ON THE BUCKLING OF A LONG ELASTIC PLATE UNDER EDGE THRUSTS. *

By Katsutada Sezawa, Member, and Genrokuro Nishimura.

Abstract.

The buckling of a long elastic plate which is clamped or supported at the edges and subjected to the boundary thrusts is studied in a similar manner as that which one of the authors employed in a paper read in 1925. The solution of the problem of the critical stability is deduced from the differential equation of a plane plate and the existence of the solution is critically examined. The relation between $P_1b^2/4D$ and $k^2$ for different values of $P_2/P_1$ is shown in diagrams, where $P_1$ and $P_2$ are the thrusts in $x$- and $y$-directions respectively, $2b$ is the breadth of the plate, and $D=\frac{2Eh^3}{(1-\nu^2)}$ in which $2h$ is the thickness of the plate and $E$ and $\nu$ give Young's modulus and Poisson's ratio respectively. The distribution of the corrugated deflection of the plate is also illustrated. To compare the result of the mathematical calculation with some experimental data of a cylindrical shell, a modified differential equation and its solution in which the effect of the curvature is involved are devised.

From the results of the mathematical calculation it is confirmed that the critical thrust acting in one direction takes different values by adjusting the magnitude of the thrust acting transversely. The corrugation of the buckled plate has a uniform pitch depending upon the ratio of the thrust $P_2/P_1$ and edge conditions of that plate. The contour curves of any bulge of the corrugation take different forms in accordance with the conditions at the edges. The authors find a good agreement between the result of calculation and that of Viscount Tokugawa's experiment and they conclude from the comparison of these results that the criterion of stability applicable to the experiments is too complex to be defined in any simple manner.

1. A few years ago one¹ of us studied some problems of elastic stability of thin plates and it was confirmed from the result of a certain example of calculation that, when a long rectangular plate supported or fixed at four edges is subject to edge thrusts acting in and parallel to the plane of a plate, the curve of deflected surface at the critical loading is of a corrugated form and also the magnitude of that critical load depends upon the breadth of the plate besides its thickness and elasticity but not on the length of the plate. In spite of its importance in connection with naval and aeronautical engineering, the problem has not attracted the attention of the investigators. Professor Wagner² in Charlottenburg seems to be only one who published a paper which gives a similar result as that above mentioned. Lately it

* The calculation contained in this paper has been mainly carried out by G. Nishimura. (K. Sezawa.)
occurred us that the problem is rather significant in relation to the buckling of a plate between transverse frames of a submarine hull due to hydrostatic external pressure. Although extensive researches of a submarine hull have already been carried out by Dr. Viscount Tokugawa 3) and others, yet we think that their mathematical analysis is not complete. Because the boundary conditions at the frames, i.e. the clamped or semi-clamped states at these boundaries cannot be obtained à priori in the analysis in cylindrical coordinates. Viscount Tokugawa, however, appeared to aim at the model experiments of the submarine hull as will be seen from his brilliant report; and, indeed, all his experimental results have become good guidances to our mathematical calculation.

Even though in the previous paper 4) which was published by one of us the buckling of a long plate in the corrugated form is dealt with, the solution is approximate in a theoretical sense and hence we must acquire a more accurate form of the solutions. The present paper, we think, seems to be mathematically complete, since the solutions satisfy the differential equation of the equilibrium of a plate and all the boundary conditions thoroughly. Somewhat similar conception on the evaluation of the solutions seemed to be taken up by Professor Reissner, 4) though we have not aware of his paper until recently; his case, nevertheless, is connected with the problem of a plate subjected to the thrust acting in one direction only and having the length of the plate comparable to its breadth like that which was dealt with by Bryan. 5) The nature of the corrugation of a long plate due to shearing forces have recently been studied by many mathematicians and engineers in aeronautics: Southwell and Skan, 6) Bollenrash, 7) Reissner and Bergmann, 8) Seydel, 9) and Schmieden 10) are most notable among them.

In spite of the enormous results of study on the stability of a plate under the above conditions of forces and boundaries, the similar problem of a plate under edge thrusts attracted, as already cited, very little attention of the investigators. Timoshenko\(^{11})\) considered the case of edge thrust, yet his problem is restricted to a plate of short length and also his method is merely to apply the idea of strain energy.

In the previous paper,\(^{11})\) the criterion of the stability was to take the limiting case where the period of vibrations of a plate subjected to the thrusts or other forces becomes infinite. This criterion of stability is valid if the material of the plate has no any damping nature in vibrations.\(^{12})\) Even when the vibrations of the plate are of damping nature, the limiting case of vanishing stability is determined only by the sign of the restitutive forces of the assumed vibrations. The assumption of minimum energy is nothing else but an alternative expression of the above criterion in the case of undamped oscillations. In the present case the critical point of stability, in which the restitutive force of vibrations vanishes or the effect of inertia force becomes zero, has been considered for the sake of simplicity.

Although the present paper treats of a two-dimensional problem, the result of calculation may be applied with a small modification to the examples of cylindrical coordinates. If the length of a circular shell which is acted on by a uniform external pressure is small compared with its diameter, even the unmodified case of the two-dimensional calculation is available from the nature of the problem to the determination of the buckling limit of the shell. We have thus compared the experimental results of the elastic stability of a circular shell, which Viscount Tokugawa published in WEC paper, directly with our mathematical calculation and found that there is a certain conformity between the experiments and our theory. It may be doubted that the solution of the stability problem of a plane plate or its modified solution may be compared reasonably with the result of experiments of a circular shell. A little consideration on the differential equations of a plane plate and those of a circular shell of short length will enable us to know that such a conception is not violent to a certain extent. A slight error accompanying the difference of the employed equations is out of question, when we think of the difficulty of adjusting the boundary conditions in the case of applying cylindrical coordinates.


Apart from the buckling of a cylindrical shell, the present two-dimensional calculation is not without significance on the practical problem of construction, such as the plate-construction of a ship\textsuperscript{13} and the fuselage of an aircraft.\textsuperscript{14}

2. We shall take a two-dimensional problem in which the axis of $x$ is directed parallel to the edge of the plate and the axis of $y$ perpendicular to $x$. Let $w$ be the deflection of the plate, $P_1$ and $P_2$ be the thrusts per unit breadth of the plate acting parallel to $x$- and $y$-axes respectively, and $D$ is defined by

$$D = \frac{2}{3} \frac{Eh^3}{(1-\nu^2)}.$$  \hspace{1cm} (1)

where $E$ is Young’s modulus, $\nu$ is Poisson’s ratio and $2h$ is the thickness of the plate. The equation of motion of a plate at the critical stability is then expressed by

$$D \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + P_1 \frac{\partial^2 w}{\partial x^2} + P_2 \frac{\partial^2 w}{\partial y^2} = 0.$$  \hspace{1cm} (2)

When both edges of the plate are clamped, we have

$$y = \pm b, \quad w = \frac{\partial w}{\partial y} = 0,$$  \hspace{1cm} (3)

where $2b$ is the breadth of the plate.

Writing

$$w = W \sin \frac{kx}{b},$$  \hspace{1cm} (4)

in (2), we obtain

$$\frac{d^4 W}{dy^4} + \left( \frac{P_2}{D} - \frac{2k^2}{b^2} \right) \frac{d^2 W}{dy^2} - k^2 \left( \frac{P_1}{D} - \frac{k^2}{b^2} \right) W = 0.$$  \hspace{1cm} (5)

Denoting

$$\frac{P_2 b^2}{D} - 2k^2 = -2\alpha, \quad k^2 \left( \frac{P_1 b^2}{D} - k^2 \right) = \beta^2, \quad W \propto \sin \frac{\lambda y}{\cos \lambda y},$$  \hspace{1cm} (6)

we find

$$\lambda^4 + \frac{2\alpha}{b^2} \lambda^2 - \frac{\beta^2}{b^2} = 0,$$  \hspace{1cm} (7)

the solutions of which are expressed by

$$\lambda_1, \lambda_2 \bigg| = \pm \frac{1}{b} \sqrt{\alpha^2 + \beta^2} - \alpha, \quad \lambda_3, \lambda_4 \bigg| = \pm \frac{i}{b} \sqrt{\alpha^2 + \beta^2} + \alpha = \pm i\lambda_4'.$$  \hspace{1cm} (8)

Taking appropriate values of $\lambda$, we find

$$w = \left\{ \Lambda \cos \frac{\sqrt{\alpha^2 + \beta^2} - \alpha}{b} y + B \cosh \frac{\sqrt{\alpha^2 + \beta^2} + \alpha}{b} y \right\} \sin \frac{kx}{b}.$$  \hspace{1cm} (9)


\textsuperscript{14} Th. v. Kármán’s lecture in Tokyo (1927).
Substituting this expression in the condition (3), we get
\[ \sqrt[3]{\alpha^2 + \beta^2} - \alpha \tan \sqrt[3]{\alpha^2 + \beta^2} - \alpha + \sqrt[3]{\alpha^2 + \beta^2} + \alpha \tanh \sqrt[3]{\alpha^2 + \beta^2} + \alpha = 0. \] (10)

If we denote
\[ \alpha = \frac{P_1 h^2}{4D}, \]
then, in virtue of (6), we have
\[ \alpha = k^2 - 2 \left( \frac{P_2}{P_1} \right)^2, \quad \beta^2 = k^2(4 \alpha^2 - k^2). \] (12)

From (10) and (12), we obtain the relation between \( \alpha \) and \( k^2 \) for a certain specified value of \( P_2/P_1 \). The compiled result is shown in the following figure.

![Figure 2](image-url)

Clamped, \( P_2 = \varepsilon P_1 \).

The solution in (9) takes different forms according with the variation of the values of \( \alpha \) and \( \beta^2 \). If \( \beta^2 > 0 \) and \( \alpha > 0 \), we get the solution of the type:
\[ w = \left\{ A \cos \frac{\sqrt[3]{\alpha^2 + \beta^2} - \alpha}{b} y + B \cosh \frac{\sqrt[3]{\alpha^2 + \beta^2} + \alpha}{b} y \right\} \sin \frac{kx}{b}, \] (91)

If $\beta^2 < 0$, $|\alpha^2| > |\beta^2|$ and $\alpha > 0$, we obtain the type:

$$w = \left\{ A' \cosh \frac{\sqrt{\alpha - \sqrt{\alpha^2 + \beta^2}}}{b} y + B' \cosh \frac{\sqrt{\alpha + \sqrt{\alpha^2 + \beta^2}}}{b} y \right\} \sin \frac{h x}{b},$$  \hspace{1cm} (9')

with the stability equation

$$\sqrt{\alpha - \sqrt{\alpha^2 + \beta^2}} \tanh \sqrt{\alpha - \sqrt{\alpha^2 + \beta^2}} - \sqrt{\alpha + \sqrt{\alpha^2 + \beta^2}} \tanh \sqrt{\alpha + \sqrt{\alpha^2 + \beta^2}} = 0.$$  \hspace{1cm} (10')

If $\beta^2 < 0$, $|\alpha^2| > |\beta^2|$ and $\alpha < 0$, we find the following solution:

$$w = \left\{ A'' \cos \frac{\sqrt{\alpha^2 + \beta^2 - \alpha}}{b} y + B'' \cos \frac{\sqrt{\alpha^2 + \beta^2 + \alpha}}{b} y \right\} \sin \frac{h x}{b},$$  \hspace{1cm} (9'')

together with the stability equation:

$$\sqrt{\alpha^2 + \beta^2 - \alpha} \tan \sqrt{\alpha^2 + \beta^2 - \alpha} - \sqrt{\alpha^2 + \beta^2 + \alpha} \tan \sqrt{\alpha^2 + \beta^2 + \alpha} = 0.$$  \hspace{1cm} (10'')

If $\beta^2 < 0$, $|\alpha^2| < |\beta^2|$ and $\alpha \geq 0$, we obtain

$$w = \left\{ A''' \cos \frac{\sqrt{\alpha^2 - \beta^2 - \alpha}}{b} y + B''' \cos \frac{\sqrt{\alpha^2 + \beta^2 + \alpha}}{b} y \right\} \sin \frac{h x}{b},$$  \hspace{1cm} (9''')

besides the stability equation

$$\sqrt{\alpha^2 - \beta^2 - \alpha} \tan \sqrt{\alpha^2 - \beta^2 - \alpha} + \cot \sqrt{\alpha^2 - \beta^2 - \alpha}$$

$$- \sqrt{\alpha^2 + \beta^2 + \alpha} \tan \sqrt{\alpha^2 + \beta^2 + \alpha} - \coth \sqrt{\alpha^2 + \beta^2 + \alpha} = 0.$$  \hspace{1cm} (10''')

To get the diagram in Fig. 2, we have employed all the relations in (10), (10'), (10''), (10''').

It may be seen from Fig. 2 that, almost in each case of $P_2/P_1$, there is a minimum of $j^2(=P_1b^2/4D)$ at a certain value of $k^2$. This relation shows that, when the ratio of $P_2/P_1$ is specified and the thrust is increased gradually from zero, the stability of the plate vanishes at the minimum of $j^2$ and the subsequent corrugation of the plate has the wave length $2\pi/k$ in which $k^2$ is the abscissa corresponding to the minimum value of $j^2$. If there is no minimum point of the curve of $j^2$, the above evidence will take place at the least value of $j^2$.

Again, in the case of large ratio of $P_2/P_1$, the minimum of $j^2(=P_1b^2/4D)$ takes a small value and the corresponding wave of the corrugation becomes long; while a small ratio of $P_2/P_1$ gives us a relatively high critical value of $P_1$ and the corrugation waves of short length. When $P_2$ has the nature of traction, the critical value of $P_1$ may be exceedingly enlarged, the wave length being correspondingly diminished. It is perhaps the most favourable fact in the problem of construction of hull structure.
that the critical load which acts in one direction in the plane of a plate is raised to any extent by a suitable adjustment of a transverse tension acting in the same plane.

We shall here take two examples of the deflection curves of a long plate to as-
certain the form of the corrugated surface: in one example we have taken the case \( k^2 = 3.0, j^2 = 3.17 \) and in the other we have chosen the case \( k^2 = 12, j^2 = 4.6 \), while the ratio of \( P_2 \) to \( P_1 \) is taken to be \( 1/2 \) in both examples. The former of these corresponds to the case of critical stability of a plate, in which \( P_2/P_1 = 1/2 \) is specified and the thrust is raised gradually from its zero state. The latter example may seem impos-
sible to be realised in the ordinary experiments. We think, however, that this state of the deformation together with those of the lower harmonics must frequently be observed in the actual experiments. Because the category of the stability of a plate, especially in a two-dimensional problem, is merely significant in its initial stage of deformation and has no much meaning in the stage of the curved deformation where the configuration of the plate enters another stable state of vibrations. The contour curves of the deflection of both cases are shown in the following drawing.

It may be seen from this sketch that the positive and negative bulges are ar-
ranged alternately in succession with a uniform pitch. In the case of the critical
loading, the length of the bulge (half wave length) is approximately equal to the
breadth of the plate.

3. We have now to proceed to the problem of a long plate having supported
edges. The boundary conditions are then expressed by
\[ y = \pm b, \quad w = \frac{\partial^2 w}{\partial y^2} + \sigma \frac{\partial^2 w}{\partial x^2} = 0. \]  
(13)

The solution in this case is denoted simply in the form:
\[ w = A \cos \left( \sqrt{\alpha^2 + \beta^2 - \alpha} \right) \frac{y}{b} \sin \frac{kx}{b}, \]  
(14)

provided \( \beta^2 > 0 \) and \( \alpha \geq 0 \), or \( \beta^2 < 0 \), \( |\alpha| > |\beta| \) and \( \alpha < 0 \). The stability equation is written by
\[ j^2 = \frac{(k^2 + \pi^2/4)^2}{4k^2 + \pi^2(P_1/P_2)}. \]  
(15)

When \( \beta^2 < 0 \), \( |\alpha| > |\beta| \) and \( \alpha > 0 \), or \( \beta^2 < 0 \), \( |\alpha| < |\beta| \) and \( \alpha \geq 0 \), we cannot get solutions of the deflection and that of the stability criteria. It is of a great fortune that the ordinary case of a plate undergoing various thrusts does not reside in these domains.
We must here notice that, when $\beta^2 < 0$, $|\alpha^2| > |\beta^2|$ and $\alpha < 0$, the expression of the deflection in the form:

$$w = B \cos \frac{\sqrt{1 - \alpha^2 + \beta^2 - \alpha}}{b} y \sin \frac{kx}{b},$$

(16)

with the same criterion of the stability as that in (15), may also be permissible in its possible existence. We must, however, adopt the expression (14) in lieu of the fact that the deflection cannot change its form at $\beta^2 = 0$, though the critical values of the thrust are the same in both cases of the deflection curves.

The compiled result of the present case is shown in the above diagram. It appears from this diagram that the value of $\eta^2$ for a relatively large range of $P_2/P_1$ is smallest at $k^2 = 0$. This tells us the fact that, in the majority of cases of the supported plate subject to the thrust acting at boundary edges, the corrugated deflection of the plate is not enabled to take place, but a single bulge extending through the length of the plate is liable to be formed. It may also be remarked that the critical load of a supported plate is much less than that of a clamped plate. Though this fact may be quite established from the general tendency of the buckling problem, yet this gives us rather a great advantage in the direction of finding the stability of a semi-clamped plate.

![Diagram](image_url)

Supported. $P_2 = \frac{1}{2} P_1$, $k^2 = 3.0$, $\eta^2 = 1.8$.

Supported. $P_2 = \frac{1}{2} P_1$, $k^2 = 12.0$, $\eta^2 = 3.9$.

Fig. 5
To obtain a conclusive idea on the distribution of the deflection of a plate supported at edges, two examples of the corrugated form of a plate have been shown in the above figure. In one case we have taken $k^2=3.0$, $j^2=1.8$ and in the other $k^2=12$, $j^2=3.9$.

It may be noticed that in some cases the length of the bulge is comparable to the breadth of the plate as in the case of clamped edges, though neither of the present examples corresponds with the deformation at the critical stability which is known from the theory of elasticity. Again, the contour curves of the deflection of a supported plate indicate a somewhat different feature, when they are compared with those of clamped edges. In the present case the bulge has the similar distribution of deflections in longitudinal as well as in transverse directions of the plate, while in the clamped plate this similarity of distributions is not possible.

4. We shall now consider the effect of the curvature of a plate (if the plate is curved,) on the differential equation of the equilibrium and on its solution. If the shell is a circular cylinder of radius $R$, then the differential equation of equilibrium in cylindrical coordinates, $(y, \phi)$, may be denoted by

$$D \left[ \frac{\partial^2 w}{\partial y^2} + \frac{2}{R^2} \frac{\partial^2 w}{\partial y^2 \partial \phi^2} + 1 \frac{\partial^2 w}{\partial y^2} \right] + P_2 \frac{\partial^2 w}{\partial y^2} + P_1 \left( \frac{1}{R^2} \frac{\partial^2 w}{\partial \phi^2} + \frac{1}{R} \right) = p,$$

where $w$ is the radial displacement of the shell, $P_2$ and $P_1'$ the axial and hoop thrusts, $P_1'$ being variable along $\phi$, and $p$ is a uniform external pressure. Let $F$ denote the shearing force acting at section $\phi$ of the cylindrical shell, then we find without any difficulty the relations:

$$\frac{\partial P_1'}{\partial \phi} = -F \partial \phi; \quad F = -\frac{D \partial^2 w}{R^2 \partial \phi^2}.$$  

Eliminating $F$ between these two equations, we get

$$\frac{\partial P_1'}{\partial \phi} = -\frac{D \partial^2 w}{R^2 \partial \phi^2}.$$  

Integrating this equation with respect to $\phi$, we obtain

$$P_1' = \frac{D \partial^2 w}{R^2 \partial \phi^2} + P_1.$$  

The constant $P_1$ has entered this integral on account of the reason that, when $\partial^2 w/\partial \phi^2 = 0$, we should have $P_1' = P_1$. Substituting (20) in (17), we find

$$D \left[ \frac{\partial^2 w}{\partial y^2} + \frac{2}{R^2} \frac{\partial^2 w}{\partial y^2 \partial \phi^2} + 1 \frac{\partial^2 w}{\partial y^2} \right] + P_2 \frac{\partial^2 w}{\partial y^2} + P_1 \left( \frac{1}{R^2} \frac{\partial^2 w}{\partial \phi^2} + \frac{1}{R} \right) = p.$$  

It must be borne in mind that, even though we may assume from the start $P_2$ to be
variable in $x$, $\phi$ and $P'$ to be variable in $x$ besides $\phi$, their effects on the differential equation of the type (21) are of the second order compared with that of $P'$ which is a certain function of $\phi$. When the radius of the cylinder is large compared with the deformation $w$, we get

$$D\left[ \frac{\partial^2 w}{\partial y^2} + \frac{2}{R^2} \frac{\partial^4 w}{\partial y^2 \partial \phi^2} + \frac{1}{R^4} \frac{\partial^6 w}{\partial \phi^6} \right] + P_2 \frac{\partial^2 w}{\partial y^2} + \left[ P_1 + \frac{D}{R^2} \frac{1}{R^2} \frac{\partial^2 w}{\partial \phi^2} \right] = \left( p - \frac{P_1}{R} \right). \quad (22)$$

In the case of a cylindrical shell subject to the external pressure as in a submarine hull, we may put

$$P_1 = pR, \quad P_2 = \frac{pR}{2}, \quad (23)$$

so that we find

$$D\left[ \frac{\partial^2 w}{\partial x^2} + \frac{2}{R^2} \frac{\partial^4 w}{\partial x \partial y \partial \phi^2} + \frac{1}{R^4} \frac{\partial^6 w}{\partial \phi^6} \right] + \left( P_1 + \frac{D}{R^2} \right) \frac{\partial^2 w}{\partial x^2} + \frac{pR}{2} \frac{\partial^2 w}{\partial y^2} = 0. \quad (24)$$

This is the differential equation of the critical state of a cylindrical shell under external hydrostatic pressure $p$ acting in both directions of the radius and the axis of the cylinder. To solve this equation we put $R \, d\phi = dx$, then (24) becomes

$$D\left[ \frac{\partial^2 w}{\partial x^2} + \frac{2}{R^2} \frac{\partial^4 w}{\partial x \partial y \partial \phi^2} + \frac{1}{R^4} \frac{\partial^6 w}{\partial \phi^6} \right] + \left( pR + \frac{D}{R^2} \right) \frac{\partial^2 w}{\partial x^2} + \frac{pR}{2} \frac{\partial^2 w}{\partial y^2} = 0. \quad (25)$$

Writing

$$w = W \sin \frac{kx}{b}, \quad (4')$$

we obtain

$$\frac{d^4 W}{dy^4} + \left( \frac{pR}{2D} - \frac{2k^2}{b^2} \right) \frac{d^2 W}{dy^2} - \frac{k^2}{b^2} \left( \frac{pR}{D} + \frac{1}{R^2} - \frac{k^2}{b^2} \right) W = 0. \quad (26)$$

Denoting

$$\frac{pRb^2}{2D} - 2k^2 = -2\alpha, \quad \kappa \kappa \left( \frac{pR}{D} \frac{1}{b^2} - \frac{\kappa^2}{R^2} - k^2 \right) = \beta^2, \quad (27)$$

we find

$$W \propto \sin \frac{\lambda y}{\cos \lambda y}, \quad (28)$$

$$\lambda^4 + \frac{2\alpha}{b^2} \lambda^2 - \frac{\beta^2}{b^4} = 0. \quad (7')$$

If we write

$$4j^2 = \frac{pRb^2}{D} + \frac{b^2}{R^2}, \quad (29)$$

15) The case, where the terms $2D \frac{\partial^2 w}{\partial x \partial y^2}$ and $\frac{pRb^2}{2} \frac{\partial^2 w}{\partial y^2}$ are neglected, was already shown by one of us in the paper, loc. cit., K. Sezawa, Jour. Soc. Nav. Arch., 38 (1926), p. 95.
then, in virtue of (27), we obtain,

\[ \alpha = k^2 - \left( \beta^2 - \frac{b^2}{4e^2} \right); \quad \beta^2 = k^2(4\beta^2 - k^2). \]  

(30)

We have thus arrived at the similar relation as that in Section 2 and 3. The computed results, in which \( b^2/4e^2 = 0 \) and both boundaries \( y = \pm b \) are clamped or supported are shown in the figure below, though these two curves are the same as those in Fig. 2 and Fig. 4. We have omitted the term \( b^2/4e^2 \) intentionally on account of the reason that the resulting error involved in the ordinary case of models is less than a few per cent. The accurate curves of \( j^2 \) occupy their positions slighter higher than those in Fig. 6.

![Fig. 6.](image)

To compare the results of the theoretical calculation with experimental analysis, we have introduced the data of Viscount Tokugawa’s model experiments\(^{16}\) of H-series. The diameter \( d \) of the model cylinder is 10 cm, its thickness \( 2h \) is \( \frac{1}{2} \) mm, while Young's modulus of the material of the shell is \( 1.07 \times 10^6 \) kg/cm\(^2\). We have assumed Poisson’s ratio to be 0·4 and have selected six cases of H-series, namely H-13, 14, 15, 16, 17, 18. The selection of these is mainly due to the applicability of our calculation and also to the reliability of the experimental results. The data of the dimensions of the models have been employed in our calculation and the

\(^{16}\) Viscount Tokugawa, loc. cit.
resulting critical pressure \( (p_0) \) and the number of lobes \( (n) \) have been compared with those of the experiments, as seen from the following table.

<table>
<thead>
<tr>
<th>2b (cm)</th>
<th>( p_0 ) (clamped)</th>
<th>( p_0 ) (supported)</th>
<th>( n )</th>
<th>( p_0 )</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>H-13</td>
<td>3.2</td>
<td>13.15 ( \text{kg/cm}^2 )</td>
<td>5.32 ( \text{kg/cm}^2 )</td>
<td>6</td>
<td>15.49 ( \text{kg/cm}^2 )</td>
</tr>
<tr>
<td>H-14</td>
<td>2.7</td>
<td>18.40</td>
<td>7.55</td>
<td>7</td>
<td>14.89</td>
</tr>
<tr>
<td>H-15</td>
<td>2.2</td>
<td>27.70</td>
<td>11.36</td>
<td>8</td>
<td>19.19</td>
</tr>
<tr>
<td>H-16</td>
<td>1.7</td>
<td>46.45</td>
<td>19.00</td>
<td>11</td>
<td>24.12</td>
</tr>
<tr>
<td>H-17</td>
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<td>59.80</td>
<td>24.48</td>
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<td>25.59</td>
</tr>
<tr>
<td>H-18</td>
<td>1.2</td>
<td>93.50</td>
<td>28.33</td>
<td>15</td>
<td>33.61</td>
</tr>
</tbody>
</table>

Since we have not adopted any artificial modification like Viscount Tokugawa's frame factor on the theoretical calculation, we will never anticipate that the experimental result well conforms with that of our calculation; yet we find that both results have no much disagreement between them. As already mentioned in Section 3, it is probable that the higher harmonics of the corrugation may be formed together with long waves of bulging in the case of a finite deflection, so that some inconsistency of the theory and experiment may be unavoidable.

Secondly, if \( k^2 \) and \( j^2 \) are deduced from the numerical data of \( p_0 \) and \( n \) which come out from Viscount Tokugawa's experiments, and if they are plotted in Fig. 6 by applying the formulae (10), (15) and (11), (12) instead of (29), (30), we find that the experimental condition of the shell nearly corresponds to the case of supported edges of a plane plate. We cannot, however, conclude simply that in the above experiment the edges of the shell were in a supported conditions. We think that this evidence may have been caused partly by the incomplete clamping of the edge and partly by some other nature like the complex state of the critical stability. The numerical values of \( k^2 \) and \( j^2 \) are tabulated below:

<table>
<thead>
<tr>
<th>2b</th>
<th>( k^2 ) calculated from actual ( n )</th>
<th>( j^2 ) calculated from actual ( p_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>H-13</td>
<td>10.25</td>
<td>3.72</td>
</tr>
<tr>
<td>H-14</td>
<td>8.85</td>
<td>2.38</td>
</tr>
<tr>
<td>H-15</td>
<td>7.00</td>
<td>2.20</td>
</tr>
<tr>
<td>H-16</td>
<td>5.28</td>
<td>1.65</td>
</tr>
<tr>
<td>H-17</td>
<td>4.41</td>
<td>1.33</td>
</tr>
<tr>
<td>H-18</td>
<td>2.45</td>
<td>1.14</td>
</tr>
</tbody>
</table>

When we apply the relations (29), (30), we shall find that two curves in Fig. 6
as well as the plotted points of H-series are raised upwards less than a few per cent.

As we are not aware of the results of the large scale experiments carried out by Viscount Tokugawa, it is doubtful that the above comparison may have its validity of determining the general nature of the problem. In large scale experiments, however, the clamping at the edges and the symmetry of the shell as well as the acting pressure must have been maintained in more favourable conditions, and hence we believe that the result of such experiments and that of calculation will then have a better agreement.

1930 October 15.

討 論

○会長（末廣治二君） 唯今の大澤博士の御講演に対して御質問又は御意見のある御方は御述べを願ひます。

○徳川武定君 大澤博士の例に依り鮮かな論文を邦発出来ました事を幸に存じます。edges から thrust を受ける薄長板の buckling mode 及 critical pressure を所謂 vibration method で解かれたのは失礼ながら本論文が初めてであると考えます。勿論 two lobed buckling の場合はありましたのが、斯様に同法を radial external pressure と end thrust に倣われる thin circular cylinder の stability の場合に extend され、自分が常で W. E. C. に出ました実験の data を用なられ、自分の取った displacement assumption では clamped edge の場合は不可能であるのを斯かる場合にも解けると云ふ事を示して下さいました。自分として大変に面白く最も有益な御示教を得た事を深く御謹拝申上げます實は昨日の午後に paper の前篇を頂いたので未だ熟読出来ませんが、此、三事付け申した点を申上げます。

（1）自分が所謂 frame factor と云ふ artificial coefficient を displacement assumption の中に取入れたのは、stiffeners で補強された cylinder の frame line に於ける boundary condition を満足させると妙でありますまして、frameless cylinder の両端では皆 support と見て良く合ひます（大澤博士の引用された H series は此場合であります、所謂 frame factor を合んで居りません）。H-13～18 で calculated values より実験値の方が下つて居るのは、stress が既に linear elastic range を超過した勢でありますまして、boundary condition の取方の不完全に至るものとは思ひません。

（2）frame で補強された場合も

（a）shell と stiffener を厳密に fit するか又は thick cylinder から rib を machine cut した
On the Buckling of a long Elastic Plate under edge Thrusts.

様々な場合、frameless の場合と同様、実験値は理論よく一致します。

(1) 然し、実際の潜水艦の建造時等の場合は、内壁の境界条件が考え難しい。rivet する際に、上の boundary condition を考慮しなかった。displacement assumption の中に、cosine factor 中に \( k \) と cloud frame factor を導入した結果、実験結果が一致したが、(当初、frame が shell と共にある deform すると、cloud から frame factor を導入したのは誤りのようです。

(2) 確認評価の問題では、critical 以上が見られると、同じ stress condition に対して無限の equilibrium state があると云うと、critical pressure に対する deformation の absolute amount を出す事は出来ない。然し、実際問題として潜水艦等では、いざ壊れると雲状に何方からの前兆が無いと操縦者が非常によく云う事がありますから、何とかして例へば collapse の前には shell wall を hammer すると雲が異って来るとか何かとよく变形方法は無いものでしょうか。

(3) 最後に、一寸気付きましたが、plain plate から extend された theory であるから length (2b) が小であると shell の数が段山になる場合でも実験値と良く合致して来るであろうと思われます。

○妹尾克授君 今御説示に就ては、大変同意見であります、御返答の必要も無いかと思いまが、御説示に際して再び気付いた事、別に論文として丁寧に読んでも居ります。一番初めに仰言された frame factor の一々の場合は論文を詳しく読んでも居ります。読まないで引用したのは失礼でございますが、私の計算の support とした状態は徳川子爵の御実験と合せてみたのは幸に思ひます。Fig. 6 中にある points が support の curve の下に外れて居りますが、高さだけを比較した方が reasonable で、は等の點を少々左へ移すと support の line と一致します。次に large scale の model で shell と frame の fit が rigid でなく shell 及 frame が一緒に变形するとの事でありますかが、frame factor と共に理論的に如何なる関係がありますか、何かの機会に承知度いと思います。向は航空研究所の佐々木達次郎博士の同様な模型実験に依っても一緒に deform して居ります。そして比較計算は、Timoshenko 氏の energy の方法でされたのであります。

次に徳川子爵の御説の cylinder の近い程合はの少し前であるという事を私は同意であります。初め夫だけで済まそうと思ってもましたが、おект出して御質問の結果を plate の計算の危い所自分でも御借りした想であります。

その次に deformation の assumption \( w \) は徳川子爵の御説の通り清々すまでもありますが、私が Fig. 3 及 4 に示したのは \( w \) を消さないように変形の mode を示した積りであります。共 absolute value は何らでも宜しいが curve の distribution の proportion は式で判る通り何処迄でも消えません。この mode が buckle の問題で infinite の変形まで適用するかといふに、initial stability は少し経つと異った stability となって condition も異って来ますから多少怪いのではありますけれども、
K. Sezawa and G. Nishituna:

Fig. 2 の minimum point の stability で壁れるのではなく此 curve の勝手な点が出て来ることも考えられます。子爵の actual experiment の結果を見ると、deformation の initial stability の mode を示すものか、pressure の increase と共に異った system の buckle が起こったのであるかという事は判断が出来ません。勿論実験を見たのではないから良くは判りませんが……

大不完全な説明ではありましたけれども、御討論に対する御説として御答申上します。

〇會長（末廣 慶二君） 外に御質問又は御意見を述べられる方はありませんか。御座いませんですねば妹澤君の有益なる paper に対して諸君子と共に拍手して謝意を表したいと思います。（一同拍手）

德川子爵の討論に関しての質疑

昭和五年十一月十日 妹澤克悟

昨日の議論会に於て御議論を賜り更めて御礼申し上げます。筆者は非常に記憶力が悪く自分で一度考究した事でも二三日経つと共順序を忘れてしまうのが常であり、従つて昨日の御討論に就ても共席で聴明すべき事が種々あったのにも拘らず、大切な事を大抵忘却して要領ある回答が出来なかった事を遺憾に思います。

筆者の論文の数表の下に 附加へた言葉は 初めは表の中の数字の比較のみでなく、寧ろ貴下の御理論と筆者の計算とのあらさを比較する根拠であったのですが、知らず々々の間に一例に附加へた所の表の中の数字の比較のみの様になってしまった譯であります。而して筆者の計算の分は 全の為に clamped と supported との両方を出してありますけれども、貴下の実験は確かに clamped の場合であると考えてしまったので御座います。従つて frame factor なる言葉も出て来た譯であります。何故共様に考へたかと云へば、始め貴下の工業御論文の論証の處はあまり熟読せずずに H series の collapsing の実験のみを拜見した處で、殆ど完全に clamped ends の状態に在ったので、斯く考へてしまつたのであります。其次態は恐らく initial で且つ elastic の stage でも矢張り共様であったと充分考慮されたのであります。尚又筆者が熟考する所では cylindrical shell の場合に仮令に如何なる装置にして supported の状態に保たんとしても、ends に curvature of deformation の生す以上、必ず clamped の状態になって居るものと見做さなければなりません。又 cylinder の端があると云ふ条件からもそうあるべきである。然るに計算的理論に依る時は、間接が外観を受けても軸軸縮を受けても、supported の場合しかやれないと言ふ事は如何にも不可思議な事であります。筆者の論文は共目的に向って多少役に立つものと思ひます。cylindrical coordinate の場合の ends を無理に clamp しようとすると、一つの ring の中の次の lobe の處で end 夫身の変位が生ずるので、support 以
外に直すこと無理な事柄で御座います。

そこで今貴下に御伺い致し度い事は貴下の御論文に在る所の ends の support された時理論と上述の clamp された実験とが何故に極端に良く一致するかと云ふ事であります。若し結果同志が良く一致して居つても、其 mechanism が異って居るのでは quality の問題として疑ふ餘地がありはせぬかとも思はれます。貴下の場際には勿論 deformation の分布に至る迄ではつまり correspondence があるとは想像致しますが、stability の計算では普通単に critical thrust や lobes の数のみを結果として出してある様ですけれども、deformation の mode 迴も誰でも頭に入れてもある事は勿論です。何卒上述の一一致の御指摘を御教示下さい。

次に伺い度い事は計算で判る様な initial stability の壊れる stage は貴下の御実験に於て如何なる場合を御考へ入れて御振りになつたか、又夫れが完全に判定がつくものとしても如何なる割合の誤差に於て正確であるかと云ふ事であります。是れは小生の経験から簡単な柱の様な場合の実験でも、或 load で多少急に deflection の大きくなる處があるにはあるが様々な點で ambiguous であるのに、殊更二次元以上の問題で面かも外壁で曲曲する様な場合には尚更そうでないかと想像されるからであります。此點も御教へ下さい。

最後に large scale experiment の二三の例を折示御示し下さいましたが、筆者の場合の様に長板の方法では除り良く當然自然に入れ dimension となって居るので、未だ比較として居りません。掌ろ ends が clamp され且つ短間筒の場合を厳し通い位で御座います。

以上矢言の點が若し有れば平に御容赦下さい。

昭和五年十一月十日の妹澤博士の質疑に対し

昭和五年十一月十八日 徳川武定

(1) 自分の analysis の道理では u, v, w に symmetrical assumption を入れなくてはならなかった関係上少し無理な所のある事は確かであります。然し H series にしろ large scale の実験の場合にしろ上的「チェンライン」の如く薄板である為め cover の edge の處から腰が折れまして clamp の様な問題を呈しません（途中の deformation の計測から云つても、collapse 後の状況から見ても）。上の例は又 bulkhead plate の deflection test の時にも同様に於いて clamp の様な状況を示し

On the Buckling of a long Elastic Plate under edge Thrusts.

H-Series の Soldered End.
Large Scale の場合

最近直径 600 mm 位の短 cylinder を 10 節計り圧壊実験をやりますから共 data で又御考へ下さい。

(2) 第二の御質問に対しては自分にも全く見当がつきません。deformation の方に前兆も現れて来ません。

之は前の議論の時に申し上げた様に何とか将来捕へなくてはならぬと考へて居ります。

以上是だ粗雑な御返事ですが御答へします。