Fractional calculus-based compression modeling of soft clay

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ABSTRACT

As a popular tool for characterizing the rheological properties of materials, fractional calculus can be applied to describe soil creeping. This study attempted to analyze the one-dimensional (1D) compression behavior of clay by using fractional calculus. A fractional Merchant model (FMM) was established based on viscoelastic-elastic correspondence principle. This model was successfully applied to fit the oedometer test results of Nansha clay, which is widely spread in the Pearl River Delta, China. It is demonstrated that the fractional derivative-based model was able to characterize the compression behavior of soft clay with fewer parameters than classical rheological models. The relationship between the order of fractional derivative and the compression process of clay was discussed. The order of fractional derivative is observed to decrease as load increases, which implies that the creep behavior of clay became less pronounced with increasing load levels. These findings may open up new avenues for theoretical and empirical modeling of clay compression in the framework of fractional calculus.

Keywords: fractional calculus, clay, one-dimensional compression, oedometer test, creep

1 INTRODUCTION

It is well known that the time-dependent behaviour is a fundamental mechanical properties of clayey soils. For instance, the total settlement of clay caused by loads can be divided into elastic settlement, primary consolidation settlement and secondary consolidation settlement. In the past eight decades, many creep models of clay has been proposed in literature. For every model, there exists certain limitations, which considerably restricts the scope of its applicability. Basically, there are three types of such models, i.e. empirical models, rheological models, and general stress-strain-time models (Liingaard et al. 2004). A variety of rheological models have been established and successfully used to predict clay compression (e.g. Taylor and Merchant 1940; Taylor 1942; Ishii 1951; Gibson and Lo 1961; Barden 1965; Suklje 1969). Compared with the other two types of models, rheological models can perform simplified simulations of the viscosity of soil with unique material visualization in graphical forms. For footings resting on soft clay, a straightforward yet close estimation of the long-term settlement can be obtained in the framework

of rheology. However, the conventional Newtonian dashpot can hardly describe the viscous behavior of clay with a satisfactory degree of accuracy (Liingaard et al. 2004). Recently a new elementary model, which allows us to capture the complicated viscous behaviour of natural and artificial materials, has emerged from fractional calculus.

Fractional calculus deals with the generalization of differentiation and integration operator to all real orders. In many fields of science and engineering, such as rheology, fluid flow, diffusive transport, electrical networks, electromagnetic theory and probability, fractional-order derivatives and integrals have been successfully utilized to solve problems. Compared with classical theories, fractional calculus can accurately describe memory-intensive and path-dependent phenomena by using fewer parameters (Caputo and Mainardi 1971; Bagley and Torvik 1983; Koeller 1984). This is very attractive for the discipline of geotechnical engineering. To date, there have been only a few studies on fractional calculus-based modeling of soil behavior (Zhu et al. 2011, 2012; Yin et al. 2013; Zhang et al. 2014). In particular, the physical meaning of the fractional parameters and their relationships with soil

properties, are still unknown.

In this paper, a fractional derivative-based model is derived to describe the one-dimensional (1D) compression of clay. Based on the analysis of oedometer test results, the predictive ability of the fractional model is evaluated. Finally, the variation of the order of fractional derivative during the compression process is discussed.

2 FRACTIONAL MERCHANT MODEL

The fractional derivative-based models are established through the use of a fractional derivative element as shown in Fig. 1. This element is sometimes called “intermediate model”. The constitutive equation of this element is

$$\sigma(t) = E\tau^\alpha D^\alpha \varepsilon(t), \quad 0 \leq \alpha \leq 1 \quad (1)$$

where σ is the stress; ε is the strain; η is the coefficient of viscosity; E is the elastic modulus; τ is a characteristic time defined as $\tau = \eta / E$; α is the order of fractional derivative; and D^α denotes the Riemann–Liouville’s fractional derivative.

Fig. 1. Fractional Merchant model (FMM).

For the standard Merchant model (SMM), if the Newtonian dashpot is replaced by the fractional derivative element, the fractional Merchant model (FMM) is established (Fig. 1). In this paper, this model will be used to predict the compression of clay.

For 1D compression of a clay, the total vertical strain ε_v is given by

$$\varepsilon_v = \varepsilon_v^e + \varepsilon_v^{ve} \quad (2)$$

where ε_v^e and ε_v^{ve} are strains of the elastic and the viscoelastic bodies, respectively.

Under a vertical stress σ_{v0} , the strain of the elastic body is $\varepsilon_v^e = \sigma_{v0} / E_0$, where E_0 is the elastic modulus. For the viscoelastic body, we have

$$\sigma_{v0} = E_1 \tau_1^\alpha (D^\alpha + 1 / \tau_1) \varepsilon_v^{ve}(t), \quad 0 \leq \alpha \leq 1 \quad (3)$$

where $\tau_1 = \eta / E_1$ is a characteristic time. This differential equation can be solved by Laplace transform. The solution $\hat{\varepsilon}_v^{ve}(s)$ of Equation (3) in Laplace domain involves the order of fractional derivative, i.e.

$$\hat{\varepsilon}_v^{ve}(s) = \frac{\sigma_{v0}}{E_1 \tau_1^\alpha} \frac{1}{s(s^\alpha + 1 / \tau_1^\alpha)}, \quad 0 \leq \alpha \leq 1 \quad (4)$$

and by making the inverse Laplace transform we obtain

$$\varepsilon_v^{ve}(t) = \frac{\sigma_{v0}}{E_1} \left[1 - E_\alpha \left[-\left(\frac{t}{\tau_1} \right)^\alpha \right] \right], \quad 0 \leq \alpha \leq 1 \quad (5)$$

where $E_\alpha(\cdot)$ is the well-known generalized Mittag-Leffler function defined as

$$E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + 1)} \quad (6)$$

Plugging Equation (5) into Equation (2), we have

$$\varepsilon_v(t) = \frac{\sigma_{v0}}{E_0} \left\{ 1 + \frac{E_0}{E_1} \left[1 - E_\alpha \left[-\left(\frac{t}{\tau_1} \right)^\alpha \right] \right] \right\}, \quad 0 \leq \alpha \leq 1 \quad (7)$$

Equation (7) is the constitutive equation of the FMM. When the order of fractional derivative $\alpha = 1$, Equation (7) becomes the stress-strain relationship of the SMM.

3 MODEL VALIDATION

To validate the proposed model, oedometer tests on soft clay were carried out. The proposed FMM is used to fit the test results, together with two classical rheological models—the SMM and the generalized Kelvin model (GKM). It should be noted that the numbers of model parameters of SMM, FMM and GKM are 3, 4, 5, respectively.

The soft clay used here was taken from Nansha Development Zone of Guangzhou, Pearl River Delta, southern China. The basic physical and mechanical properties of the specimens are as follows: $G_s = 2.71$, $w = 69.0\%$, $\rho = 15.7 \text{ kN/m}^3$, $w_L = 51.2\%$, $w_P = 25.2\%$, $c' = 30.6 \text{ kPa}$, and $\phi' = 22.2^\circ$. The oedometer tests were carried out under a double drainage condition. Multi-loadings were applied on the specimens in stages. For each stage, the loading lasted for 3 days to ensure the primary consolidation was absolutely completed and the secondary compression have taken place for a sufficient period.

Fig. 2 shows the experimental results and the fitting results from the FMM. It is observed that the predicted results shows good agreement with the test data, demonstrating the model’s predictive ability. As indicated by the root-mean-square error (RMSE) values in Fig. 3, the FMM provides far better approximations than the SMM and, more importantly, this model fits the experimental data better than the GKM except for the first load increment. The two strain-time curves under the third and the sixth load increments clearly show that the SMM almost fails to fit the experimental data, while the upgraded one (i.e. the FMM) is able to provide better predictions (Figs. 4(a) and (b)). It can be concluded that, although the GKM has one more parameter than the FMM, the latter has made prediction more accurately.

Fig. 2. Comparison between experimental and predicted results for oedometer tests. The solid lines indicate results predicted by the FMM.

Fig. 3. RMSE values of least squares fitting under different load levels.

(a) Strain-time curve under the third load increment

(b) Strain-time curve under the sixth load increment

Fig. 4. Comparison of the predicted results between different component models.

4 DISCUSSION

Although the advantage of the fractional derivative-based model in describing the 1D compression behavior of soft clay is verified, a clear understanding of the physical meaning of the fractional parameters is required. Here some preliminary findings are presented.

The relationship between the fractional parameters and the load levels is shown in Fig. 5. The coefficient of viscosity η is found to be much larger under a higher load level (Fig. 5(a)), which is consistent with the phenomena described by Wong and Varatharajan (2014). In contrast, the increase in load level is accompanied by a decrease in the order of fractional derivative (Fig. 5(b)). It seems that α is not a constant during the soil compression process. This is reasonable because that the fractional derivative element is intermediate between ideal solid and Newtonian fluid. When $\alpha=0$, the fractional derivative element is collapsed into a spring (ideal solid). While $\alpha=1$, it will be a dashpot (Newtonian fluid). When the load on the clay increases, the pore water in clay will be gradually discharged and the clay becomes denser, which will result in less pronounced creep behavior of clay and consequent decrease of α .

Fig. 5. Relationships between fractional parameters and load levels.

5 CONCLUSIONS

In this paper, the compression behavior of clay is characterized using a fractional derivative-based model, i.e. the FMM. This model is established by replacing the Newtonian dashpot of the standard Merchant model with a fractional derivative element. By using Laplace transform, the vertical strain solution of clay under 1D compression condition is obtained.

To validate the proposed fractional model, oedometer tests on soft clay were conducted. Through a comparison with classical rheological models, the FMM is found to be able to provide better prediction results with only four parameters.

The order of fractional derivative is observed to decrease as load increases. This phenomenon is consistent with the fact that under a higher load level, the soil structure becomes denser, which is accompanied by less significant viscosity.

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