(昭和 41 年 11 月造船協会秋季講演会において講演)

On the Transverse Strength of Ore Carriers

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Summary

When a large ore carrier is floating in the full load condition, a large amount of ore load is acting on the double bottom downwards and a large amount of buoyancy is acting on the wing tank upwards. In this case, it is expected that the relative deflection between the side shell and the longitudinal bulkhead increases and induces large stresses in the transverse rings. According to this way of thinking, the authors try to calculate the transverse strength of ore carriers including the relative deflection and the stresses induced by it, and then try to make some considerations upon the effects of shear rigidity of swash bulkheads and of transverse rings.

From these calculations following things are found :

- (1) Induced stresses due to relative deflection between side shells and longitudinal bulkheads can not be neglected.
- (2) For the purpose of making relative deflection small, it is not effective to set swash bulkheads of reinforced transverse ring type, but is effective to set swash bulkheads of perforated plane type.

1. Introduction

When a large ore carrier is floating in the full load condition, a large amount of ore load is acting on the double bottom downwards and a large amount of buoyancy is acting on the wing tank upwards. In this case it is expected that the relative deflection between the side shell and the longitudinal bulkhead increases and induces large stresses in the transverse rings. The longer the length of wing tank becomes, the larger those induced stresses will become. It is, therefore, necessary to take into consideration not only stresses due to external forces such as ore load and buoyancy but also stresses induced by relative deflection at the same time, when the transverse strength of large ore carriers is examined. As to tankers, this problem has been already discussed^{1),2)}.

In this paper, according to this way of thinking, authors try to estimate the followings :

- (1) the amount of these induced stresses in transverse rings,
- (2) the effects of shear rigidity of swash bulkheads and of transverse rings on relative deflection.

2. Process of Calculation

2.1 General Considerations

In order to make an exact calculation of three dimensional strength of ore carriers, it is necessary to decide the unknown reaction forces between longitudinal and transverse members on the condition to make the deformation of longitudinal members equal to that of transverse members on the conjunction points. In this paper the authors consider this problem as follows :

- (1) Deformation of Longitudinal Members

Share the external forces reasonably among longitudinal members, then calculate the deformation

原稿受付 昭和 41 年 7 月 10 日

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of those members as composed parallel beams with spring actions of transverse rings, watertight bulkheads, swash bulkheads, deck plates and bottom plates. This calculation is carried out over all the holds.

(2) Deformation of Transverse Members

Consider, at first, the transverse ring supported by longitudinal members, and then calculate the deformation of transverse members subjected to external loads of 1-transverse spacing.

Next give the forced displacement obtained in section (1) to each support point, and then calculate the deformation of those members. The final deformation of the transverse ring is obtained by superposing above two deformations.

(3) External Loads

Take ore load, hull weight and buoyancy as the loads for longitudinal members. Take ore load and statical water pressure in still water as the loads for transverse members.

2.2 Method of Calculation of Transverse Rings

(1) Deformation of Transverse Rings

Transverse ring deforms as shown in Fig.1 (a), when it is in the full load condition in still water. This state is the superposition of two states; one is the state where transverse ring is subjected to external loads with no displacement at all the support points, as shown in Fig.1 (b), and the other is that where forced displacements equal to those of longitudinal member are given to each support point one by one as shown in Fig.1 (c), (d), (e), (f).

Fig.1 Deformation of Transverse Ring

Forced displacements of each support point are influenced by the rigidity and the length of span of the corresponding longitudinal members.

Let us consider next longitudinal member corresponding to the support point A, namely, the center girder. It seems to have hardly influence on the transverse members except those near the watertight bulkheads, when the distance of watertight bulkheads is long.

Let us consider next the longitudinal member corresponding to the support point E, namely, the deck plate with effective breadth of the side shell and longitudinal bulkhead. Usually there are several hatches between two transverse bulkheads when those distance is long. This member, therefore, can be considered as a continuous beam with support points at deck struts and transverse bulkheads, so that it has much ability to support transverse members.

The above mentioned things seem to be reasonable from the results of an actual ship test and a model test which have been done by authors before.

Consequently we assume the following boundary conditions for transverse rings.

(a) The reaction forces from center girder is zero and $\theta_A = 0$ at the point A.

Fig.2 Condition of Calculation of Transverse Ring

- (b) The horizontal displacement at the point E is zero.
- (c) The displacement at the point B is y_B .
- (d) The displacement at the point C is y_C .

Thus the condition of calculation of transverse rings becomes as shown in Fig.2 (a), (b), (c).

(2) Calculation of transverse rings

From preceding considerations, we can say that it is enough for investigating the strength of transverse rings to carry out two cases of calculation; one is the case shown in Fig.2 (a), and the other is the case shown in Fig.2 (b) or (c). Ordinary rigid frame theory³⁾ is used for this calculation. The assumptions in this calculation are as follows:

- (a) Both bending deflection and shear deflection are taken into consideration, but the elongation of structural members is neglected.
- (b) The effect of variable cross-sections of structural members are taken into consideration by using the conception of spanpoint.
- (c) The neutral axis of structural members lies at the middle of both neutral axes when each neutral axis does not coincide at the joint.
- (d) The effective breadth for bottom plates, side shells, longitudinal bulkheads and deck plates is calculated by using H. A. Shades theory.
- (e) The cross sectional area of web plate is considered as the effective cross sectional area for shear deflection.

2.3 Method of Calculation of Relative Deflection

(1) Division of Structures

In discussing the relative deflection between longitudinal bulkheads and side shells, we divide the longitudinal members of a ship into two parts; one is the side shell part, and the other is the longitudinal bulkhead part (Fig.3). The coordinates are taken as shown in Fig. 3. Let the positive direction of Y axis be downward. As to the deflection of structural members, we consider the elongation of the neutral axes of structural members.

(2) Spring Action of Transverse Rings

When the deflections of the side shell part and the longitudinal bulkhead part are different, transverse rings act as springs, so that reaction forces act between those parts in proportion to their relative deflection.

Denote by K_{11} and K_{22} the forces which are necessary to bring about unit displacement to the point C and the point B respectively, and then denote by K_{12} and K_{21} the reaction forces at the point B and the point C respectively. From the equilibrium of forces we obtain $K_{11} - K_{12} = 0$, $K_{22} - K_{21} = 0$. By using reciprocal theorem, we obtain $K_{12} = K_{21}$. Hence the force P_{12} and the force P_{21} which are necessary to bring about the displacement y_1 to the point C and the displacement y_2 to the point B respectively can be represented in the following forms.

$$\left. \begin{aligned} P_{12} &= K_{11}y_1 - K_{21}y_2 = K_{12}(y_1 - y_2) \\ P_{21} &= -K_{12}y_1 + K_{22}y_2 = -K_{12}(y_1 - y_2) \end{aligned} \right\} \quad (1.1)$$

These forces mean the spring action of transverse rings on longitudinal members. For the sake

Fig. 3 Longl. Members

of simplicity we assume that these forces act between transverse rings as uniformly distributed forces.

(3) Spring Action of Deck Plates and Bottom Plates

Shearing forces act in deck plates and bottom plates in proportion to the difference of longitudinal displacements of two longitudinal members shown in Fig. 4.

Denote by q_{12U} , q_{12L} the shearing forces acting in deck plates and bottom plates in the x -direction. Taking into account of the shear deformation only, we obtain,

$$\left. \begin{aligned} q_{12U} &= G \frac{t_U}{b_U} (u_{1U} - u_{2U}) \\ q_{12L} &= G \frac{t_L}{b_L} (u_{1L} - u_{2L}) \end{aligned} \right\} \quad (1.2)$$

where u_{1U} and u_{1L} are the longitudinal displacements at the deck line and the bottom line of the side shell part respectively, and u_{2U} and u_{2L} are those of the longitudinal bulkhead part respectively.

(4) Equation of Equilibrium

Let us consider an element cut out as shown in Fig. 4. Considering the equilibrium of all the forces and moments, we obtain the following equations:

$$\left. \begin{aligned} \frac{dS_1}{dx} &= p_{12} - f_1 \\ \frac{dM_1}{dx} &= S_1 - e_{1U}q_{12U} + e_{1L}q_{12L} \\ \frac{dN_1}{dx} &= q_{12U} + q_{12L} \\ \frac{dS_2}{dx} &= -p_{21} - f_2 \\ \frac{dM_2}{dx} &= S_2 + e_{2U}q_{12U} - e_{1L}q_{12L} \\ N_2 &= -N_1 \end{aligned} \right\} \quad (1.3)$$

Fig. 4 Equilibrium of Elements

Fig. 5 Boundary Condition

The stress resultants expressed in terms of the displacement components y_1 , y_2 , φ_1 , φ_2 , u_1 and u_2 are all given as follows:

$$\left. \begin{aligned} S_1 &= GA_{s1} \left(\frac{dy_1}{dx} - \varphi_1 \right) \\ M_1 &= -EI_1 \frac{d\varphi_1}{dx} \\ N_1 &= EA_{N1} \frac{du_1}{dx} \\ S_2 &= GA_{s2} \left(\frac{dy_2}{dx} - \varphi_2 \right) \\ M_2 &= -EI_2 \frac{d\varphi_2}{dx} \\ N_2 &= -N_1 \end{aligned} \right\} \quad (1.4)$$

Substituting (1.1), (1.2) and (1.4) into (1.3), we can get the fundamental differential equation shown in Appendix 1.

(5) Boundary Condition

The authors consider the following boundary conditions for the fundamental differential equation:

- (a) The side shell is supported at the engine room front bulkhead and the fore peak bulkhead. In other words, the longitudinal

bulkhead is hanged by the transverse bulkhead at those points.

- (b) Each longitudinal member is connected at the position of transverse bulkheads (transverse water-tight bulkheads and swash bulkheads), which deform by shearing forces due to the relative deflection.
- (c) All the normal forces, deflections, rotations of cross section and elongation of members are continuous at the position of transverse bulkheads in wing tanks.
- (6) Numerical Calculation

Numerical Calculation of the fundamental differential equation is carried out by the electronic computer. The method of calculation is shown in Appendix 2.

3. Examples of Numerical Calculations

Using the method of calculation as shown in Chapter 2, we calculated the transverse strength of a large ore carrier. The results are presented in this Chapter. The load condition used in these examples is the full load condition in still water.

3.1 Shear Deformation of Wing Tanks

The values of rigidity of structural members and loads are listed in Table 1, and the results are shown in Fig.6. From these results the following things are found :

Table-1. Structural & Load Elements (I)

Structural Elements			Load Elements (Kg/cm)					
Elements	Values		Load	Member	Side Shell Part	Long ^L B ^{HD} Part	Weight	Buoyancy
			Cond.	Region	Weight	Buoyancy		
			Even Loading	No. 1	82.4	5936	19488	1313.2
Shear Rigidity				2	"	"	"	"
K T.W.T B ^{HD}	kg/cm	2.151x10 ⁶		3	"	"	"	"
K ORD. TRANS	"	0.1872 "		4	"	"	"	"
				5	"	"	"	"
				6	"	"	"	"
Shear Rigidity				7	"	"	"	"
Side Shell Part	kg	2.336x10 ⁹		8	"	"	"	"
Long ^L B ^{HD} Part	"	1.580 "		9	"	"	"	"
Bending Rigidity				10	"	"	"	"
Side Shell Part	kg/cm ²	7.302x10 ¹⁵	Eccentric Loading	No. 1	82.4	6369	1910.7	1408.8
Long ^L B ^{HD} Part	"	12.24 "		2	"	6332	28040	1400.7
Tenside Rigidity				3	"	6222	14647	1376.5
Side Shell Part	kg	16.07x10 ⁹		4	"	6112	28040	1352.1
Long ^L B ^{HD} Part	"	2440 "		5	"	5915	12180	1308.6
				6	"	5792	10358	1281.2
β_U	—	3264x10 ³		7	"	5682	11037	1256.9
β_L	—	2475 "		8	"	5572	14477	1232.6
e_{1U}/D	—	0.5303		9	"	5462	27704	1208.3
e_{2U}/D	—	0.5977		10	"	5352	14477	1184.0

- (a) The ratio of the shear deformation of the transverse ring in the middle part of a wing tank to that of the transversverse water-tight bulkhead is from 1.6 to 1.7 in the case of this calculation. The shear deformation of transverse watertight bulkhead, therefore, must be taken into consideration, when relative deflection is calculated in way of cutting out 1 tank part.
- (b) Relative deflection becomes about 40 percent larger at the position of transverse rings in the middle part of a wing tank and 20 percent larger at the position of watertight bulkheads in eccentric loading than in uniform loading.

3.2 Bending Moment Diagram and Shearing Force Diagram

The values of rigidity and scantlings of structural members and loads are listed in Table 2. The

Fig.6 Shearing Deformation

Table-2. Structural & Load Elements (2)

Item		Members							
Elements	Unit	AB	BC	CD	DE	EF	FG	GB	DG
Length of Spar	cm	608	947	735	870	635	868	745	797
Effective Breadth of Plate	"	240	210	176	191	154	154	206	—
Shear Rigidity	10^9 kg	0.3279×10^9	0.1478×10^9	0.1082×10^9	0.0929×10^9	0.0695×10^9	0.0929×10^9	0.0929×10^9	0.0703×10^9
Bending Rigidity	10^{13} kg-cm ²	7.606×10^{13}	0.5968×10^{13}	0.4133×10^{13}	0.4017×10^{13}	0.1459×10^{13}	0.3158×10^{13}	0.3652×10^{13}	0.0761×10^{13}
Load acting on Left End	kg/cm	-533	450	413	131	0	0	227	0
Load acting on Right End	"	-533	450	131	0	0	227	441	0

Note: Load acting inward is taken as a positive.

results are shown in Fig.7 (a), (b). They correspond to the transverse ring in the middle part of a wing tank in the eccentric load condition. From these results the following things are found:

- (1) Relative deflection has great influence on bending moments and shearing forces.
- (2) The shapes of bending moment diagram and shearing force diagram vary remarkably according to the magnitude of the relative deflection.

3.3 Stresses in Joints of Structural Members

In investigating the transverse strength we need ultimately the stress distribution of transverse rings.

It is easy to know normal stresses and shearing stresses in the middle part of structural members, but is very difficult to know stresses in joints of structural members. There are many methods for calculating the stresses in joints, for example, Osgood's theory⁴⁾, Terada's theory⁵⁾, etc. But these theories do not consider distributed loads on the joints directly and, moreover, are made for compara-

Fig.7 Results of Calculation of Transverse Ring
—Full Load Condition in Still Water (Eccentric Loading)—

tively simple joints. So there are many problems when we apply these theories on actual ships. We think, therefore, that further researches on his problem should be done hereafter.

Taking into consideration these problems in this paper, we developed Terada's theory and made an approximate method of calculation of stresses in joints.

The stresses in joints estimated by this method are shown in Fig.8(a), (b). From these results we know that relative deflection has great influence on the stresses, too.

Fig.8 Stress Distribution at Joints (Full Load Condition in Still Water)

4. Some Considerations on Relative Deflection

We knew from the results of numerical calculations shown in Chapter 3 that stresses induced by relative deflection between side shells and longitudinal bulkheads has great influence on the strength of transverse rings. Consequently it becomes necessary to make the relative deflection small in order to preserve the transverse strength. In this Chapter the effect of shear rigidity of swash bulkheads and transverse rings on relative deflection are discussed.

4.1 Effect of Shear Rigidity of Swash Bulkheads

Swash bulkheads are set in wing tanks at some intervals in order to restrain the motion of inner fluid and preserve the transverse strength. There are two types of swash bulkheads; one is reinforced ordinary transverse ring what is called strong transverse ring type, and the other is what is called

Table - 3. Shear Rigidity of a Swash Bulkhead

Case No.	$\frac{K_{SWASH\ BHD}}{K_{W.T.\ BHD}}$	$\frac{K_{SWASH\ BHD}}{K_{ORD.T.\ RING}}$	Remarks
1.	0.0875	[1.000]	No Swash Bulkhead
2.	0.175	[2.000]	Strong Transverse Ring Type
3.	0.262	[3.000]	Strong Transverse Ring Type
4.	0.750	[8.571]	Perforated Plane Type
5.	1.000	[11.43]	Water Tight Bulkhead

Fig. 9 Effects of Shear Rigidity of Swash Bulkheads

Fig. 10 Effects of Shear Rigidity of Ordinary Trans. Rings.

Table - 4 Shear Rigidity of an Ord. T. Ring.

Case No.	$\frac{K(\text{ORD. T. RING})}{K(\text{ORD. T. RING})_0}$	Remarks.
1	1.00	No. Swash B ^{HD}
6	1.25	"
7	1.50	"
8	1.70	"
9	2.00	"

Note: $K(\text{ORD. T. RING})_0 = 0.1872 \times 10^6 \text{ Kg/cm}$

perforated plane type which has many small holes. In order to examine the effect of swash bulkheads upon relative deflection, some numerical calculations were done in such condition as shown in Table 3, where the shear rigidity of swash bulkheads is varied systematically.

The results are given in Fig. 9. In Fig. 9 the Case No. 5 corresponds to the type which was used usually before long tanks are adopted. From these results we can observe that perforated plane type is effective for the purpose of preserving the transverse strength.

4.2 Effect of Shear Rigidity of Transverse Rings

Let us consider to reinforce ordinary transverse rings for making relative deflection small. For the purpose of investigation this effect, some numerical calculations were done in such condition as shown in Table 4, where the shear rigidity of ordinary transverse rings is varied systematically.

The results are given in Fig. 10.

From these results we can observe the following thing: In order to make the relative deflection of long tanks with no swash bulkhead as small as that of previous wing tanks, it is necessary to reinforce the transverse rings so as to have shear rigidity two or three times as strong.

What we must notice in this case is that reinforcing a transverss ring brings about the increase of bending moments and shearing forces induced by unit relative deflection, though it increases the ultimate strength of the transverse ring itself. Consequently it becomes necessary, ultimately, to estimate the stress distribution in discussing the transverse strength.

5 Conclusion

In this paper the transverse strength of a large ore carrier which is in the full load condition in still water was researched. As the results of this research the followign

things were found :

- (1) Relative deflection between side shell and longitudinal bulkhead has great effect on bending moments, shearing forces and stresses of transverse rings, so it must be considered in discussing the transverse strength.
- (2) About half of the stress in joint of the transverse ring is what induced by relative deflection.
- (3) For the purpose of making relative deflection small, to set swash bulkheads of perforated plane type is effective.

The authors wishes to acknowledge to Mr. Okabe (Nagasaki Shipyard, M.H.I. Co., Ltd.) and Mr. Ooi (Yokohama Shipyard, M.H.I. Co., Ltd.) for their valuable advices and discussion and also to Mr. Hagiwara and Mr. Nakajima (Hiroshima Technical Institute, M.H.I. Co., Ltd.) and others for their kind assistance in this works.

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Appendix-1. Fundamental Differential Equation for Longitudinal Members

Substituting (1.1), (1.2) and (1.4) to (1.3), we obtain following formula as the fundamental differential equation for this problem.

$$\left. \begin{aligned}
 &GA_{s1} \frac{d}{dx} \left(\frac{dy_1}{dx} - \varphi_1 \right) = k(y_1 - y_2) - f_1 \\
 &-EI_1 \frac{d^2 \varphi_1}{dx^2} = GA_{s1} \left(\frac{dy_1}{dx} - \varphi_1 \right) \\
 &\quad - G \left(\frac{e_{1U} t_U}{b_U} - \frac{e_{1L} t_L}{b_L} \right) u_1 + G \left(\frac{e_{1U} t_U}{b_U} - \frac{e_{1L} t_L}{b_L} \right) u_2 \\
 &\quad - G \left(e_{1U}^2 \frac{t_U}{b_U} + e_{1L}^2 \frac{t_L}{b_L} \right) \varphi_1 + G \left(e_{1U} e_{2U} \frac{t_U}{b_U} + e_{1L} e_{2L} \frac{t_L}{b_L} \right) \varphi_2 \\
 &EA_{N1} \frac{d^2 u_1}{dx^2} = G \left(\frac{t_U}{b_U} + \frac{t_L}{b_L} \right) u_1 - G \left(\frac{t_U}{b_U} + \frac{t_L}{b_L} \right) u_2 \\
 &\quad + G \left(e_{1U} \frac{t_U}{b_U} - e_{1L} \frac{t_L}{b_L} \right) \varphi_1 - G \left(e_{2U} \frac{t_U}{b_U} - e_{2L} \frac{t_L}{b_L} \right) \varphi_2 \\
 &GA_{s2} \frac{d}{dx} \left(\frac{dy_2}{dx} - \varphi_2 \right) = -k(y_1 - y_2) - f_2 \\
 &-EI_2 \frac{d^2 \varphi_2}{dx^2} = GA_{s2} \left(\frac{dy_2}{dx} - \varphi_2 \right) \\
 &\quad + G \left(e_{2U} \frac{t_U}{b_U} - e_{2L} \frac{t_L}{b_L} \right) u_1 - G \left(e_{2U} \frac{t_U}{b_U} - e_{2L} \frac{t_L}{b_L} \right) u_2 \\
 &\quad + G \left(e_{1U} e_{2U} \frac{t_U}{b_U} + e_{1L} e_{2L} \frac{t_L}{b_L} \right) \varphi_1 - G \left(e_{2U}^2 \frac{t_U}{b_U} + e_{2L}^2 \frac{t_L}{b_L} \right) \varphi_2
 \end{aligned} \right\} \quad (A1.1)$$

where,

y : total deflection

φ : angle of rotation of cross section due to bending

u : displacement of neutral axis in the longitudinal direction

A_s : effective cross sectional area for shear deflection

I : moment of inertia of cross section

A_N : effective cross sectional area for elongation

k : spring constant divided by 1-transverse ring spacing

f : external force, downward positive

$e_U, e_L, t_U, t_L, b_U, b_L$: scantlings shown in Fig. 4 (b).

E & G : Young's modulus and modulus in shear and suffix 1 and 2 denote those of side shell part and longitudinal part, respectively.

Appendix-2. Method of Calculation of Fundamental Differential Equation

General Theory... Here we consider a problem of determining numerical solution of a general boundary value problem of linear differential equation in (x_0, x_n) .

$$\frac{dY}{dx} = A(x) \cdot Y + F(x) \quad (A2.1)$$

Practically, with suitable subdivision of whole interval, we can assume that $A(x)$ and $F(x)$ are piecewise constant, then we have

$$\frac{dY}{dx} = A_i \cdot Y + F_i, \quad (A2.2)$$

where A_i is a non-singular constant matrix and F_i is a constant column vector in each subinterval (x_{i-1}, x_i) .

Let $y_i = y(x_i)$, then solution in (x_{i-1}, x_i) has the form

$$Y = e^{A_i(x-x_{i-1})} Y_{i-1} + A_i^{-1}(e^{A_i(x-x_{i-1})} - I) F_i \quad (A2.3)$$

where I is an identity matrix and e^{Ax} is a matrix defined as

$$e^{Ax} = \sum_{j=0}^{\infty} \frac{A^j x^j}{j!} \quad (A2.4)$$

In ordinary engineering problems, boundary conditions can quite generally be formulated at each x_i with suitable matrices E_i, H_i, T_i, B_i and vectors G_i, C_i as

$$\begin{cases} E_i Y_{i-1} = G_i + H_i \tilde{Z}_i \\ Y_{i+0} = T_i Y_{i-1} + B_i Z_i + C_i \end{cases} \quad (i=1, 2, \dots, n-1) \quad (A2.5)$$

$$Y_0 = B_0 Z_0 + C_0 \quad (i=0) \quad (A2.6)$$

$$Y_n = B_n Z_n + C_n \quad (i=n) \quad (A2.7)$$

$$E_n Y_n = G_n \quad (i=n) \quad (A2.8)$$

where Z_i is a possible unknown vector at $x=x_i$ and $\tilde{Z}_i = \begin{pmatrix} \tilde{Z}_{i-1} \\ Z_i \end{pmatrix}$ with $\tilde{Z}_0 = Z_0$ and Z_n is void.

Above formulation of boundary conditions is able to involve the problem with changing the dimension of the solution space.

Now let

$$R_i = e^{A_i(x_i - x_{i-1})} \quad (A2.9)$$

$$K_i = A_i^{-1}(e^{A_i(x_i - x_{i-1})} - I) \quad (A2.10)$$

and let

$$\tilde{B}_i = (T_i \cdot R_i \cdot \tilde{B}_{i-1}, B_i) \quad (A2.11)$$

$$\tilde{C}_i = T_i \cdot R_i \cdot \tilde{C}_{i-1} + T_i \cdot K_i \cdot F_i + \tilde{C}_i \quad (A2.12)$$

with $\tilde{B}_0 = B, \tilde{C}_0 = C_0,$

then we obtain

$$Y_i^{+0} = \tilde{B}_i \tilde{Z}_i + \tilde{C}_i, \quad (A2.13)$$

$$Y_{i+1}^{-0} = R_{i+1}(\tilde{B}_i \cdot \tilde{Z}_i + \tilde{C}_i) + K_{i+1} \cdot F_{i+1} \quad (A2.14)$$

By (A2.5) we have

$$E_{i+1} \cdot Y_{i+1}^{-0} = E_{i+1}(R_{i+1}(\tilde{B}_i \cdot \tilde{Z}_i + \tilde{C}_i) + K_{i+1} \cdot F_{i+1}) \quad (A2.15)$$

$$=(E_{i+1} \cdot R_{i+1} \cdot \tilde{B}_i, 0) \tilde{Z}_{i+1} + E_{i+1} \cdot R_{i+1} \cdot \tilde{C}_i + E_{i+1} \cdot K_{i+1} \cdot F_{i+1} \quad (\text{A } 2.16)$$

$$=G_{i+1} + H_{i+1} \cdot \tilde{Z}_{i+1} \quad (\text{A } 2.17)$$

hence

$$[(E_{i+1} \cdot R_{i+1} \cdot \tilde{B}_i, 0) - H_{i+1}] \cdot \tilde{Z}_{i+1} = G_{i+1} - E_{i+1} \cdot R_{i+1} \cdot \tilde{C}_i - E_{i+1} \cdot K_{i+1} \cdot F_{i+1} \quad (\text{A } 2.18)$$

By solving equation (A 2.18) ($i=0, 1, 2, \dots, n-1$) simultaneously, we have $\tilde{Z}_{n-1}$, i. e. all Z_i ($i=0, 1, \dots, n-1$), then our boundary value problem is changed into an initial value problem and it is solved by (A 2.3).