 (1954)