Fundamental Formula for Wave Motion caused by Moving Bodies

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Summary

In this paper, considering the fact that accompanying with fundamental studies on the motion of ships, for instance on their steering faculty or manoeuvrability, the theory of wave resistance of ships moving non-uniformly claims its significance, the author gives a fundamental formula for the surface disturbance of water in some general forms. Then some problems are studied as its applications. One of them is a calculation of additional virtual mass of spheres and spheroids floating on the surface of water. Another application is a calculation of wave resistance of a submerged circular cylinder moving along a straight line.

§ 1. Introduction

The wave resistance of a ship moving with uniform speed has been investigated theoretically by various researchers since Michell (1) established his famous theory, and a great number of problems concerned have already been solved up to the present. However, for the fluid motion caused by nonuniform motion of a ship, only a few works have been performed in the past, in spite of the importance for the correlation with the manoeuvring of ships.

Now the present author tries to introduce a fundamental formula for wave motion caused by moving bodies, which is applicable for general cases.

§ 2. Fundamental formula for the surface disturbance of water

To establish the theory we take following assumptions as usual.

(1) Fluid is non-viscous and incompressive.
(2) The surface disturbance is comparatively small, so that the second and higher powers of fluid velocity may be neglected.
(3) At first the depth of water is assumed infinite and the effect of the sea bottom will be considered afterwards.

Take the origin on the still water level, axes of x and y horizontal, and axis of z vertical upwards. Denoting the velocity potential by $\phi$, then the condition of the water surface is given by the following equation.

$$ \frac{\partial \phi}{\partial t} - \gamma \frac{\partial \zeta}{\partial x} + \mu \phi = \text{const.} $$ (1)

where $\zeta$ is the surface elevation and $\mu$ is a fictitious frictional coefficient introduced by Rayleigh which should vanish ultimately. By the condition of small inclination of the water surface, the following relation exists.

$$ \frac{\partial \zeta}{\partial t} = -\frac{\partial \phi}{\partial x}. $$ (2)

The equations (1) and (2) are satisfied at $z=\zeta$, but it is permissible for us to consider they are valid at $z=0$, because $\zeta$ is small. From (1) and (2) we derive

$$ \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} + \mu \frac{\partial \phi}{\partial t} = 0 $$ (3)

Now, consider a velocity potential $\phi_0$ due to a certain distribution of singularities of antisymmetry about the $x-y$ plane, which becomes zero on the plane $z=0$ and vanishes at infinity. Then we may write the complete velocity potential in the form,

$$ \phi = \phi_0 + \phi_1, $$ (4)

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where \( \phi_1 \) is to be determined so as to satisfy the condition (3). As \( \phi_1 \) is a harmonic function, \( \nabla^2 \phi_1 = 0 \) (5) in the lower half domain and may be expressed by the integral representation as follows.

\[
\phi_1 = \iiint_{-\infty}^{\infty} F(\alpha, \beta, \xi, \eta) \cos \{\alpha(\xi - x)\} \cos \{\beta(\eta - y)\} \exp(\sqrt{\alpha^2 + \beta^2} z) \, d\alpha \, d\beta \, d\xi \, d\eta. \tag{6}
\]

Put

\[
(-\partial \phi_0 / \partial x)_{z=0} = w_0(x, y, t), \tag{7}
\]

and assume the function \( w_0 \) to be expressed by the Fourier integral such as

\[
w_0 = \frac{1}{4\pi^2} \iiint_{-\infty}^{\infty} \omega_0(\xi, \eta, \tau) \cos \{\alpha(\xi - x)\} \cos \{\beta(\eta - y)\} \, d\alpha \, d\beta \, d\xi \, d\eta. \tag{8}
\]

Substituting (4), (6) and (8), in (3), we have a linear differential equation determining the function \( F \).

\[
d^2F/dt^2 + \mu \, dF/dt + g \sqrt{\alpha^2 + \beta^2} F = (1/4 \pi^2) \, g \omega_0. \tag{9}
\]

This equation can easily be solved, and applying the condition that the fluid is initially at rest, we can obtain \( \phi_1 \) in the form,

\[
\phi_1 = \frac{1}{4\pi^2} \int_{-\infty}^{t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \omega_0(\xi, \eta, \tau) \cos \{\alpha(\xi - x)\} \cos \{\beta(\eta - y)\} \exp(\sqrt{\alpha^2 + \beta^2} z) \, d\alpha \, d\beta \, d\xi \, d\eta \times \sin \left\{ \frac{1}{g} (\alpha^2 + \beta^2) \frac{1}{4} (t - \tau) \right\} \frac{1}{g} (\alpha^2 + \beta^2) - \frac{1}{4} \, d\alpha \, d\beta, \tag{10}
\]

or putting

\[
\alpha = \kappa \cos \theta, \quad \beta = \kappa \sin \theta, \tag{11}
\]

we have

\[
\phi_1 = \frac{1}{4\pi^2} \int_{-\infty}^{t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \omega_0(\xi, \eta, \tau) \, d\xi \, d\eta \int_{-\infty}^{\infty} \exp(\kappa z + i\kappa \tilde{w}) \, \sin \left\{ \frac{1}{g\kappa} \frac{1}{4} (t - \tau) \right\} \frac{1}{g\kappa} \, d\kappa, \tag{12}
\]

where \( \tilde{w} = (x - \xi) \cos \theta + (y - \eta) \sin \theta \).

If we denote the velocity potential due to the singularity distribution in the lower half domain by \( \bar{\phi}_0 \),

\[
\bar{\phi}_0 = \bar{\phi}_0(x, y, z) - \bar{\phi}_0(x, y, -z). \tag{14}
\]

Writing

\[
(-\partial \bar{\phi}_0 / \partial x)_{z=0} = \bar{w}_0(x, y, t), \tag{15}
\]

we have

\[
\bar{w}_0 = 2 \, \bar{w}_0, \tag{16}
\]

and (12) becomes

\[
\phi_1 = \frac{1}{2\pi^2} \int_{-\infty}^{t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \omega_0(\xi, \eta, \tau) \, d\xi \, d\eta \int_{-\infty}^{\infty} \exp(\kappa z + i\kappa \tilde{w}) \, \sin \left\{ \frac{1}{g\kappa} \frac{1}{4} (t - \tau) \right\} \frac{1}{g\kappa} \, d\kappa. \tag{17}
\]

\section{§ 3. Various cases}

(1) point source

A source of strength \( \sigma \) is situated at a submerged point whose coordinates are \( x_0, y_0, z_0 \). Then

\[
\bar{\phi}_0 = \frac{\sigma}{\pi} = \frac{\sigma}{2\pi} \int_{-\pi}^{\pi} \exp[-\kappa(z - z_0) + i\kappa((x - x_0) \cos \theta + (y - y_0) \sin \theta)] \, d\kappa, \quad z > z_0, \tag{18}
\]

where

\[
\kappa = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2},
\]

and consequently
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\[ w_0 = \frac{\sigma}{2\pi} \int_{-\infty}^{\infty} d\theta \int_{0}^{\infty} \exp[kx_0 + ik \{(x-x_0) \cos \theta + (y-y_0) \sin \theta\}] k \, dx, \]

(20)

\[ \phi_0 = \left( \frac{1}{r_1} - \frac{1}{r_2} \right)^\gamma \]

(21)

where

\[ r_2 = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z+z_0)^2}. \]

(22)

Assume \( x_0, y_0, z_0 \) and \( \sigma \) to be functions of \( t \),

\[ \phi_1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \sigma(\tau) e^{\frac{\mu}{2}(\tau-t)} d\tau \int_{-\pi}^{\pi} d\theta \int_{0}^{\infty} \exp[k(z+z_0(\tau)) + ik \tilde{\omega}_0(\tau)] \sin \left\{ \sqrt{\frac{g}{\kappa}} (t-\tau) \right\} \sqrt{\frac{g}{\kappa}} \, dx, \]

(23)

where

\[ \tilde{\omega}_0(\tau) = (x-x_0(\tau)) \cos \theta + (y-y_0(\tau)) \sin \theta. \]

(24)

(2) doublet

For a doublet of moment \( m \) at \( x_0, y_0, z_0 \), whose axis has direction cosines \( n_1, n_2, n_3 \),

\[ \phi_0 = -\frac{m}{n} \left( \frac{1}{r_1} + \left( \frac{1}{r_1} \right)^\gamma \right) \]

(25)

from which

\[ w_0 = -\frac{im}{\pi} \int_{-\infty}^{\infty} (n_1 \cos \theta + n_2 \sin \theta + n_3) \, d\theta \int_{0}^{\infty} \exp[k(z+z_0(\tau)) + ik \tilde{\omega}_0(\tau)] k^2 \, dx, \]

(26)

\[ \phi_1 = -\frac{i}{\pi} \int_{-\infty}^{\infty} m(\tau) e^{\frac{\mu}{2}(\tau-t)} d\tau \int_{-\pi}^{\pi} n_1(\tau) \cos \theta + n_2(\tau) \sin \theta + i n_3(\tau) \]

\[ \times \exp[k(z+z_0(\tau)) + ik \tilde{\omega}_0(\tau)] \sin \left\{ \sqrt{\frac{g}{\kappa}} (t-\tau) \right\} \sqrt{\frac{g}{\kappa}} \, dx, \]

(27)

(3) moving body

A body moving under the water surface is replaced by a certain distribution of singularities over the surface of the body. In order to find the singularity distribution, we apply Green's formula to a point between two closed surfaces \( S_1 \) and \( S_2 \), of which the latter contains the former.

\[ \phi = \frac{1}{4\pi} \int_{S_1} \left( \phi \frac{\partial}{\partial n} \frac{1}{r_1} - \frac{\partial \phi}{\partial n} \frac{1}{r_1} \right) dS - \frac{1}{4\pi} \int_{S_2} \left( \phi \frac{\partial}{\partial n} \frac{1}{r_1} - \frac{\partial \phi}{\partial n} \frac{1}{r_1} \right) dS, \]

(28)

where \( n \) is the outward normal to the surface \( S_1 \) or \( S_2 \), and the differentiation should be performed with respect to \( \xi, \eta, \) and \( \zeta \), which are substituted for \( x_0, y_0, \) and \( z_0 \) in \( r_1 \). The surface \( S_1 \) may be chosen as the body's surface \( S \) and \( S_2 \) as the free surface. Then the first term of (28) is no other than the velocity potential \( \phi_0 \), while the second term corresponds to the remainder of \( \phi \). Hence we find the density of the source distribution on the surface \( S \) to be \(- (1/4\pi) \left( \partial \phi / \partial n \right)_S \) and that of the doublet distribution to be \((1/4\pi) \phi_0 \). Though the complete velocity potential can easily be derived from (21), (23), (25), and (27) in which \( n_1, n_2, n_3 \) are taken to be the direction cosines of the outward normal of the surface \( S \), it is extremely difficult to determine the singularity distribution so as to satisfy the boundary condition on the surface of the body, i.e.

\[ - \frac{\partial \phi}{\partial n} = n v, \]

(29)

where \( v \) is the velocity of any point on the surface and \( n \) is the unit vector along the outward normal. For a submerged body the effect of the fluid motion excluding \( \phi_0 \) is comparatively small on the body's surface, and we are permitted to take the first approximation by putting

\[ \phi = \phi_0 \quad \text{on } S \]

(30)

and regarding that \( \phi_0 \) satisfies the boundary condition (29) by itself. For a slender ship, the surface \( S \) is taken as the centre line plane \( S_0 \) according to the same hypothesis as that adopted by Michell.

\[ S = 2 S_0. \]

(31)
If the ship is moving horizontally within her plane of symmetry with instantaneous velocity $V(t)$ in the direction of positive $x$, the normal velocity on the ship's surface is $-V \frac{\partial y}{\partial x}$, as the ship's form is represented by the equation.

$$y = f(x, z). \quad (32)$$

Then the doublet distributions on both sides cancel each other and we get the following expression for the fluid motion.

$$\phi = -\frac{V}{2\pi} \int \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \frac{\partial y}{\partial x} dS - \frac{1}{2\pi^2} \int_0^t \int_{-\infty}^\infty \int_{-\infty}^\infty V(t') \int \frac{\partial y}{\partial x} dS d\tau d\theta \times \int_0^{2\pi} \exp \left[ k(x + t') + ik\bar{w}(\tau) \right] \sin \left[ \sqrt{\frac{g\kappa}{r}} (1 - \gamma) \right] \frac{\gamma}{g\kappa} dx. \quad (33)$$

In the above, we have put

$$r_2 = \sqrt{(x - X - \xi)^2 + y^2 + (z - \zeta)^2}, \quad r_1 = \sqrt{(x - X - \xi)^2 + y^2 + (z + \zeta)^2},$$

$$\bar{w}(\tau) = (x - X - \xi) \cos \theta + y \sin \theta$$

$$X(\tau) = \int V(t') dt. \quad (34)$$

§ 4. Resistance

The force acting on the body is given by the integral as

$$F = \iint p \mathbf{n} dS. \quad (35)$$

The resistance to the motion is the component of $F$ inverse to the direction of motion. For the translatory motion, $v$ is the same at any point, and the resistance becomes

$$R = -Fv/V = -\left( \frac{1}{V} \right) \int \int p \frac{\partial \phi}{\partial n} dS, \quad (36)$$

where

$$V = |v|$$

On the other hand, in the case of rotation the resistance takes the form of resisting couple. Denoting the rotation of the body by $\omega$, the velocity at any point is $\omega \times r$, where $r$ is the radius vector of that point about the axis of rotation. Then the resisting couple becomes

$$M = -\int \int \frac{\omega \times r}{|\omega|} dF = -\frac{1}{|\omega|} \int \int p \frac{\partial \phi}{\partial n} dS \quad (37)$$

where

$$\omega = |\omega|.$$

The pressure $p$ is given by the equation,

$$p = \frac{\partial \phi}{\partial t} - \frac{1}{2} q^2 - gz + c(t), \quad (38)$$

$$q^2 = (\text{grad } \phi)^2.$$

Substituting (38) in (35), we can evaluate the force. The third term of (38) gives the statical buoyancy whose amount is equal to the weight of the displaced liquid, and the last term contributes nothing to the force.

Now we take up the first term of (38). First of all we consider a velocity potential which satisfies the boundary condition on the surface $S$ and becomes zero at $z=0$. As in the translatory motion this potential can be written as $V\psi$, we put

$$\phi = V\psi + X. \quad (39)$$

Then the boundary condition which is satisfied by $X$ on the surface of the body becomes

$$\frac{\partial X}{\partial n} = 0. \quad (40)$$
We can determine the function $X$ by a successive approximation. First we take $X_1$ which has the same form as $p'$ in (10) or (12), though in $X_1$ the function $w_1$ defined by the equation

$$w_1 = -\left(\nabla \phi / \nabla z\right)_{z=0}$$

is introduced in place of $w_0$. Next we give $X_2$ which is due to a singularity distribution on the surface $S$ so as to extinguish the normal velocity on $S$ induced by $X_1$. The next stage of approximation $X_3$ is similar to $X_1$, with $w_2$

$$w_2 = \left(-\partial X_3 / \partial z\right)_{z=0}$$

in place of $w_1$, and so on. We can easily be shown from (10) or (12) that the function $X_1$, $X_2$, $X_3$ . . . . . . are all independent of the present state of the motion of the body, but are determined by the history of the motion which the body has described, and so does the function $X$. After due consideration, we find that the resistance which is directly proportional to the acceleration comes from $V\psi$ alone. We can define the inertial reaction as the resistance which is proportional to the acceleration of the body, and from (36) we obtain

$$R_0 = -\rho V \int \psi \phi \partial \phi / \partial n \, dS.$$  

(43)

The force coming from the second term of (38) may be transformed into a somewhat simpler form by Gauss' theorem, and it can be shown that the horizontal component is determined only by the history of the motion.

§ 5. Inertial reaction of a prolate spheroid floating on the surface of water

There are a few cases in which we can determine the function $\psi$ directly. One example is a prolate spheroid, including a sphere and a circular cylinder, with its axis on the free surface.

We use the spheroidal coordinates such as

$$x = k \mu \nu, \quad y = k \nu' \left(\mu^2 - 1\right) \left(1 - \nu^2\right) \cos \varphi, \quad z = k \nu' \left(\mu^2 - 1\right) \left(1 - \nu^2\right) \sin \varphi.$$  

(44)

Denoting the eccentricity of the ellipse $\mu = \mu_0$ by $e$, major axis by $2a$, and minor axis by $2c$, we have

$$k = ae, \quad \mu_0 = 1/e, \quad k^2 \mu_0^2 - 1 = c.$$  

(45)

We may put $k=1$ for simplicity without injuring generality. The velocity potential, which becomes zero at $z=0$, takes the form,

$$\psi = \sum \sum A_n^m C_n^m (\mu) P_n^m (\nu) \sin m\varphi,$$  

(46)

where $P_n^m$ and $C_n^m$ are the associated Legendre functions of the first and second kinds. For the translation in $x$ direction the boundary condition on the spheroid may be written as

$$(-\partial \psi / \partial \mu)_{\mu = \mu_0} = (4/\pi) \nu \left(\sin \varphi + \sin 3 \varphi / 3 + \sin 5 \varphi / 5 + \ldots\right),$$  

(47)

from which the coefficients $A_n^m$ are determined. For odd $m$ and even $n$,

$$A_n^m = -\frac{(2n)!(n-m)!(m+1)!}{2^{n+1} C_n^m (\mu_0) m(n+m)!(m+1)!} \sum_{r=0}^{n-m} (-1)^r \frac{2^{2r}(n-m-2r+1)(2n-2r)!}{r!(n-r)!}\left[\frac{n-m+1}{2} - r\right] \left[\frac{n}{2} - r + 1\right]$$  

(48)

where dot means the derivative of $C_n^m$ and otherwise

$$A_n^m = 0.$$  

For the translation in $y$ direction, the boundary condition on the spheroid is

$$(-\partial \psi / \partial \nu)_{\mu = \mu_0} = (2/\pi) \mu_0 \nu / \sqrt{1 - \nu^2} / \sqrt{\nu^2 - 1} \cdot \left[\frac{4}{3} \sin 2 \varphi + \frac{8}{15} \sin 4 \varphi + \ldots\right],$$  

(49)
from which the coefficients, when \( m \) and \( n \) are both even, become

\[
A_n^m = - \frac{\mu_0 m (2n+1)(n-m)!(m+2)!}{2^{n+1} Q_n^m (\mu_0) \sqrt{\mu_0^2 - 1} (m^2 - 1)(m+n)! \left( \frac{m}{2} + 1 \right)!} \sum_{r=0}^{n-m} (-1)^r 2^r (2n-2r)! r! (n-r)! \left( \frac{n-m}{2} - r \right)! \left( \frac{n+3}{2} - r \right)! ,
\]

and otherwise

\[
A_n^m = 0.
\]

In the last we consider the turning about the vertical axis through its centre. In this case the velocity potential is \( \omega \phi \) and the boundary condition on the spheroid becomes

\[
\left( -\frac{\partial \phi}{\partial \mu} \right)_{\mu=0} = \frac{2}{n} \left( \frac{4}{3} \sin 2\varphi + \frac{8}{15} \sin 4\varphi + \ldots \right) .
\]

From the above relation the coefficients \( A_n^m \) are determined as

\[
A_n^m = - \frac{m (2n+1)(n-m)!(m+2)!}{2^{n+1} Q_n^m (\mu_0) \sqrt{\mu_0^2 - 1} (m^2 - 1)(n+m)! \left( \frac{m}{2} + 1 \right)!} \sum_{r=0}^{n-m-1} (-1)^r 2^r (n-m-2r+1)(2n-2r)! r! (n-m+1 \cdot n+3)! \left( \frac{n-2r}{2} \right)! \left( \frac{n+3}{2} - r \right)! ,
\]

for even \( m \) and odd \( n \), and otherwise

\[
A_n^m = 0.
\]

The inertial reaction is expressed by the inertia coefficient \( \Lambda \), which is the ratio of the force given by (43) to the mass inertia of the displaced liquid for the translation or the ratio of the resisting couple to the moment of inertia of the displaced liquid for the rotation. In the case of the longitudinal translation, the inertia coefficient is given as follows.

\[
K_1 = - \frac{3 \varepsilon}{2} \sum_{n=1}^{\infty} \frac{(n+m)!}{(2n+1)(n-m)!} Q_n^m (1/\varepsilon) \left( \frac{m}{2} \right)! A_n^m .
\]

For the lateral translation

\[
K_2 = - \frac{3 \varepsilon^2}{2} \sum_{n=1}^{\infty} \frac{(n+m)!}{(2n+1)(n-m)!} Q_n^m (1/\varepsilon) Q_n^m (1/\varepsilon) (A_n^m)^2 ,
\]

and for the horizontal turning

\[
K' = - \frac{15 \varepsilon^3}{2(2-\varepsilon^2)} \sum_{n=1}^{\infty} \frac{(n+m)!}{(2n+1)(n-m)!} Q_n^m (1/\varepsilon) Q_n^m (1/\varepsilon) (A_n^m)^3 .
\]

The inertia coefficient of a spheroid in the unbounded liquid is given by Lamb [2], while it is reduced considerably for a floating solid. According to the numerical calculations of the above formulae, the inertia coefficient of a floating sphere is about 0.54 of that in the unbounded liquid.

In the longitudinal translation of a prolate spheroid this ratio is about 0.43 for length beam ratio 5, and about 0.36 for length beam ratio 10. In the lateral translation these become about 0.43 and 0.41 respectively, and in the horizontal turning these are about 0.46 and 0.42 respectively.

\section{6. Horizontal motion of a submerged circular cylinder and a sphere}

As an application of the fundamental formula, we consider a circular cylinder of radius \( a \) starting horizontally at a definite depth \( f \) under the surface of water in the direction perpendicular to its axis.
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For the first approximation, the velocity potential \( \phi_0 \) which satisfies the boundary condition on the cylinder is a two-dimensional horizontal doublet.

\[
\phi_0 = Va^2 \frac{(x-x_0)}{\sqrt{(x-x_0)^2 + (z+f)^2}}.
\]

(56)

from which

\[
\phi_1 = 2a^2 \int_0^\infty V(\tau) \, d\tau \int_0^\infty e^{-k(\tau-\tau')} \sin \kappa \{x-x_0(\tau')\} \sin \{\sqrt{\frac{g}{\kappa}} (t-\tau')\} \sqrt{\frac{g}{\kappa}} \, d\kappa.
\]

(57)

Putting

\[
A = \int_0^\infty V(\tau) \cos \kappa \{x_0(t) - x_0(\tau')\} \sin \{\sqrt{\frac{g}{\kappa}} (t-\tau')\} \, d\tau,
\]

\[
B = \int_0^\infty V(\tau) \sin \kappa \{x_0(t) - x_0(\tau')\} \sin \{\sqrt{\frac{g}{\kappa}} (t-\tau')\} \, d\tau,
\]

(58)

we have

\[
\phi_1 = 2a^2 \int_0^\infty e^{-k(\tau-\tau')} \sqrt{\frac{g}{\kappa}} \left( A \sin \kappa x_1 + B \cos \kappa x_1 \right) \, d\kappa.
\]

(59)

Next we consider the second approximation by adding \( \phi_2 \) so as to satisfy the boundary condition on the cylinder. Putting

\[
x = r \cos \theta, \quad z + f = r \sin \theta,
\]

(60)

\[
\phi_2 = -Va^2 \int_0^\infty \exp \left( -2 \kappa f + \kappa a^2 \sin \theta / r \right) \sin \left( \kappa a^2 \cos \theta / r \right) \, d\kappa
\]

\[
+ 2a^2 \int_0^\infty \exp \left( -2 \kappa f + \kappa a^2 \sin \theta / r \right) \left\{ A \sin \left( \kappa a^2 \cos \theta / r \right) + B \cos \left( \kappa a^2 \cos \theta / r \right) \right\} \, d\kappa
\]

\[
= \psi_1 + \psi_2.
\]

(61)

The resistance of the cylinder is

\[
R = \int_0^{2\pi} \rho \cos \theta \, d\theta.
\]

(62)

The resistance proportional to the acceleration is

\[
R_a = \pi \rho a^2 \tilde{V}(1 - a^2/2f^2).
\]

(63)

As the contribution of \( \psi_1 \) is involved in \( R_a \), the remaining part of the resistance comes from the pressure

\[
p_1 |_p = -V \frac{\partial}{\partial x_1} (\phi_1 + \phi_2) - \frac{1}{a^2} \frac{\partial \phi_0}{\partial \theta} \frac{\partial (\phi_1 + \phi_2)}{\partial \theta} + \frac{\partial (\phi_1 + \phi_2)}{\partial t},
\]

(64)

from which

\[
R_1 = \int_0^{2\pi} \rho \cos \theta \, d\theta
\]

\[
= 4\pi \rho a t \int_0^\infty e^{-2\kappa f} \left( \kappa V B + \partial \frac{\partial}{\partial t} A \right) \sqrt{\frac{g}{\kappa}} \, d\kappa
\]

\[
= 4\pi \rho \, ga \int_0^\infty V(\tau) \, d\tau \int_0^\infty e^{-2\kappa f} \cos \kappa \{x_0(t) - x_0(\tau')\} \cos \left[ \sqrt{\frac{g}{\kappa}} (t - \tau') \right] \kappa^2 \, d\kappa.
\]

(65)

Writing

\[
t - \tau = u, \quad x_0(t) - x_0(\tau) = X(u)
\]

(66)

and using the expansion

\[
\int_0^\infty e^{-2\kappa f} \cos \left( \kappa X(\cos \sqrt{\kappa} u) \right) \kappa^2 \, d\kappa
\]

\[
= \sum_{n=0}^\infty \frac{(-1)^n g^{\frac{3}{2}n} u^{2n}}{(2n)!} \frac{(n+2)1 \cos \left[ \left( \frac{(n+3) \tan^{-1}(X/2f)}{4f^2 + X^2} \right)^{n+1}/2 \right]}{(4f^2 + X^2)^{\frac{n+1}{2}}},
\]

(67)
we have
\[ R_1 = 4\pi \rho g a^4 \sum_{n=0}^{\infty} \frac{(-1)^n(n+2)!}{(2n)!} \frac{g^n}{(4f^2+X^2)^{n+3/2}} u^{2n} du. \]  
(68)

From the above we can compute the resistance of the cylinder soon after it has started. As the simplest example, consider a circular cylinder starting suddenly from rest with a uniform speed \( v \).

In this case, putting
\[ a_n(x) = f \cos(n+3) \sin \alpha \cos \theta \, da, \]  
(69)
we find
\[ R_1 = \pi \rho g a^4 \sum_{n=0}^{\infty} \frac{(n+2)!}{(2n)!} \left( \frac{-2gf}{c^2} \right)^n S_n \left( \tan^{-1} \frac{ct}{2f} \right). \]  
(70)

The equation (70) is suitable for small \( t \), but as \( t \) increases the convergence of the series becomes worse. For the value long after the beginning of the motion, it is convenient for us to perform the integral about \( \tau \) at first and then integrate with respect to \( \kappa \) by means of the principle of stationary phase. This being done, we have the final results as follows.
\[ R_1 = 4\pi^2 \rho g a^4 \frac{a^4}{2} e^{-2g/ct} \sin \left( \frac{1}{4} (gt/c - \pi) \right). \]  
(71)

The first term of (71) is the wave resistance of a cylinder in uniform motion and the second term represents the deviation from it.

The resistance of a cylinder soon after it has started is shown in curves of the ratio to the steady value in the uniform motion for two cases in which \( c/\sqrt{g}f = 1 \) and \( c/\sqrt{g}f = 2 \). We can see the remarkable difference between two cases. In the case of \( c/\sqrt{g}f = 1 \), the increase of resistance is rather slow, but in the case of \( c/\sqrt{g}f = 2 \), the resistance increases rapidly after the starting and exceeds the steady value in a moment.

A similar method may be applied to the motion of a submerged sphere. The first approximation becomes
\[ \phi_0 = V a^3(x - x_0) \left( x^2 + y^2 + (x + f)^2 \right)^{1/2}, \]  
(72)
from which
\[ \phi_1 = \frac{i a^3}{2} \frac{d}{d \tau} \int_{0}^{\infty} \frac{V(\tau) \cos \theta \, d\theta}{-x} \, \exp \left[ -\kappa (f + x) + i \kappa \left( x - x_0(\tau) \right) \cos \theta + y \sin \theta \right] \times \sin \left( \frac{r}{\sqrt{g} \kappa} (t - \tau) \right) \frac{r}{\sqrt{g} \kappa} \, d\kappa. \]  
(73)

Taking up to the second approximation, we obtain the resistance which is proportional to the acceleration as follows.
\[ R_2 = \frac{\pi}{3} \rho a^3 V (2a^3/f^3). \]  
(74)

Putting
\[ C = \int_{0}^{\infty} V(\tau) \cos \left[ \kappa \left( x_0(t) - x_0(\tau) \right) \cos \theta \right] \sin \left[ \frac{r}{\sqrt{g} \kappa} (t - \tau) \right] \, d\tau, \]  
(75)
\[ D = \int_{0}^{\infty} V(\tau) \sin \left[ \kappa \left( x_0(t) - x_0(\tau) \right) \cos \theta \right] \sin \left[ \frac{r}{\sqrt{g} \kappa} (t - \tau) \right] \, d\tau, \]  
(75)
we have the remainder of the resistance.

\[ R_1 = 2 \rho \sigma \int_0^\infty \int_0^\infty e^{-2\xi \eta} \kappa V d\kappa \theta d\theta, \]  

(76)

where \( J_0, J_2 \) are Bessel functions of the first kind.

Though the evaluation of (77) is more troublesome than (65), it seems that the results show a similar feature to the cylinder.

§ 7. Restricted water depth

Formulae hitherto introduced are valid when the depth of water is infinite, while in the shallow water we should take into account the effect of the sea bottom.

At the bottom the vertical velocity vanishes, and with uniform depth \( h \) the boundary condition is given as

\[ \partial \phi / \partial z = 0 \quad \text{at} \quad z = -h. \]  

(78)

Then the second function \( \Phi_1 \) which satisfies the above condition takes the following form in place of (12).

\[ \Phi_1 = \frac{1}{4 \pi^2} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \exp (i\kappa \xi) \cosh \{\kappa (z+h)\} \times \sin \{\sqrt{\kappa \eta} \tan \kappa h (\tau - \tau')\} \sinh \kappa h \cosh \kappa h \frac{1}{\sqrt{\kappa \eta}} d\kappa. \]  

(79)

Needless to say, the function \( \phi_0 \) should also satisfy the condition at the bottom.

Postscript:

This paper was first presented to the spring meeting of the Society of N.A. of Japan held in April 1950, but the publication has been delayed for four years. Quite recently the author has a chance of reading Prof. Havelock's papers in which similar problems are worked. [3], [4], [5]. The papers [3] and [4] correspond to §6 of the present paper, while [5] corresponds to §5. For a circular cylinder, Havelock's results expressed by equations (15) and (23) of his paper [3] differ from (65) and (71) of the present one. Consequently the numerical results shown in curves are also different. The reason lies in the fact that Havelock left out the term \( \rho \partial \phi / \partial t \) in the evaluation of the resistance. In fact, we shall have the same result, if we omit \( \partial A / \partial t \) in the equation (65).

References:

[1] J. H. Michell, Phil. Mag. 45 (1898) p. 113