The Optimal Steering Calculation Method of Passing through Several Waypoints

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Summary

The authors made the analytical solution of ship trajectory by simple zig-zag maneuver with equal time-alternative steering in their previous study. It is found, in this study, that 1) Lateral deviation remains even after the helm returns to midship, 2) The bigger the T (index of stability on course and quickness in responding to steering) becomes, the longer the maximum lateral deviation is delayed. These facts indicate it is hard to predict ship trajectory even by the simple steering. It is more difficult, consequently, to search the steering of passing through some waypoints. This report describes how to compute the optimal steering for passing through several waypoints by linear theory with constant velocity. The steering function is expressed as Fourier sine series with coefficients which are fixed by calculus of variations under the restriction on passing through all waypoints. A frigate and a container, as examples, are analyzed along with the target course represented by three waypoints. The analysis shows that ship length and Nomoto's steering quality indices have strong influence on both capability of passing through waypoints and lateral deviation from the target course.

1. Introduction

In the previous study¹⁾, authors applied Nomoto's 1st order model²⁾ for maneuverability to the analysis of a ship response and trajectory, and developed analytical solutions for the simple zig-zag maneuver with equal time-alternative steering.

It is impressive, in the study, that lateral deviation remains even after the helm returns to midship. Moreover, the bigger the T' (index of stability on course and quickness in responding to steering) becomes, the longer the maximum lateral deviation is delayed. These facts indicate it is difficult to predict the trajectory even after the simple zig-zag steering as mentioned above. It must be more difficult, consequently, to find out the steering function of passing through several waypoints with continuous operation. At present, such a study is not found as far as authors know.

This paper describes how to calculate the optimal steering function of passing through several waypoints by linear theory with constant velocity. The steering function is expanded into Fourier sine series with undetermined coefficients. The coefficients are determined by calculus of variations to minimize the linear combination of the square angular velocity integration and square heading angle one with parameter ε . The ε means a weighting factor between angular velocity and deviation of a course.

The target course, as an example, represented by 3 waypoints is adopted from the sample course which is used in the study on ship-handling limits of unstable ships from the viewpoint of position control³⁾. The optimal steering computation is carried out with both a frigate⁴⁾ and a container ship⁵⁾ for research as examples. Their steering quality indices are obtained from the analysis of the model zig-zag experiments in the papers.

As the result of the calculation, it is cleared that the value of the parameter ε becomes the index of lateral deviation from the target course and also found that ship length and steering quality indices have the strong influence on both the capability of passing through

waypoints and the lateral deviation.

2. Mathematical formulation

Symbols with dimension are defined as follows; length: L (m), breadth: B (m), draft: d (m), velocity: V (m/s), maximum helm angle in model zig-zag experiment: δ_0 (deg.), neutral helm correction⁶⁾: δ_r (deg.), the 1st turn angle of a target course: θ (deg.).

Symbols nondimensionalized are shown as below; block coefficient: C_b , Froude number: F_n , imaginary unit: i , time: t , the maximum straight distance to be considered on x axis: x_∞ , the time when a ship reaches to x_∞ : t_∞ , steering function: $\delta(t)$ (port>0), heading angle: $\psi(t)$ (turning to port>0), the steering quality indices: K' (index of turning ability), T' , T'_1 , T'_2 , T'_3 (index of stability on course and quickness in responding to steering), trajectory: $P(t) = [x(t), y(t)]$, the m th waypoint: $P_m = (x_m, y_m)$, polygonal line representing target course: $\eta(x)$, lateral deviation from a target course: $R(x)$, undetermined coefficient of the n th Fourier sine series: a_n , the n th term's period: τ_n , the weighting factor linearly combining the integration of square heading angle with that of square angular velocity: ε . A quantity related to time is nondimensionalized by L/V , and one related to distance is done by L , and one related to angle is done by δ_0 . When the symbols are expressed with dimension, if it is a steering quality index, a mark (') is removed, and if else, a mark ($\sim$) is placed.

Let us consider a steering function is expanded into Fourier sine series with undetermined coefficients which are fixed by calculus of variations. Though some complex functions are used in order to simplify the analysis, the real part is taken in all the functions.

The domain of the problem is defined as follows,

$$0 \leq t < \infty. \quad (1)$$

Let us expand a steering function into Fourier sine series of N terms in total with unfixed coefficients a_n . Hence,

$$\delta(t) = -i \sum_{n=1}^N a_n e^{inpt} \quad , n = 1, 2, \dots, N \quad , p = \frac{\pi}{t_\infty}, \quad (2)$$

Here, t_∞ is the time when a ship reaches x_∞ . x_∞ is the maximum straight distance to be considered on x axis. $\delta(t)$ is clearly

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periodic and $2t_\infty$ is the period. The Laplace transformation is

$$\mathcal{L}(\delta) = \int_0^\infty \delta(t) e^{-st} dt. \quad (3)$$

Then, the integration can be carried out easily,

$$\mathcal{L}(\delta) = -i \sum_{n=1}^N \frac{a_n}{s - inp}. \quad (4)$$

First, let us express the 1st order quantities by some series. The basic equation²⁾ is

$$T' \frac{d\psi}{dt} + \psi = K' \delta. \quad (5)$$

The Laplace transform of ψ becomes

$$\mathcal{L}(\psi) = \frac{K'}{T's + 1} \mathcal{L}(\delta). \quad (6)$$

Hereafter, the suffix (j) in the right shoulder of functions and values represents the order of basic equations, that is, $j=1$ corresponds to the 1st order functions and values, and $j=2$ does the 2nd order ones. ψ is obtained by the inverse Laplace transform after substituting (4) into (6) and by using the function explained in Appendix A,

$$\psi(t) = K' \sum_{n=1}^N a_n O_n^{(1)}(t). \quad (7)$$

ψ is obtained by integrating (7) from 0 to t shown in Appendix A,

$$\psi(t) = K' \sum_{n=1}^N a_n \hat{O}_n^{(1)}(t). \quad (8)$$

Linearized trajectory¹⁾ is expressed as follows,

$$x(t) = t, y(t) = \omega \int_0^t \psi(u) du, \quad \omega = \delta_0 \pi / 180, \quad (9)$$

By using the function explained in Appendix A,

$$y(t) = K' \omega \sum_{n=1}^N a_n \hat{O}_n^{*(1)}(t). \quad (10)$$

Next, let us consider the 2nd order quantities. Basic equation²⁾ is

$$T_1' T_2' \frac{d^2 \psi}{dt^2} + (T_1' + T_2') \frac{d\psi}{dt} + \psi = K' \left(\delta + T_3' \frac{d\delta}{dt} \right). \quad (11)$$

The Laplace transform of ψ is

$$\mathcal{L}(\psi) = \frac{K'(1 + T_3's)}{T_1'T_2's^2 + (T_1' + T_2')s + 1} \mathcal{L}(\delta). \quad (12)$$

ψ is obtained by the inverse Laplace transform after substituting (4) into (12) and by using the function explained in Appendix B,

$$\psi(t) = K' \sum_{n=1}^N a_n O_n^{(2)}(t), \quad (13)$$

By integrating (13) as well as the 1st order functions,

$$\psi(t) = K' \sum_{n=1}^N a_n \hat{O}_n^{(2)}(t), \quad (14)$$

$$y(t) = K' \omega \sum_{n=1}^N a_n \hat{O}_n^{*(2)}(t). \quad (15)$$

We consider both (i) the integration of the square angular velocity and (ii) that of the square heading angle as the object for minimization. (i) means minimization of kinetic energy while a ship turns and (ii) means that of overshoot angle while a ship goes along the target course. If the number of waypoint is assumed M and the restriction on passing all waypoints is added,

$$E = (1 - \varepsilon) \int_0^{t_\infty} \dot{\psi}^2 dt + \varepsilon \int_0^{t_\infty} \psi^2 dt + \sum_{m=1}^M \lambda_m [y(t_m) - y_m]. \quad (16)$$

ε is the weighting factor linearly combining (i) and (ii), hence,

$$0 \leq \varepsilon \leq 1. \quad (17)$$

λ_m is the Lagrange multiplier corresponding to the m^{th} waypoint. The following integrations are newly introduced in order to express (16) by a series,

$$\gamma_{v\mu}^{(j)} = K'^2 \int_0^{t_\infty} O_v^{(j)}(t) O_\mu^{(j)}(t) dt, \quad (18)$$

$$\hat{\gamma}_{v\mu}^{(j)} = K'^2 \int_0^{t_\infty} \hat{O}_v^{(j)}(t) \hat{O}_\mu^{(j)}(t) dt. \quad (19)$$

The calculating method of these integrations is shown in Appendix C. Moreover, we introduce next values as follows,

$$\Gamma_{v\mu}^{(j)} = (1 - \varepsilon) \gamma_{v\mu}^{(j)} + \varepsilon \hat{\gamma}_{v\mu}^{(j)}, \quad (20)$$

$$I_{vm}^{(j)} = \omega K' \hat{O}_v^{*(j)}(t_m). \quad (21)$$

Then, (16) is represented with the series shown as below,

$$E = \sum_{v=1}^N \sum_{\mu=1}^N a_v a_\mu \Gamma_{v\mu}^{(j)} + \sum_{m=1}^M \lambda_m \left[\sum_{v=1}^N a_v I_{vm}^{(j)} - y_m \right]. \quad (22)$$

Finally, we can obtain the simultaneous equation to be solved by differentiating E with respect to a_v and λ_m ,

$$\left. \begin{aligned} \frac{\partial E}{\partial a_v} &= 2 \sum_{\mu=1}^N a_\mu \Gamma_{v\mu}^{(j)} + \sum_{m=1}^M I_{vm}^{(j)} \lambda_m = 0, \\ \frac{\partial E}{\partial \lambda_m} &= \sum_{v=1}^N a_v I_{vm}^{(j)} - y_m = 0. \end{aligned} \right\} \quad (23)$$

Although this calculation is under the restriction of both linear theory and constant velocity, it seems to be effective for giving an overview of the steering function and the response of a ship.

3. The target course

The target course is adopted, as an example, from the course used in the study on ship-handling limits of unstable ships from the viewpoint of position control³⁾. The outline is shown in Fig. 1.

Fig.1 The target course.

Three waypoints are placed in two corners (P_1 and P_2) and course end (P_3). $\bar{R}$ in Fig.1 represents ship's lateral deviation from a target course in the direction of the breadth shown as following equation.

$$\bar{R}(\tilde{x}) = \begin{cases} \bar{\eta}(\tilde{x}) & , 0 \leq \tilde{x} \leq \tilde{x}_1, \\ [\bar{\eta}(\tilde{x}) - \bar{y}_1] \cos \theta - (\tilde{x} - \tilde{x}_1) \sin \theta, & \tilde{x}_1 \leq \tilde{x} \leq \tilde{x}_2, \\ \bar{\eta}(\tilde{x}) - \bar{y}_2 & , \tilde{x}_2 \leq \tilde{x} \leq \tilde{x}_3. \end{cases} \quad (24)$$

If $\bar{R} > 0$, a ship deviates to port side. Here, θ is the turn angle of the target course at P_1 .

4. Ships to be analyzed

We selected two ships as the analysis examples, that is, Italian research frigate⁴⁾ and Japanese research container ship, SR108⁵⁾. The steering quality indices are obtained from the experiments in the papers with the minimal residue method⁷⁾. The outline of analysis is shown in Fig.2, and the principal dimensions of the ships and the obtained indices are tabulated in Table 1.

Fig.2 Outline of zig-zag experiments.

Table 1 Principal dimension and steering quality indices.

Ship	kind		Container		Frigate			
	name		Model	SR108	Model	IF1	IF2	IF3
Principal Dimensions	L	m	3.5	175	7.2	100	142	184
	B	m	0.508	25.4	0.96	13.3	18.9	24.5
	d	m	0.19	9.5	0.30	4.17	5.92	7.67
	C_b		0.573		0.50			
	$Ob/(L/B)$		0.083		0.067			
	F_n		0.15		0.25			
	V	m/s	0.88	6.21	2.10	7.83	9.33	10.6
	L/V	s	3.98	28.17	3.43	12.78	15.23	17.33
	δ_0	deg.	10	—	20	—	—	—
	1st order	K	1/s	0.27	0.038	0.26	0.070	0.059
T		s	14.2	100.4	2.41	8.98	10.70	12.18
K'			1.08		0.89			
T'			3.56		0.70			
δ_r		deg.	0.16	—	-0.015	—	—	—
K		1/s	0.33	0.047	0.29	0.078	0.065	0.057
2nd order	$T1$	s	20.8	147.1	7.95	29.6	35.3	28.2
	$T2$	s	0.34	2.40	1.58	5.89	7.02	5.61
	$T3$	s	2.04	14.4	6.00	22.4	26.6	21.3
	K'		1.32		0.99			
	$T1'$		5.22		2.32			
	$T2'$		0.085		0.46			
	$T3'$		0.51		1.75			

The steering quality indices of actual ships are converted from those of models by L/V . The actual size frigates, that is, IF1, IF2 and IF3 have different ship length supposed by us.

5. Analytical accuracy and the order of equations

In order to confirm the influence of the equations' order on the analytical accuracy, the steering functions of Frigate model and Container model are calculated with both order's functions under the condition of $\varepsilon = 0$. In the case, the optimization is carried out only concerning angular velocity in turning. The series-coefficients of Container model is shown in Fig.3 and the steering function is shown in Fig.4.

Fig.3 Comparison of Container model's series coefficients between the 1st order and the 2nd one.

Fig.4 Comparison of Container model's steering functions between the 1st order and the 2nd one.

In Fig.3, $|a_n|$ of the 1st order becomes larger as τ_n gets closer to the short period side, on the other hand, that of the 2nd order keeps low level even in the shortest period. The period of the n^{th} term is expressed as follows,

$$\tau_n = \frac{2t_\infty}{n}. \quad (25)$$

The period, from this equation, becomes shorter as n grows larger. The shortest period becomes 0.35 in $n=200$. Comparing between τ_n and T' , τ_n is less than T' in almost $n \geq 20$. This is inconsistent with the requirement of the 1st order approximation to have to be composed of long period components. This influence is found remarkably in the steering functions. The 1st order function is two or three times as large as the 2nd one around origin shown in Fig.4.

The same computation of Frigate is carried out for the purpose of the comparison with Container. Most of τ_n are larger than T' shown in Fig.5 (See next page.). It seems to show better agreement with the requirement of the 1st order approximation than Container.

The steering functions of both orders, accordingly, show good agreement with each other as shown in Fig.6.

Fig.5 Comparison of Frigate model's series coefficients between the 1st order and the 2nd one.

Fig.6 Comparison of Frigate model's steering functions between 1st order and 2nd one.

When the ship having large T' is analyzed, the accuracy seems not to rise without the 2nd order analysis in this way. The results, hereafter, are all shown from the 2nd order analysis.

6. Comparison between SR108 and IF1($\varepsilon = 0, 1$)

First, let us compare SR108 and IF1 with dimension when $\varepsilon = 0$. The comparison of steering functions is shown in Fig.7.

Fig.7 Comparison of steering functions between SR108 and IF1 when $\varepsilon = 0$.

The helm angle of SR108 is large, but not so considerably. It is noticeable that both functions have large value in initial action. SR108 has large helm angle of around 38 degrees but IF1 has small

one of about 7 degrees. It seems to be caused by the difference of ship length and T' . Although both ships are able to pass through the same waypoints described in Fig.9, IF1 requires only small helm angle. Judging from this fact, IF1 seems to have better capability to pass through the waypoints than SR108.

The comparison of series coefficients is shown in Fig.8.

Fig.8 Comparison of series coefficients between SR108 and IF1 when $\varepsilon = 0$.

The coefficients of SR108 are generally larger than those of IF1. This is the reason why SR108's helm angle grows larger than IF1. The other quantities induced by the steering functions in Fig.7 are plotted to the graph in Fig.9.

Fig.9 comparison of $\tilde{y}$, $\tilde{\psi}$, $\tilde{y}$, $\tilde{R}$ between SR108 and IF1 when $\varepsilon = 0$.

The Heading angle and the trajectory of SR108 match very well with those of IF1. The angular velocity with dimension, in general, is in proportion to velocity and helm angle, and in inverse proportion to ship length. This is why there is the slight difference in $\tilde{\psi}$ curves. The $\tilde{\psi}$ curves are almost straight lines between adjacent waypoints. It agrees with the consideration in Appendix E and also shows the ships turn with constant angular acceleration when $\varepsilon = 0$.

The trajectory curves show the sigmoid shape having one inflection point and the ships deviate in the direction of portside nearly 200m maximum between P_2 and P_3 . When $\varepsilon = 0$, it is impressive that the trajectory and the heading angle are determined by the arrangement of waypoints.

Next, we carry out the calculation when $\varepsilon = 1$. In the case, the optimization is carried out only concerning deviation of a course. The comparison of both ships' $|a_n|$ is shown in Fig.10 (See next page.). The coefficients grow large as they approach short period side. It is considerably different from the case of $\varepsilon=0$ that the shorter the period is the larger the amplitude becomes. It seems to

be because $\hat{O}_n^{(j)}$ varies more widely than $O_n^{(j)}$ by $1/np$.

This property has great influence on the steering functions. The curve of the steering function is shown in Fig.11.

Fig.10 Comparison of series coefficients between SR108 and IF1 when $\varepsilon = 1$.

Fig.11 Comparison of steering functions between SR108 and IF1 when $\varepsilon = 1$.

As shown in Fig.11, the amplitude of short period terms is unthinkable large, especially in the keen peaks taking place at P_1 and P_2 . Though the steering functions do not have possibility at all for actual maneuver, they cause the interesting property in the heading angle and the trajectory. These are plotted to Fig.12.

Fig.12 Comparison of $\tilde{\psi}$ and $\tilde{y}$ between SR108 and IF1 when $\varepsilon = 1$.

Both $\tilde{y}$ and $\tilde{\psi}$ lines of SR108 match well with those of IF1 though both ships have different indices and ship lengths. It is interesting that the trajectory matches almost with the target course. The curves of heading angle become nearly rectangular and there is little overshoot in the curves. The heading angle from P_1 to P_2 is around 10% larger than the turn angle of the course at P_1 . This is caused by the linearization of trajectory defined by eq. (9). When $\varepsilon=1$, the steering function does not have reality for actual maneuver but it makes the lateral deviation almost 0. The trajectory and the heading angle, namely, are also determined by the arrangement of waypoints.

7. Calculation by the change of ship length

As mentioned before, the magnitude of the steering function is in a trade-off relation with the lateral deviation. When $\varepsilon \rightarrow 0$, the steering function becomes small but the lateral deviation becomes large. When $\varepsilon \rightarrow 1$, they have reverse relations.

In this chapter, we discuss how trajectory changes by the change of ship length. The ships to be analyzed are IF1, IF2 and IF3. ε is determined by the convergent calculation under the restriction that the helm angle does not exceed 20 degrees. We evaluate it is steerable from safety point of view.

As the result of the convergent calculation, ε becomes 0.36 in IF1, 0.31 in IF2, 0.0 in IF3. The dimensionless distance of the target course gets short according to dividing it by long ship length. This is why the longer length the ship has, the smaller ε becomes. Hence, IF3 has the largest lateral deviation in the three ships as described later. The comparison of $|\tilde{a}_n|$ is shown in Fig.13.

Fig.13 Series coefficients with the change of ship length.

The values in long period side are generally large but IF1 and IF2 are also large in the range of $n=10-20$. The term-number having the large amplitude changes according to ship length in spite of the same dimensionless steering quality indices. The steering functions and the angular velocities are shown in Fig.14 and Fig.15.

Fig.14 The steering functions with the change of ship length.

Fig.15 The angular velocities with the change of ship length.

The maximum helm angle takes place in the initial action of IF3 shown in Fig.14. IF3's maximum helm angle already exceeds 20 degrees before convergent calculation, therefore, ε becomes 0. The angular velocity of IF3 in initial action is about twice as large as IF1 shown in Fig.15. The partial course keeping the helm midship becomes longer as ε grows larger. The angular velocity in this course, accordingly, becomes nearly 0. The angular velocity, moreover, has the keen peaks at P_1 and P_2 as ε grows larger. To evaluate the size of the peaks, the model's initial angular velocity is estimated by averaging from the origin to the 1st maximum heading angle using the experiment curve shown in Fig.2. We convert it into IF1, IF2 and IF3 with dimension. These averaged angular velocities are written in Fig.15. The peaks of the ships are almost equal respectively to them.

The comparison of heading angles is shown in Fig.16.

Fig.16 The heading angles with the change of ship length.

It is found that the heading angle gets closer to the turn angle at P_1 as ε becomes larger. The overshoot angle of IF3 gets large to around 10 degrees in the partial course between P_1 and P_2 because $\varepsilon = 0$. We think the large overshoot angle like this is not permitted by crews when they must pass the course having comparatively narrow width which is almost same as ship length³⁾.

The outline of trajectories is shown in Fig.17.

Fig.17 Trajectories and lateral deviations with the change of ship length.

IF1 and IF2 go nearly along the target course but they deviate laterally 15m-20m in the partial course from P_2 to P_3 . This deviation is almost equal to ship breadth. IF3 deviates laterally very long distance of 200m in the same partial course. This is almost equal to the ship length. In this way, the ability of passing, which means the ability of passing through waypoints with minimum lateral deviation, gets worse by lengthening ship even though the ships have the same steering quality indices nondimensionalized. In order not to get the ability down, it seems to be necessary, in initial design, to take following measures for improving indices⁸⁾

without dimension.

- extending skeg or/and reshaping stern frame line
- increasing rudder area or adopting high-lift rudder

In the actual ship handling, the operation lowering ship speed seems to be considered, but with a view to discussing it, it seems to be important to study the steering quality indices at low Froude number by zig-zag experiment and so on.

8. Conclusions

As the result of the present study, following points are concluded.

- (1) By expanding the steering function into Fourier sine series with unfixed coefficients, we can solve the problem of the variational approach which consists of the linear combination connecting the integration of the square angular velocity and that of square heading angle with the weighting factor ε , and which has the restriction on passing thorough all waypoints.
- (2) It is necessary to use the 2nd order functions in order to analyze the ship having large T' .
- (3) When ε is 0 or 1, the curve shapes of trajectory and heading angle are determined by the arrangement of waypoints in the condition the sine series are 200 terms in total.
- (4) When ε approaches 1, the lateral deviation becomes small but the helm angle grows large. when ε approaches 0, adversely, they show an opposite trend. It will be a next work to reveal the extent to which ε grows according to the steering quality indices.
- (5) When we make the restriction of the steerable helm angle from the safety point of view, the parameter ε is obtained by the convergence calculation. Then, $\dot{\psi}$, ψ , y , δ can be calculated from the series coefficients fixed by the obtained ε .
- (6) The ability of passing through waypoints gets worse if a ship is lengthened, even though the steering quality indices nondimensionalized do not change.

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Appendix A $O_n^{(1)}, \hat{O}_n^{(1)}, \hat{O}_n^{*(1)}$ function

The inverse Laplace transformation of (6) is

$$\psi(t) = \frac{1}{2\pi i} \int_{\beta-\infty i}^{\beta+\infty i} \mathcal{L}(\psi) e^{qt} dq. \quad (A1)$$

Here, β is arbitrary positive value larger than the poles on the real axis. Let us suppose that ψ is expressed as (7). Hence,

$$O_n^{(1)}(t) = \frac{-1}{2\pi} \int_{\beta-\infty i}^{\beta+\infty i} \frac{e^{qt} dq}{(T'q + 1)(q - inp)}. \quad (A2)$$

The integration is obtained easily from the sum of residues.

$$O_n^{(1)}(t) = \frac{e^{inpt} - e^{-\frac{t}{T'}}}{i - npT'} \xrightarrow{t \rightarrow 0} 0. \quad (A3)$$

Then, the new function is defined shown as below,

$$\hat{O}_n^{(1)}(t) = \int_0^t O_n^{(1)}(u) du. \quad (A4)$$

By carrying out the integration of (A3),

$$\hat{O}_n^{(1)}(t) = \frac{1}{i - npT'} \left[\frac{e^{inpt} - 1}{inp} + T' \left(e^{-\frac{t}{T'}} - 1 \right) \right]. \quad (A5)$$

Moreover, the new function is defined again as below,

$$\hat{O}_n^{*(1)}(t) = \int_0^t \hat{O}_n^{(1)}(u) du. \quad (A6)$$

By carrying out the integration of (A5),

$$\hat{O}_n^{*(1)}(t) = \frac{1 + inpt - e^{inpt}}{(np)^2(i - npT')} + \frac{T' [T'(1 - e^{-t/T'}) - t]}{i - npT'}. \quad (A7)$$

Appendix B $O_n^{(2)}, \hat{O}_n^{(2)}, \hat{O}_n^{*(2)}$ function

Let us consider the 2nd order functions referring to Appendix A.

$$O_n^{(2)}(t) = \frac{-1}{2\pi} \int_{\beta-\infty i}^{\beta+\infty i} \frac{(T'_3q + 1)e^{qt} dq}{(T'_1q + 1)(T'_2q + 1)(q - inp)}. \quad (B1)$$

The integration is obtained from the sum of residues.

$$O_n^{(2)}(t) = \frac{-i(1 + inpT'_3)e^{inpt}}{(1 + inpT'_1)(1 + inpT'_2)} + \frac{i(T'_1 - T'_3)e^{-t/T'_1}}{(T'_1 - T'_2)(1 + inpT'_1)} - \frac{i(T'_2 - T'_3)e^{-t/T'_2}}{(T'_1 - T'_2)(1 + inpT'_2)} \xrightarrow{t \rightarrow 0} 0. \quad (B2)$$

With reference to (A4), (B2) is integrated with respect to t .

$$\hat{O}_n^{(2)}(t) = \frac{-(1 + inpT'_3)(e^{inpt} - 1)}{np(1 + inpT'_1)(1 + inpT'_2)}$$

$$- \frac{iT'_1(T'_1 - T'_3)(e^{-t/T'_1} - 1)}{(T'_1 - T'_2)(1 + inpT'_1)} + \frac{iT'_2(T'_2 - T'_3)(e^{-t/T'_2} - 1)}{(T'_1 - T'_2)(1 + inpT'_2)}. \quad (B3)$$

With reference to (A6), (B3) is integrated again with respect to t ,

$$\hat{O}_n^{*(2)}(t) = \frac{i(1 + inpT'_3)(e^{inpt} - inpt - 1)}{(np)^2(1 + inpT'_1)(1 + inpT'_2)} + \frac{iT'_1(T'_1 - T'_3)(T'_1 e^{-t/T'_1} + t - T'_1)}{(T'_1 - T'_2)(1 + inpT'_1)} - \frac{iT'_2(T'_2 - T'_3)(T'_2 e^{-t/T'_2} + t - T'_2)}{(T'_1 - T'_2)(1 + inpT'_2)}. \quad (B4)$$

Appendix C The integration, $\gamma_{v\mu}^{(f)}, \hat{\gamma}_{v\mu}^{(f)}$

First, let us consider the 1st order integration. Four functions are introduced as follows.

$$\left. \begin{aligned} F_1(v, t) &= 1, \quad F_2(v, t) = \cos vpt, \\ F_3(v, t) &= \sin vpt, \quad F_4(v, t) = e^{-t/T'} \end{aligned} \right\} \quad (C1)$$

Moreover, the new coefficients are defined as follows,

$$\left. \begin{aligned} c_1(v) &= 0, \\ c_2(v) &= -c_4(v) = -vpT'c_3(v) = \frac{-vpT'}{1 + (vpT')^2} \end{aligned} \right\} \quad (C2)$$

The real part of $O_n^{(1)}$ is represented by (C1), (C2) as follows,

$$O_v^{(1)}(t) = \sum_{q=1}^4 c_q(v) F_q(v, t). \quad (C3)$$

Furthermore, the new integration are defined as below,

$$J_{QU}(v, \mu) = \int_0^{t_\infty} F_q(v, t) F_u(\mu, t) dt. \quad (C4)$$

Then, $\gamma_{v\mu}^{(1)}$ is expressed as follows using J_{QU} ,

$$\gamma_{v\mu}^{(1)} = K'^2 \sum_{q=1}^4 \sum_{u=1}^4 c_q(v) c_u(\mu) J_{QU}(v, \mu). \quad (C5)$$

The calculation method of J_{QU} is described later in this appendix.

New coefficients are introduced by integrating (C3),

$$\left. \begin{aligned} \hat{c}_1(v) &= \frac{c_3(v)}{vp} + T'c_4(v), \quad \hat{c}_2(v) = -\frac{c_3(v)}{vp}, \\ \hat{c}_3(v) &= \frac{c_2(v)}{vp}, \quad \hat{c}_4(v) = -T'c_4(v) \end{aligned} \right\} \quad (C6)$$

Accordingly, the real part of $\hat{O}_v^{(1)}$ is expressed as follows,

$$\hat{O}_v^{(1)}(t) = \sum_{q=1}^4 \hat{c}_q(v) F_q(v, t). \quad (C7)$$

It is very useful to be able to compute with F_q in the same way as (C3). Then, $\hat{\gamma}_{v\mu}^{(1)}$ becomes

$$\hat{\gamma}_{v\mu}^{(1)} = K'^2 \sum_{q=1}^4 \sum_{u=1}^4 \hat{c}_q(v) \hat{c}_u(\mu) J_{QU}(v, \mu). \quad (C8)$$

J_{QU} is computed by following equations using the integrations in Appendix D,

$$\left. \begin{aligned} J_{11}(v, \mu) &= t_\infty, J_{12}(v, \mu) = \frac{\sin \mu p t_\infty}{\mu p}, J_{13}(v, \mu) = \frac{1 - \cos \mu p t_\infty}{\mu p}, \\ J_{14}(v, \mu) &= T' \left(1 - e^{-\frac{t_\infty}{T'}} \right), J_{22}(v, \mu) = f_{cc}(t_\infty, vp, \mu p), \end{aligned} \right\} \quad (C9)$$

$$\left. \begin{aligned} J_{23}(v, \mu) &= f_{cs}(t_{\infty}, vp, \mu p), J_{24}(v, \mu) = f_{ec}\left(t_{\infty}, \frac{1}{T'}, vp\right), \\ J_{33}(v, \mu) &= f_{ss}(t_{\infty}, vp, \mu p), J_{34}(v, \mu) = f_{es}\left(t_{\infty}, \frac{1}{T'}, vp\right), \\ J_{44}(v, \mu) &= \frac{T'}{2}\left(1 - e^{-\frac{2t_{\infty}}{T'}}\right), \text{generally, } J_{QU}(v, \mu) = J_{UQ}(\mu, v). \end{aligned} \right\} \quad (C10)$$

Second, let us consider the 2nd order functions. New coefficients are defined as follows,

$$\left. \begin{aligned} N_1(v) &= vpT'_1, N_2(v) = vpT'_2, N_3(v) = vpT'_3, \\ T_{12} &= T'_1 - T'_2, T_{13} = T'_1 - T'_3, T_{23} = T'_2 - T'_3, \end{aligned} \right\} \quad (C10)$$

$$\left. \begin{aligned} d_1(v) &= 0, \\ d_2(v) &= \frac{N_3(v) - N_1(v) - N_2(v) - N_1(v)N_2(v)N_3(v)}{[1 + N_1^2(v)][1 + N_2^2(v)]}, \\ d_3(v) &= \frac{1 - N_1(v)N_2(v) + N_3(v)[N_1(v) + N_2(v)]}{[1 + N_1^2(v)][1 + N_2^2(v)]}, \\ d_4(v) &= \frac{T_{13}N_1(v)}{T_{12}[1 + N_1^2(v)]}, d_5(v) = \frac{-T_{23}N_2(v)}{T_{12}[1 + N_2^2(v)]}. \end{aligned} \right\} \quad (C11)$$

Moreover, the five functions are introduced as below,

$$\left. \begin{aligned} G_1(v, t) &= 1, G_2(v, t) = \cos vpt, G_3(v, t) = \sin vpt, \\ G_4(v, t) &= e^{-\frac{t}{T'_1}}, G_5(v, t) = e^{-\frac{t}{T'_2}}. \end{aligned} \right\} \quad (C12)$$

Then, the real part of $O_v^{(2)}$ is expressed as follows,

$$O_v^{(2)}(v) = \sum_{Q=1}^5 d_Q(v)G_Q(v, t). \quad (C13)$$

The new integration is introduced with reference to (C4),

$$\hat{J}_{QU}(v, \mu) = \int_0^{t_{\infty}} G_Q(v, t)G_U(\mu, t)dt. \quad (C14)$$

$\gamma_{v\mu}^{(2)}$ is, accordingly, expressed as follows,

$$\gamma_{v\mu}^{(2)} = K'^2 \sum_{Q=1}^5 \sum_{U=1}^5 d_Q(v)d_U(\mu)\hat{J}_{QU}(v, \mu). \quad (C15)$$

The new coefficients are defined by integrating (C13),

$$\left. \begin{aligned} \hat{d}_1(v) &= \frac{d_3(v)}{vp} + T'_1 d_4(v) + T'_2 d_5(v), \\ \hat{d}_2(v) &= -\frac{d_3(v)}{vp}, \hat{d}_3(v) = \frac{d_2(v)}{vp}, \\ \hat{d}_4(v) &= -T'_1 d_4(v), \hat{d}_5(v) = -T'_2 d_5(v). \end{aligned} \right\} \quad (C16)$$

The real part of $\hat{O}_v^{(2)}$ is represented by G_Q with reference to (C7),

$$\hat{O}_v^{(2)}(t) = \sum_{Q=1}^5 \hat{d}_Q(v)G_Q(v, t). \quad (C17)$$

$\hat{\gamma}_{v\mu}^{(2)}$ is obtained in the same way as the 1st order integration,

$$\hat{\gamma}_{v\mu}^{(2)} = K'^2 \sum_{Q=1}^5 \sum_{U=1}^5 \hat{d}_Q(v)\hat{d}_U(\mu)\hat{J}_{QU}(v, \mu). \quad (C18)$$

We can use $J_{QU} = \hat{J}_{QU}$ in the range of $1 \leq Q, U \leq 4$, if we suppose $T' = T'_1$. The others are as follows,

$$\left. \begin{aligned} \hat{J}_{15}(v, \mu) &= T'_2\left(1 - e^{-\frac{t_{\infty}}{T'_2}}\right), \hat{J}_{25}(v, \mu) = f_{ec}\left(t_{\infty}, \frac{1}{T'_2}, vp\right), \\ \hat{J}_{35}(v, \mu) &= f_{es}\left(t_{\infty}, \frac{1}{T'_2}, vp\right), \hat{J}_{45}(v, \mu) = \frac{T'_1 T'_2}{T'_1 + T'_2}\left(1 - e^{-\frac{T'_1 + T'_2}{T'_1 T'_2} t_{\infty}}\right), \end{aligned} \right\}$$

$$\hat{J}_{55}(v, \mu) = \frac{T'_2}{2}\left(1 - e^{-\frac{2t_{\infty}}{T'_2}}\right), \text{generally, } \hat{J}_{QU}(v, \mu) = \hat{J}_{UQ}(\mu, v). \quad (C19)$$

Appendix D The integrations using in $\gamma_{v\mu}^{(j)}, \hat{\gamma}_{v\mu}^{(j)}$

Let us define new symbols as follows,

$$p = a + b, q = a - b, \sin x = S\bar{x}, \cos x = C\bar{x}. \quad (D1)$$

The integrations using in $\gamma_{v\mu}^{(j)}$ and $\hat{\gamma}_{v\mu}^{(j)}$ are

$$f_{ss}(\tau, a, b) = \int_0^{\tau} S\bar{a}uS\bar{b}udu = \frac{1}{2}\left[\frac{S\bar{q}\tau}{q} - \frac{S\bar{p}\tau}{p}\right], \quad (D2)$$

$$f_{cc}(\tau, a, b) = \int_0^{\tau} C\bar{a}uC\bar{b}udu = \frac{1}{2}\left[\frac{S\bar{q}\tau}{q} + \frac{S\bar{p}\tau}{p}\right], \quad (D3)$$

$$f_{cs}(\tau, a, b) = \int_0^{\tau} C\bar{a}uS\bar{b}udu = \frac{1}{2}\left[\frac{1 - C\bar{p}\tau}{p} - \frac{1 - C\bar{q}\tau}{q}\right]. \quad (D4)$$

Here, we must pay attention to following limiting values.

$$S\bar{q}\tau/q \xrightarrow{q \rightarrow 0} \tau, (1 - C\bar{q}\tau)/q \xrightarrow{q \rightarrow 0} 0. \quad (D5)$$

Moreover, we have another integrations,

$$f_{ec}(\tau, a, b) = \int_0^{\tau} e^{-au}C\bar{b}udu = \frac{a(1 - e^{-a\tau}C\bar{b}\tau) + be^{-a\tau}S\bar{b}\tau}{a^2 + b^2}, \quad (D6)$$

$$f_{es}(\tau, a, b) = \int_0^{\tau} e^{-au}S\bar{b}udu = \frac{b(1 - e^{-a\tau}C\bar{b}\tau) - ae^{-a\tau}S\bar{b}\tau}{a^2 + b^2}. \quad (D7)$$

Appendix E A consideration of ψ curve when $\varepsilon = 0$

Let us regard a ψ curve as an exponential function with the base $t - t_0$ keeping heading angle constant between adjacent two points which are placed in the short time Δt apart.

$$\psi(t) = \frac{\xi + 1}{\Delta t^{\xi+1}}(\psi_0 - \dot{\psi}_0 \Delta t)(t - t_0)^{\xi} + \dot{\psi}_0, \quad t_0 \leq t \leq t_0 + \Delta t. \quad (E1)$$

Here, ψ_0 is the integration of $\dot{\psi}$ from $t=t_0$ to $t_0 + \Delta t$, that is, the heading angle from $t=t_0$ to $t_0 + \Delta t$. $\dot{\psi}_0$ is the angular velocity at $t = t_0$. (E1) clearly satisfies

$$\dot{\psi}(t_0) = \dot{\psi}_0, \quad \int_{t_0}^{t_0 + \Delta t} \dot{\psi}(t)dt = \psi_0. \quad (E2)$$

According to the differentiation of (E1) with respect to t , the angular acceleration becomes

$$\ddot{\psi}(t) = \frac{\xi(\xi + 1)}{\Delta t^{\xi+1}}(\psi_0 - \dot{\psi}_0 \Delta t)(t - t_0)^{\xi-1}. \quad (E3)$$

If we suppose $\xi < 1$, $\ddot{\psi}$ diverges to infinity when $t \rightarrow t_0$. The turning motion must have finite angular acceleration in the initial response, hence, ξ must satisfy the following condition,

$$\xi \geq 1. \quad (E4)$$

The integration of the square angular velocity is

$$E = \int_{t_0}^{t_0 + \Delta t} \dot{\psi}^2(t)dt = \frac{(\xi + 1)^2}{2\xi + 1} \frac{(\psi_0 - \dot{\psi}_0 \Delta t)^2}{\Delta t} + \dot{\psi}_0(2\psi_0 - \dot{\psi}_0 \Delta t). \quad (E5)$$

Since E is clearly monotonic increase with respect to ξ , the shape giving the minimal E has $\xi=1$, accordingly, it is a straight line.