A Simple Mean-variance Model for Equities and Equity Options under Inflation Rate Dynamics

Yizhou Ren†1 and Koichi Miyazaki†1

Abstract: Recently, portfolio managers have to optimize their portfolios taking the inflation rate into consideration because many central banks have adopted inflation targeting policies and the asset returns should be influenced by the target inflation rate. In such an investment environment, we model the equity process so that it can depend on an inflation rate that follows the OU process. Accordingly, we derive the expected returns, the volatilities and the correlation matrix, which are key ingredients for the use of the mean-variance model. We also provide an option pricing formula based on our equity model under the hedge-neutral probability measure and generate the future paths of not only the equity prices and the inflation rate but also the option prices. Using the proposed model, it is possible to optimize portfolios including the equity options under several inflation scenarios.

Key words: Finance, Portfolio Management, Inflation Rate Dynamics, Equity Options, Mean-Variance Model, Real Risk Free Rate, Real Excess Growth Rate, Ornstein-Uhlenbech Process

1 INTRODUCTION

The mean-variance model (Markowitz[1]) is a model to minimize the risk (variance) of portfolios under the constraint of setting its return to some target level. Since the mean-variance model was introduced, tremendous numbers of its extensions have been developed.

In the financial industry, progress related to the mean-variance model has been focused on how to utilize the mean-variance model in actual fund management. In the old days, it was quite difficult to practice large-scale optimization for the optimal portfolio. To attain the optimal portfolio staying away from the computational difficulties, the capital asset pricing model (CAPM) that adopts the beta risk for convenient asset allocation was proposed by Sharpe (Sharpe [2]). The CAPM is based on the mean-variance model and the equilibrium theory in economics. Since the technological constraints for large-scale optimization have been mostly removed since the 1980s, the model such as the Black Litterman model [3] that provides non extreme optimal portfolio utilizing the expected return through a combination of the investors’ view and the equilibrium return of the CAPM under the mean-variance framework have also been invented. Additionally with the development of the financial market, the asset classes included in the portfolio are not limited to equities but also extended to bonds (Hoevenaars et al. [4]) and commodities (Bekkers et al. [5], Jensen et al. [6]).

In this research, we adopt the one period mean-variance model, and provide a method to generate the expected return and the variance covariance matrix by simulating the future asset return depending on the inflation rate, a factor that has recently been important in the financial industry. In generating these matrices, we expand the asset class from the underlying assets (equities) into that including the options (equity options) in addition to the underlying assets. Even though the optimization model is the one period mean-variance model, in order to make the expected returns and the variance covariance matrices correspond to several investment horizons consistently, we describe both the equity price and the inflation rate via continuous time stochastic processes.

We choose the inflation rate as an important factor in the modeling of the equity return process because many empirical literatures suggest that there is a
relationship between the inflation rate and the equity return and also, recently many central banks are adopting the inflation targeting policy. Among the empirical literatures on the relationship between the inflation rate and the equity return, we list Fama and Schwert [7], Pindyck [8], Kaul [9], Boudoukh and Richardson [10], Campbell et al. [11], Sato et al. [12], Lee et al. [13] among others. Many of the literatures provide the negative correlation between them, while Boudoukh and Richardson [10] suggest a positive correlation and an inflation hedging effect of long horizon equity investments. Reviewing the inflation rates in 19 OECD countries, Lee et al. [13] found the mean reverting tendency of the inflation rate.

To represent the mean reverting tendency, we describe the inflation rate process by the mean reverting stochastic process whose diffusion part is correlated with that of the equity process. The inflation targeting policies adopted by many central banks recently (Bernanke et al. [14], Rudebusch and Svensson [15], Truman [16]) announce the target inflation rate quantitatively as the goal of the monetary policy and activate the monetary policy so as to guide the inflation rate around the target. When the target inflation rate is announced, the average inflation rate will converge to the target in the future and thus it is useful to provide the optimal asset allocation model under some inflation rate scenarios.

The purpose of this paper is toward the construction of optimal portfolio using the mean-variance model, to provide the method to generate the appropriate expected returns and the variance covariance matrices not only for equities but also for equity options corresponding to several investment horizons with simulation in the era such that many central banks adopt the inflation targeting policy. A further objective is to discuss the sensitivity of the efficient frontier and the optimal portfolio with respect to the parameters of the proposed method.

The organization of this paper is as follows. In the second section, our model is introduced. In the third section, the parameter estimation technique and the simulation method to derive the optimal portfolio are provided. In the fourth section, the empirical experiments are discussed. In the final section, summary and the concluding remarks are added.

2 EQUITY PROCESS AND OPTION PRICING UNDER THE INFLATION RATE DYNAMICS

2.1 Equity Model Incorporating Inflation Rate Dynamics

We consider a continuous time Markovian economy with a filtered probability space \( (\Omega, F, \{F_t\}, P) \) in the finite horizon \([0,T]\). Two correlated Brownian motions, \( w \) and \( w_X \) with correlation \( \rho \) generate uncertainty in the probability space. All stochastic processes are assumed to be adapted to \( \{F_t, t \in [0,T]\} \), the augmented filtration generated by \( w \) and \( w_X \). We propose the equity process whose drift part depends on the inflation rate \( X_t \) as Eq. (1). And the drift part of the risk free asset \( B_t \) is also modeled to be dependable on the inflation rate as in Eq. (2).

\[
\frac{dS_t}{S_t} = (r + \lambda X_t + \lambda^5 X_t + \mu)dt + \sigma dw_t, \quad (1)
\]

\[
\frac{dB_t}{B_t} = (r + \lambda X_t)dt, \quad (2)
\]

\[
dX_t = \alpha(\bar{X} - X_t)dt + \sigma dw_{tX}, \quad (3)
\]

where \( r + \lambda X_t \), \( r \) and \( \lambda \) are nominal risk free rate, real risk free rate and elasticity of the risk free rate to the inflation rate, respectively. In Eq. (1), \( \lambda^5 X_t + \mu \), \( \lambda^5 \), \( \mu \) and \( \sigma \) represent excess growth rate of the equity derived by subtracting the risk free rate from growth rate of the equity, elasticity of the excess growth rate to the inflation rate, real growth rate without the inflation rate and the volatility of the equity process, respectively. When the equity has positive (negative) and large \( \lambda \), the correlation between the equity and the inflation rate is positive (negative) high and it means that the equity grows in the inflationary (deflationary) period.

The inflation rate \( X_t \) is modeled by the Ornstein Uhlenbech process (here after OU process) as Eq. (3). In Eq. (3), \( \bar{X}, \alpha (>0) \) and \( v \) are average inflation rate, mean reverting speed and volatility of the inflation process, respectively.

Because our model is an incomplete model except \( \rho = \pm 1 \), in converting the drift of the equity process to be \( r + \lambda X_t \), we adopt the hedge neutral measure \( P^h \) introduced by Basak and Chabakauri [17]. Under the hedge neutral measure \( P^h \) with the market price of risk
\[ \frac{\lambda^X_t + \mu}{\sigma}, \] the two Brownian motion \( w^h \) and \( w^x \) with correlation \( \rho \) are defined by Eq. (4).

\[
dw^h_t = dw_t + \frac{\lambda^X_t + \mu}{\sigma} dt
\]

\[
dw^x_t = dw_{xt} + \rho \frac{\lambda^X_t + \mu}{\sigma} dt
\]

\[
dw^h_t dw^x_{xt} = \rho dt
\] (4)

The hedge neutral measure \( P^h \) is defined using the Radon Nikodym derivative as Eq. (5). For the detail of the hedge neutral measure, refer to Basak and Chabakauri [17].

\[
dP^h = \exp\left(-\int_0^t \left( \frac{\lambda^X_s + \mu}{\sigma} \right)^2 ds - \int_0^t \frac{\lambda^X_s + \mu}{\sigma} dw^x_s \right)
\] (5)

**Proposition 1** Under the hedge neutral measure \( P^h \), the equity and the inflation processes are given by Eq. (6) and (7),

\[
\frac{dS_t}{S_t} = \left( r + \lambda^X_t \right) dt + \sigma dw^h_t
\]

\[
dX_t = \beta (\bar{X} - X_t) dt + \nu dw^x_t
\] (6) (7)

where, \( \beta = \alpha + \rho \nu \frac{\lambda^X_t}{\sigma} \) and \( \bar{X} = \left( \alpha \bar{X} - \rho \nu \frac{\mu}{\sigma} \right) / \beta \). Under the hedge neutral measure \( P^h \) and positive \( \beta \), the inflation rate reverts back to the hedge neutral average inflation rate \( \bar{X} \) and the speed of it becomes the hedge neutral mean reverting speed \( \beta \).

The equity price is given by Eq. (8).

\[
S_t = S_0 \exp\left[ \left( -\frac{1}{2} \sigma^2 + \lambda^X_t \right) T + \frac{\lambda^X_t (X_0 - \bar{X})}{\beta} \left( 1 - e^{-\beta t} \right) \right] + \frac{\lambda^X_t}{\beta} \left( 1 - e^{-\beta (t-s)} \right) dw^h_s + \sigma \int_0^t dw^h_s
\] (8)

**Proof** Refer to Appendix A.

**Remark 1**

\( \beta \) could have both positive and negative values. However, only for positive values, the inflation rate process becomes mean reverting process under the hedge neutral probability measure.

2.2 Option Pricing Incorporating Inflation Rate Dynamics

In the Black and Scholes model (hereafter BS model), the option price is derived by modeling the equity process with the geometric Brownian motion under the risk neutral measure. In our equity model incorporating the inflation rate process, we adopt the general theory of option valuation method using the hedge neutral probability measure. The price at time \( t \) of the European call option with maturity \( T \) and strike price \( K \) is derived by computing Eq. (9).

\[
c_t = E^h e^{-r(T-t)\frac{\lambda^X_t}{\beta}} \left[ \frac{\lambda^X_t}{\beta} \right] \Phi\left( \frac{\lambda^X_t (S_t - K)}{\beta} \right)
\] (9)

where \( E^h \left[ \cdot \right] \) denotes the expectation under the hedge neutral measure.

**Proposition 2** The price at time \( t \) of the European call option with maturity \( T \) and strike price \( K \) is provided by Eq. (10).

\[
c_t = S_0 \Phi(d_1) - K \exp\left( -\frac{I_1(t)}{2} \right) \Phi(d_2)
\] (10)

where \( \Phi(y) \) denotes cumulative density function of the standard normal distribution \( \Phi(y) = \int_{-\infty}^y \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \), and

\[
\ln \left( \frac{S_t}{K} \right) + \frac{I_2(t)}{2} \left( \frac{S_t}{K} \right) + \frac{I_3(t)}{2} \left( \frac{S_t}{K} \right) - \frac{I_4(t)}{2} \left( \frac{S_t}{K} \right)
\]

\[
I_1(t) = r + \lambda^X_t \left( T - t \right) + \frac{\lambda^X_t}{\beta} \left( 1 - e^{-\beta t} \right)
\]

\[
I_2(t) = \sigma^2 \left( T - t \right) + \frac{2\lambda^X_t \nu}{\beta} \left( T - t \right) - \frac{1 - e^{-\beta (t-s)}}{\beta}
\]

\[
I_3(t) = \frac{\lambda^X_t}{\beta} \left( T - t \right) - \frac{2\lambda^X_t}{\beta} \left( 1 - e^{-\beta (t-s)} \right) + \frac{1 - e^{-2\beta (t-s)}}{2\beta}
\]

**Proof** Refer to Appendix B.

3 THE SIMPLE MEAN-VARIANCE MODEL FOR EQUITIES AND EQUITY OPTIONS UNDER INFLATION RATE DYNAMICS

3.1 Parameter Estimation Method for the Equity Process

We provide the parameter estimation method for the equity process using the historical equity price data \( S_t \).

The discrete version of Eq. (1) is the regression Eq. (11) with error term \( \epsilon_t \sim \mathcal{N}(0,1) \) that the equity return \( R_t \) is
regressed by the inflation rate $X_t$. In Eq. (11), $r$ and $\lambda'$ are the intercept and the coefficient, respectively, resulting from the regression of the risk free short rate by the inflation rate.

$$R_t^s = \frac{S_{t+1} - S_t}{S_t} = (r + \mu) + (\lambda' + \lambda^s)X_t + \alpha\epsilon_t$$

(11)

The parameter related to the inflation rate is estimated by regression Eq. (12). In Eq. (12), $\epsilon_t \sim N(0,1)$ and $\rho$ is estimated as the correlation between $\epsilon_t$ and $u_t$.

$$X_{t+1} = \alpha \bar{X} + (1-\alpha)X_t + \nu u_t$$

(12)

### 3.2 Parameter Estimation Method for the Mean-Variance Model and the Efficient Frontier

Under several inflation rate scenarios, we examine the portfolio selection problem from total $N \times 4$ assets such as $N$ kinds of underlying equities and three kinds of corresponding equity options: in the money (ITM), at the money (ATM) and out of the money (OTM). To obtain the optimal portfolio by way of the mean-variance model, we have to estimate the expected return and the variance covariance matrix of the asset returns. They are easily derived by our simulation method. The simulation method is summarized in Steps 1 to 4 below. In the simulation, the stochastic process of equity $i$ ($i = 1, 2, \cdots, N$) is modeled as Eq. (13). The correlation between equity $i$ and the inflation rate and the correlation between equity $i$ and $j$ satisfy $d\omega_i, d\omega_{i\lambda} = \rho_i dt$ and $d\omega_i, d\omega_{ij} = \rho_{ij} dt$ ($i \neq j$), respectively.

$$\frac{dS_{it}}{S_{it}} = \left(r + \lambda' X_t + \lambda^s X_t + \mu_i\right) dt + \sigma_i d\omega_{it}$$

(13)

[Step1] Using the Monte Carlo simulation method, generate 5000 sample paths of monthly time interval up to 12 months for each of the inflation rate process and the equity processes from Eq. (3) and Eq. (13), respectively. In the simulation, the diffusion parts of the processes are generated from the multivariate normal distribution satisfying $d\omega_i, d\omega_{i\lambda} = \rho_i dt$ and $d\omega_i, d\omega_{ij} = \rho_{ij} dt$ ($i \neq j$).

Setting the parameters of the average inflation rate $\bar{X}$ and the mean reverting speed $\alpha$ in Eq. (3), we are able to generate various kinds of inflation paths as the future inflation rate scenario.

[Step2] At each monthly time epoch, substituting the equity price and the inflation rate simulated in [Step1] into the call option pricing formula of Eq. (10) with the estimated parameters, we attain 5,000 sample paths of call option prices $c_i$ corresponding to each of the underlying equities. As the equity options corresponding to each underlying equity, three kinds of options such as ATM option (The strike price $K$ is equal to the spot price $S_0$), ITM option (The strike price $K$ is equal to the spot price $S_0$ multiplied by (1 minus two standard deviation of the past return)) and OTM option (The strike price $K$ is equal to the spot price $S_0$ multiplied by (1 plus two standard deviations of the past return)) are adopted for the possible investment assets. The option return is defined as the return by selling the option that we bought at time 0, at the end of the investment horizon and we adopt the same maturity (12 months at time 0) for all of the options without depending on the investment horizon (time to sell). When we evaluate the option price at future time epoch, the remaining maturity of the option at the time becomes shorter than 12 months by the months passed from time 0 and the value of the option is equal to its payoff at maturity (max($S_t - K, 0$)) at the time 12 months after time 0.

[Step3] In the empirical experiment, the investment horizons are set to be one, three, six and twelve months. Thus, the option returns corresponding to each investment horizons have to be computed for each of the 5000 $c_i$ sample paths. Once we get the option returns for each of the sample paths, it is easy to compute the expected returns, the volatilities and the correlation matrices. As the same manner, after computing the equity returns for each of the sample paths, we are able to derive the expected returns, the volatilities, the correlation matrices and further more the correlation between the option returns and the equity returns.

[Step4] The efficient frontier is derived by finding the minimum variance portfolio using the mean-variance model under the expected returns, the volatilities and the correlation matrices derived in [Step3]. The constraint of the investors’ target expected return $R_p$ and the restriction of short sale are as in Eq. (14).

$$\min: w^\top \Sigma w$$

s.t.: $w^\top E[\mathbf{R}] = R_p$

$$w^\top 1 = 1$$

$$w \geq 0$$

where $w$, $E[\mathbf{R}]$ and $\Sigma$ are the weight vector, the expected return vector and the variance covariance matrix, respectively.
Remark 2
In this paper, equity options are investable assets in the portfolio. Due to the highly skewed option return, the mean-variance-skewness model (for example, refer to Konno and Suzuki [18]), which incorporates skewness in addition to usual mean and variance is preferable. However, the application of the mean-variance-skewness model to our issue is beyond the scope of this paper and must be left for the future research.

4 EMPIRICAL ANALYSES

4.1 Data
For equity data, we use 33 sectors of industrial index data announced by the Tokyo Stock Exchange covering the period from April 2002 to March 2011. The industries adopted in the 33 sectors of the industrial index are as follows. Inside the bracket indicates code number of the industry, which is utilized to express the industries in the tables and figures in the empirical analyses. Fishery, Agriculture & Forestry (1), Mining (2), Construction (3), Foods (4), Textiles & Apparels (5), Pulp & Paper (6), Chemicals (7), Pharmaceutical (8), Oil & Coal Products (9), Rubber Products (10), Glass & Ceramics Products (11), Iron & Steel (12), Nonferrous Metals (13), Metal Products (14), Machinery (15), Electric Appliances (16), Transportation Equipment (17), Precision Instruments (18), Other Products (19), Electric Power & Gas (20), Land Transportation (21), Marine Transportation (22), Air Transportation (23), Warehousing & Harbor Transportation Services (24), Information & Communication (25), Wholesale Trade (26), Retail Trade (27), Banks (28), Securities & Commodity Futures (29), Insurance (30), Other Financing Business (31), Real Estate (32), Services (33).

Regarding the inflation rate and the risk free short rate, we use monthly the consumer price index (CPI) year on year basis and the one year Japanese government bond yield, respectively. The estimated parameters related to the inflation rate and the risk free short rate are \( r = 0.022\% \), \( \lambda^e = 0.153 \), \( \alpha = 0.074 \), \( \bar{X} = -0.01\% \) and \( \nu = 0.026\% \).

4.2 Purpose and Method of the Empirical Analyses
The purpose and the method of the empirical analyses consist of the following four parts.

(Purpose and method 1) Industry by industry, we capture the elasticity (\( \lambda^e \)) of the excess growth rate to the inflation rate and the excess real growth rate (\( \mu \)) except the inflation influence and then grasp what sectors of the industries will be selected in the optimal portfolio under the average inflation rate scenarios.

To fulfill the purpose, we estimate the parameters \( \mu \) and \( \lambda^e \) for all of the 33 sectors of industry indexes and visualize them to compare clearly among the industries.

(Purpose and method 2) Because our equity model is dependable on the inflation rate, the price of the option written on the underlying equity is different from the one derived by the usual Black and Scholes [19]. We confirm whether the option price is consistent with the elasticity (\( \lambda^e \)) of the excess real growth rate to the inflation rate or not. For the purpose, we divide the option price from our model by the one from the BS model industry by industry and plot the option price ratios for all of the industries taking the elasticity (\( \lambda^e \)) as the x-axis and the ratio as the y-axis. Based on the figure, we discuss the rationality between the sign and the magnitude of the elasticity and level of the option price ratio.

(Purpose and method 3) We derive the expected returns, the volatilities and the correlation matrix of the assets possibly included in the optimal portfolio for some inflation rate scenarios or several investment horizons (one, three, six and twelve months), and then examine their validity in relation to the results of (Purpose and method 1) and (Purpose and method 2). Their derivation is already mentioned in section 3.2. Toward the purpose, we select 10 kinds of industries out of 33 that are most likely to be included in the optimal portfolio due to space constraints, and reduce the number of investment assets to be 40 (10×4). The selected industries are Mining (2), Foods (4), Pharmaceutical (8), Rubber Products (10), Iron & Steel (12), Electric Power & Gas (20), Land Transportation (21), Air Transportation (23), Information & Communication (25) and Services (33).

(Purpose and method 4) We construct the efficient frontier and the optimal portfolio using the mean-variance model with the expected return, the volatilities and the correlation matrix of the investment assets derived in (Purpose and method 3). We examine how the efficient frontier and the optimal portfolio are influenced with respect to the average inflation rate scenario and the investment horizon and then discuss the appropriateness of the selected assets in each optimal portfolio considering
the expected return, the volatilities and the correlation matrix of the investment assets. The method for the purpose is already provided in section 3.2.

4.3 Results and Implications of the Empirical Analyses

Each of results and the implications for the empirical analyses in this section correspond to a purpose and method in the previous section.

(Result and implication 1) The movements of the Tokyo stock price index (TOPIX) and the inflation rate in the parameter estimation period are depicted in Fig. 1. Fig. 1 suggests that TOPIX surges in the period of negative inflation rate (the deflationary period) from April 2003 to 2006, while it goes down in the period of positive inflation rate (the inflationary period) from October 2007 to April 2009. Thus, in our market and periods, the correlation between the equity return and the inflation rate is thought to be negative as most of the preceding studies suggest.

Industry by industry, we see the relationship between the equity return and the inflation rate. The elasticity \( \lambda^S \) of the excess growth rate to the inflation rate and the excess real growth rate except the inflation effect (\( \mu \)) for all the industries are provided in Fig. 2 and Fig. 3, respectively. From Fig. 2, we observe a negative correlation between the equity return and the inflation rate in 30 sectors out of 33. The sectors that have positive elasticity are Pulp & Paper (6), Electric Power & Gas (20) and Air Transportation (23). In the three sectors, the elasticity \( \lambda^S \) of Pulp & Paper is too small, so the equity return of the sector has almost no influence from the inflation rate. The reason for the positive elasticity in the Electric Power & Gas and Air Transportation sectors seems to be that these sectors are close to duopoly and it could be possible for them to put the increased cost by inflation on the price. On the other hand, the elasticity \( \lambda^S \) in the sectors like Glass & Ceramics Product (11) and Wholesale Trade (26) is a negative large value. Regarding Wholesale Trade (26) sector, in the very competitive market the increase in the cost due to the inflation seems to push down the sectors profit. In Fig. 3, we see the positive excess real growth rate except inflation in around half of the sectors. Especially, the growth rate of Iron & Steel (12) sector is relatively large positive value, while that of Other Financing Business (31) is large negative value.

(Result and implication 2) Regarding the ITM, the ATM and the OTM options for the underlying assets of 33 sectors of industrial indexes, the relationships between the elasticity \( \lambda^S \) and the option price ratio under the estimated parameters (\( X_o = -0.04\% \) and the maturity is 12 months) are dotted in Fig. 4 through 6, respectively. When the price ratio is above (below) 1, it indicates that our option model price is higher (lower) than the BS model price. In all of the three figures, the price ratio is slightly above 1 when the elasticity \( \lambda^S \) is around 0, while it becomes far below 1 when the elasticity \( \lambda^S \) is negative large. And the tendency becomes salient when the “moneyness” goes to out of the
money. The difference in the two option prices comes from the difference in the equity models. Our equity model is dependable on the inflation rate and the equity price under the hedge neutral measure is given by Eq. (8) that is different from the geometric Brownian motion under the risk neutral measure in the BS model. Option price is basically determined by the magnitude of both drift and diffusion parts of the equity model. Comparing the drift and the volatility of our model with those of the BS model, we find that the drift of our model becomes larger than the one of the BS model when the elasticity $\lambda^s$ goes negative large, while the volatility of our model becomes much smaller than the one of the BS model when the absolute value of the elasticity $\lambda^s$ becomes larger. Thus, when the elasticity $\lambda^s$ is negative large, the price ratio becomes below 1 due to the reason that the effect of relatively small volatility appears much stronger than the effect of the relatively large drift. The difference of our price from the BS price depending on the elasticity level may suggest an arbitrage opportunity, when the option market price is evaluated by the BS model. Thus, in the future, our model potentially provides the framework to generate the expected returns, the volatilities and the correlation matrix incorporating the arbitrage opportunity related to the evaluation in the option market.

(3) For the ten industries selected in section 3.2, setting the investment horizon to be 6 months and the average inflation rate scenario to be -1%, 0% and 1%, we provide the expected returns and the volatilities for the equities, the ITM, ATM and OTM call options in Table 1 through 4. Focusing on the expected returns, in the industries with the negative elasticity $\lambda^s$ such as Mining (2), Foods (4), Pharmaceutical (8), Rubber Products (10), Iron & Steel (12), Land Transportation (21), Information & Communication (25) and Services (33) the expected returns both for the equities and equity options become high in the negative target inflation rate, while in the industries with positive elasticity $\lambda^s$ such as Electric Power & Gas (20), Air Transportation (23) the results are
The reason is that the inflation rate dependent part of the excess growth rate for the industries with negative elasticity $\lambda^s$ takes positive value in the negative target inflation rate. From the comparison of the three kinds of options, the absolute value of the expected return becomes large when the moneyness goes to out of the money. The result is mainly explained by the fact that the initial option price (initial investment amount) is getting small when the moneyness goes to out of the money. The expected return for Mining (2) and Iron & Steel (12) are very large and for the two industries the elasticity $\lambda^s$ is negative and $\lambda^s$ positive large value. In the optimal portfolio, the weights of the two industries are thought to be large. Regarding the volatilities, as with the same as the expected return, for the industries with negative elasticity $\lambda^s$, the lower the inflation rate (negative inflation rate), the larger the volatilities. The level of the volatility is in the order of the equities, the ITM options, the ATM options and the OTM options from the smallest.

The correlation matrix among the assets is provided in Table 5. We only list the case of 0% average inflation rate, because the correlation matrices for the average inflation rates of -1 and 1% are almost the same as the one in Table 5. Focusing on the correlation between the equity and the equity option for an industry, the level of the correlation with the equity is in the order of ITM, ATM and OTM options from the highest. And the level of the correlation between the industries is in the order of the equities, ITM, ATM and OTM options from the highest. Due to the space constraint, the influence of the investment horizon

### Table 1 Expected return and volatility for the equity returns (6 months horizon)

<table>
<thead>
<tr>
<th></th>
<th>(2)</th>
<th>(4)</th>
<th>(8)</th>
<th>(10)</th>
<th>(12)</th>
<th>(20)</th>
<th>(21)</th>
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<th>(25)</th>
<th>(33)</th>
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<tr>
<td>Expected return</td>
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</tr>
<tr>
<td>1%</td>
<td>4.4%</td>
<td>1.4%</td>
<td>-0.2%</td>
<td>1.9%</td>
<td>6.2%</td>
<td>-1.0%</td>
<td>-0.8%</td>
<td>-4.2%</td>
<td>-1.8%</td>
<td>-2.4%</td>
</tr>
<tr>
<td>0%</td>
<td>3.3%</td>
<td>0.8%</td>
<td>-0.6%</td>
<td>1.3%</td>
<td>4.5%</td>
<td>-0.6%</td>
<td>-1.3%</td>
<td>-3.4%</td>
<td>-2.2%</td>
<td>-3.0%</td>
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<tr>
<td>1%</td>
<td>2.2%</td>
<td>0.2%</td>
<td>-0.9%</td>
<td>0.8%</td>
<td>2.8%</td>
<td>-0.2%</td>
<td>-1.7%</td>
<td>-2.6%</td>
<td>-2.5%</td>
<td>-3.7%</td>
</tr>
</tbody>
</table>

| Volatility | | | | | | | | | | |
| 1%    | 23.3%| 11.3%| 11.5%| 16.9%| 23.9%| 10.9%| 10.4%| 16.7%| 12.4%| 12.8%|
| 0%    | 23.1%| 11.3%| 11.5%| 16.8%| 23.6%| 11.0%| 10.4%| 16.9%| 12.3%| 12.7%|
| 1%    | 22.9%| 11.2%| 11.4%| 16.8%| 23.2%| 11.0%| 10.4%| 17.0%| 12.3%| 12.6%|

### Table 2 Expected return and volatility for the ITM call option returns (6 months horizon)

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| Volatility | | | | | | | | | | |
| 1%    | 88.6%| 79.8%| 74.7%| 82.4%| 92.0%| 71.1%| 72.1%| 70.3%| 69.5%| 70.1%|
| 0%    | 86.9%| 78.5%| 73.9%| 81.5%| 89.6%| 72.0%| 71.0%| 71.7%| 68.7%| 68.7%|
| 1%    | 85.3%| 77.2%| 73.1%| 80.5%| 87.1%| 72.9%| 70.0%| 73.1%| 68.0%| 67.3%|

### Table 3 Expected return and volatility for the ATM call option returns (6 months horizon)

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</table>

| Volatility | | | | | | | | | | |
| 1%    | 123.8%| 111.9%| 102.8%| 114.9%| 129.9%| 96.2%| 98.2%| 93.6%| 92.8%| 93.9%|
| 0%    | 120.4%| 109.1%| 101.1%| 112.9%| 125.0%| 98.1%| 96.0%| 96.4%| 91.4%| 91.1%|
| 1%    | 117.1%| 106.4%| 99.5% | 111.0%| 120.1%| 100.0%| 93.9%| 99.2%| 89.9%| 88.2%|

### Table 4 Expected return and volatility for the OTM call option returns (6 months horizon)

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<p>| Volatility | | | | | | | | | | |
| 1%    | 172.6%| 160.2%| 141.1%| 161.0%| 186.1%| 127.9%| 132.7%| 119.7%| 120.2%| 122.2%|
| 0%    | 166.3%| 154.7%| 137.9%| 157.3%| 176.6%| 131.4%| 128.7%| 124.4%| 117.7%| 117.3%|
| 1%    | 160.2%| 149.4%| 134.9%| 153.7%| 167.4%| 135.0%| 124.8%| 129.1%| 115.2%| 112.5%|</p>
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Table 5: Correlation matrix (6 months horizon)
on the correlation is not provided here, however, the result suggests that in all of the 9 kinds of correlation matrices except the correlation matrix among the equities, the longer the investment horizon, the lower the correlation. Thus, with the increase in the investment horizon, the risk reduction effect in the portfolio becomes salient and the efficient frontier should be shifted to upper left corner in the average inflation rates of -1%.

Remark 3
In the analysis of this section and the next section, when we adopt the average inflation rate scenarios as -1, 0 and 1%, we conveniently use the values of \( \alpha \) and \( \nu \) estimated from the data period that we estimate the value of \( \overline{X} \) to be -0.01%. Strictly, when we adopt the average inflation rate scenarios as -1, 0 and 1%, we should use the values of \( \alpha \) and \( \nu \) corresponding to the value of the average inflation rate. However, due to the lack of such kinds of empirical results, we conveniently adopt the same value mentioned above. In the future research, the values of \( \alpha \) and \( \nu \) had better be estimated from the long period of data such that the average inflation rate scenario is to be the same as the estimated average inflation rate \( \overline{X} \).

(Result and implication 4) The efficient frontiers for investment horizons of 1, 6 and 12 months are provided in Fig. 7 through Fig. 9 As we suggest in (Result and implication 3), in the case of 1 month investment horizon, the influence of the level of average inflation rate on the efficient frontier is almost nothing, however the efficient frontier of the average inflation rate of -1% becomes much superior in Sharpe ratio to that of the average inflation rate of 1% with the increase in the investment horizon to 6 and 12 months. The result comes from the negative elasticity (\( \lambda^S \)) of the excess growth rate to the inflation rate in most of the industries selected in section 3.2.

The Sharpe ratio and the weight of each asset in the tangent portfolio under each average inflation rate scenario for investment horizons of 1, 6 and 12 months are listed in Table 6 through 8. In any tables, the assets not included in the tangent portfolio are omitted. First of all, we would like to focus on the fact that the assets possibly included in the tangent portfolio are the equities such as Mining (2), Iron & Steel (12), Electric Power & Gas (20), the ITM options written on the underlying assets of Mining (2) and Iron & Steel (12) and the ATM option written on the underlying asset of Iron & Steel (12). Reviewing the elasticity (\( \lambda^S \)) of the excess growth rate to the inflation rate and the excess real growth rate except the inflation effect (\( \mu \)) for the three underlying equities from Fig. 2 and 3, we see that the elasticity (\( \lambda^S \)) is negative large and the excess real growth rate except the inflation effect (\( \mu \)) is positive large in the underlying equities of Mining (2) and Iron & Steel (12), while the elasticity (\( \lambda^S \)) is slightly positive and

\[
\text{Vol.64 No.2E (2013) 281}
\]
the excess real growth rate except the inflation effect (£μ) is slightly negative in the underlying equity of Electric Power & Gas (20).

In the case of 1 month investment horizon, because the inflation rate does not approach enough to the average inflation rate within the investment horizon, the assets selected to the tangent portfolio are generally the same except the difference in either the underlying asset or the option written on it and the selected underlying equities are Mining (2) and Iron & Steel (12). The options are included in the tangent portfolio to some extent due to the attractive risk return profile such as the relatively high expected returns with not so large volatilities in the short investment horizon in addition to the lower correlations with other assets. In the long investment horizon of 12 months, the assets selected under the average inflation rate of -1% are pretty different from those under the average inflation rate of 1%. When the investment horizon is long, due to the large influence of the elasticity (£ξ) on the excess growth rate of the equity, the positive elasticity is preferable in the average inflation rate of 1%. Thus, in the tangent portfolio under the average inflation rate of 1%, the underlying equity of Electric Power & Gas (20) is included up to 51%. From the above analyses, it could be said that the features of our model such that (1) the equity process is dependable on the inflation rate and (2) options are also included in the possible investment assets, work effectively in the construction of the optimal portfolio.

5 SUMMARY AND THE CONCLUDING REMARKS

Recently, portfolio managers have to optimize their portfolios taking the inflation rate into consideration, because many central banks have adopted the inflation targeting policy and the asset returns should be influenced by the target inflation rate. In such an investment environment, modeling the equity process to be able to depend on the inflation rate that follows the OU process, we derive the expected returns, the volatilities and the correlation matrix, which are key ingredients for the use of the mean-variance model. We also provide the option pricing formula based on our equity model under the hedge neutral probability measure and generate the future paths of not only the equity prices and the inflation rate but also the option prices. By way of the proposed model, it is possible to optimize the portfolio including the equity options under some average inflation rate scenarios.

The results of the empirical analyses are mainly following four points. First, regarding the elasticity of the excess growth rate of industry index to the inflation rate, it is mostly negative and only in three sectors out of 33 the elasticity is positive. Second, for the underlying industry indexes that have negative large elasticity, our option price is smaller than those of the BS model. Third, the expected returns either for the equities or for the equity options are much larger in the average inflation rate of -1% than in that of 1%. And the volatilities of the equity options are larger in the average inflation rate of -1% than in that of 1%. The expected returns either for the equities or for the equity options are much bigger than those of the equities. Lastly, the efficient frontier under the average inflation rate of -1% is almost the same as the one under the average inflation rate of 1% in the investment horizon of one month, however, the former is getting much more superior to the latter in the Sharpe ratio when the investment horizon becomes longer. The assets selected in the tangent portfolio seem to be appropriate considering their risk return profiles.

We believe our study shed some light on the method to utilize the mean-variance model in the era of the inflation targeting policies by central banks and the rapidly expanding financial market including derivatives.

ACKNOWLEDGEMENTS
We deeply thank two anonymous reviewers for their constructive comments.
This research is supported by KAKENHI (22510143).
References


Appendix
Appendix A) Solving Eq. (7), we get $X_t$ as
$$X_t = X_0 + \left( X_0 - X_t \right) e^{-\beta t} + \int_0^t e^{-\beta(t-s)} dw_s$$ (A1)
Substituting Eq. (A1) into Eq. (A2), which is the solution of Eq. (6) and computing the integral related the inflation rate, we attain Eq. (8).
$$S_t = S_0 \exp \left[ \left( r - \frac{\sigma^2}{2} \right) t + \lambda \int_0^T X_s ds + \sigma \int_0^T dw_s \right]$$ (A2)

Appendix B) Substituting Eq. (A1) and Eq. (8) into Eq. (9), we get
$$c_t = E^h \left[ \max \left\{ S_t \exp \left[ -\frac{\sigma^2}{2}(T-t) + \sigma \int_0^T dw_s \right] - K \exp \left[ -r(T-t) - \lambda \int_0^T X_s ds \right] \right\} \right]$$ (A3)
Putting $W^1 = S_t \exp \left[ -\frac{\sigma^2}{2}(T-t) + \sigma \int_0^T dw_s \right]$ and $W^2 = K \exp \left[ -r(T-t) - \lambda \int_0^T X_s ds \right]$ in Eq. (A3) and computing the expectation in (A3) with the use of convenient formula (A4), we attain Eq. (10).
$$c_t = E^h \left[ \max \left\{ W^1 - W^2, 0 \right\} \right] = E^h \left[ W^1 \right] \Phi(t_1) - E^h \left[ W^2 \right] \Phi(t_2)$$ (A4)
$$d_1 = \ln \left( E^h \left[ W^1 \right] / E^h \left[ W^2 \right] \right) + \text{var} \left[ \ln \left( W^1 / W^2 \right) \right] / 2 \sqrt{\text{var} \left[ \ln \left( W^1 / W^2 \right) \right]}$$
$$d_2 = d_1 - \sqrt{\text{var} \left[ \ln \left( W^1 / W^2 \right) \right]}$$