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Elastic-Plastic Analysis of Framed Structures using the Matrix Method (3rd Report)

—Ultimate Strength with
Consideration of Web Buckling—

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Summary

Many types of framed structures are constructed of deep girders as the main supporting members. These girders are usually composed of deep webs. When such structures are subjected to external loads, local failures, such as buckling of webs or flanges, local yielding of a section or a part of a section etc..., are liable to occur. However, it is often experienced that structures can still carry further loads after local failures have occurred.

In the design of such structures, it is important to get information on the complete behaviour of the structure until final collapse, including the ultimate strength of the structure and the deformations of the members accompanied by different loading conditions.

In this paper the load-deformation relationship before and after local failures (such as buckling and/or plastification) is investigated for a deep girder element subjected to a combined load of shearing force and bending moment. An analytical method for evaluating the ultimate strength of statically indeterminate structures consisting of deep girders is presented. This method is based on the matrix method for nonlinear analysis of framed structures, which was developed in the 1st. and 2nd. Reports of this series.

Examples of analysis are performed on straight deep girders which are fixed at both ends. The results of analyses suggest that the ultimate strength analysis by the new method is important since the plastic analysis predicts sometimes much higher collapse load than that by the method which accounts for the effects of buckling. On the other hand, the elastic analysis estimates the carrying capacity much lower than the actual.

1 Introduction

Many types of framed structures are constructed of deep girders as the main supporting members. These girders usually have deep webs. When such structures are subjected to external loads, local failures, such as buckling of web or flanges, local yielding etc..., are liable to occur. However it is often experienced that structures can still carry further loads after local failures have occurred.

In the design of such structures, it is important to get information on the complete behaviour of the structure until final collapse, including the ultimate strength of the structure and the deformations of the members accompanied by the different loads.

Concerning the ultimate strength of deep girder element, several notable works have been done by which the ultimate strength of an element subjected to shearing force and bending moment can be evaluated with a high degree of accuracy¹⁾. These methods, as they do not account for deformations, are suitable only for statically determinate structures, in which the ratios of internal force components,

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such as the ratios of shearing force to bending moment, do not change at any section.

For statically indeterminate structures when local failures take place in one member or more, the stiffness of the members will change and the ratio of internal force components will also change. Then, not only the ultimate strength but also the load-deformation relationship after local failure is necessary for analysing those types of structures. An example for such analysis is one for non-linear behaviour of framed structures presented by one of the authors²⁾, in which a general method based on the matrix method is presented, introducing the concept of plastic hinge with the aid of the incremental theory of plasticity. Thus the elastic-plastic analysis of three dimensional frames with large deformations is possible taking account of the changes of the internal forces at the plastic hinges.

In this paper the load-deformation relationship before and after local failures is investigated for a deep girder element subjected to a combined load of shearing force and bending moment. An analytical method for evaluating the ultimate strength of statically indeterminate structures consisting of deep girders is presented. This method is based on the matrix method and accounts for the effects of local failures.

The authors consider this work as a step towards the analysis of the actual ship structures and have also the intention of taking account of axial forces.

2 Notations

A_F : Flange area	M_p : Plastic bending moment
A_w : Web area	M_u : Ultimate bending moment
B : Buckling function	M_{wp} : Plastic bending moment of the web
E : Young's modulus	M_{wy} : Yield bending moment of the web
e : Strain	M_y : Yield bending moment
F : Yield function	n : Number of elements in a girder
$\bar{F}$: Column matrix of external forces	R_i : Internal force components
f_s : Shape factor= M_p/M_y	t_w : Web thickness
h : Depth of web	t_f : Flange thickness
I : Sectional moment of inertia about the neutral axis	$\bar{U}$: Column matrix of deformation components
I_f : Sectional moment of inertia of the flanges about their neutral axis	V : Shearing force
I_w : Sectional moment of inertia of the web about its neutral axis	V_{cr} : Critical shearing force
$\bar{K}$: Stiffness matrix in the elastic region	V_p : Plastic shearing force
$\bar{K}_B$: Stiffness matrix after buckling for $V \leq V_{cr}$	V_u : Ultimate shearing force
$\bar{K}_{B1}$: Stiffness matrix after buckling for $V > V_{cr}$	v : Displacement
$\bar{K}_p$: Stiffness matrix in the plastic region	v_s : Shear displacement
L : Length of girder element	α : Angle of max. tensile stress in the web at the moment of yielding
M : Acting bending moment	γ : Shear factor
M_w : Bending moment acting on the web alone	δ : For increment
M_c : The bending moment which can be carried by the section when $V = V_u$	ν : Poisson's ratio
M_{cr} : Critical bending moment	σ_1 : Max. tensile stress at the moment of yielding
M_{crw} : Critical bending moment of the web	σ_y : Axial yield stress
M_{fp} : Plastic bending moment of the flanges	$\sigma_{t\alpha}$: Tensile stress at the angle α
	τ_{wy} : Shear yield stress
	λ : Aspect ratio= L/h

3 Basic assumptions

3.1 The following assumptions are made in developing the method of analysis for the behaviour of structural parts in different stages of loading.

- (1) Structural members are homogeneous and prismatic.
- (2) Distributed loads are replaced by a statically equivalent set of concentrated loads.
- (3) Proportional loading is to be applied.

- (4) The effect of each force is cumulative.
- (5) Material is elastic perfectly plastic.
- (6) The web of any structural element will buckle when the buckling function B is equal to zero, that is

$$B(R_i) = 0$$

- (7) Flanges are strong enough to withstand buckling.
- (8) General yielding of a section must be attained, when the yield function F is equal to zero, that is

$$F(R_i) = 0$$

and, at the yielded section, the following equation is satisfied in the course of loading.

$$\delta F = 0$$

- (9) The direction of forces applied are coincidental with those of the principal axes of the members.
- (10) Structures are not of the type which fail due to brittle fracture or loss of overall stability before plastic collapse occurs.

3.2 Girder element and acting forces

Deep girders are usually constructed of deep webs bounded by flanges. To increase the buckling strength of the web, stiffeners are usually fitted to it. These stiffeners are mostly fitted normal to the centerline of the girder, and subdivide the web into small panels.

As buckling may take place between two stiffeners, the part between each two stiffeners will be taken as an element. This is composed of a web surrounded by two flanges and two vertical stiffeners, Fig. 1.

The loading system including concentrated and distributed forces will be replaced by an equivalent system of concentrated forces acting at the ends of the elements, i.e. at the location of the stiffeners.

Fig. 1 Girder element and acting forces

For each element, shearing force V and bending moment M are acting at each end. The positive directions are as shown in Fig. 1.

4 Plastic, buckling and ultimate strength interaction curves

4.1 Plastic strength

When the web does not buckle, or when buckling is not accounted for, the girder element will continue to carry loads until it yields at one or at both ends. The values of the plastic shearing strength, plastic bending moment and the plastic condition for combined loading of shearing force and bending moment, are presented in Appendix 1.

4.2 Buckling strength

There are several methods for evaluating the buckling condition of a girder element under combined loading of shearing force and bending moment^{1),3)}. The method which is adopted in this analysis is presented in Appendix 1.

4.3 Ultimate strength with consideration of postbuckling strength of the web

Akita and Fujii¹⁾ presented the ultimate strength yield condition of girder element after web has buckled as shown in Appendix 1.

4.4 Ultimate strength without consideration of postbuckling strength of the web

Values of the ultimate shearing strength, the ultimate bending moment and the ultimate strength

Fig. 2 Plastic, buckling and ultimate strength interaction curves

in Fig. 3, and the relation $V_{cr} \leq V_u$ and $M_{cr} \leq M_u$ will always hold.

5 Maximum strength analysis of deep girders

In the course of loading, structural elements will behave in different ways, according to the local failures which may take place. These behaviors will be described in the following.

5.1 Elastic behavior of girder element before local failure

Increasing the load from zero, in the elastic range before local failure the girder element obeys the matrix equation

$$\delta \bar{F} = \bar{K} \delta \bar{U} \quad (1)$$

The well known form of $\bar{K}$, as presented in ref. (4), neglects the effects of shear deformation. As shear deformation can not be neglected in deep girders, a stiffness matrix was developed taking account of its effect and the explicit form of this matrix is shown in Appendix 2. In developing this matrix, shear deflection was considered, but shear slope was not included as it has a local effect and has no contribution to the rotation of the overall cross-section.

This matrix will hold until the element satisfies the buckling condition (web buckling), from point 0 till point 1 in Fig. 2.

5.2 Post-buckling behavior of the girder element

Increasing the load after the web has buckled, the combination of loads causing buckling of the web is assumed to satisfy the buckling interaction curve for the web.

As the web is the only member which can carry shearing forces, then some of the bending moment which was previously carried by the web at the moment of buckling will be rejected, due to the increase of the shearing force, in order to allow the web to follow the interaction curve, Fig. 4. This rejected moment will be carried by the part of the section of the girder which is still effective after buckling. This will result in a new stiffness matrix $\bar{K}_B$ expressing the relation between the increase in load and the corresponding increase in deformations.

$$\delta \bar{F} = \bar{K}_B \delta \bar{U} \quad (2)$$

Fig. 3 Critical shearing force and bending moment

under a combined loading of shearing force and bending moment, are developed and are presented in Appendix 1.

Plastic, buckling and ultimate strength conditions can be represented by the so called "interaction curves" as in Fig. 2.

Since the strain hardening is not accounted for, the adopted buckling strength curves will be as shown

Fig. 4 Rejected bending moment

The amount of bending moment rejected from the web, the effective section after buckling and the corresponding stiffness matrix are developed and shown in Appendix 3.

The analysis is carried out for two cases :

- (1) Taking account of the post buckling strength of the web
- (2) The web loses all its strength after buckling

In case (2), once the ultimate strength condition (the chain line in Fig. 2) is satisfied, point (ii), that section will yield, and Eq.(2) will not be valid any more. Load-deformation relationship after that point will be shown in 5.3. On the other hand, in case (1), when the shearing forces reach the critical value V_{cr} , point 2 in Fig. 2, the web will be free of the bending moment which was carried at the moment of buckling, and the rejection of bending moment from the web will come to an end. The original matrix presented in Appendix 2 will hold again with two differences :

1-The moment of inertia of the section will be changed to that of the flanges only, since the web will not be able to carry any further bending moment and will only sustain consistently the same distance between the flanges.

2-After the shearing force has reached the critical value V_{cr} , its increase will be balanced by tension field in the web. Then the relationship between the increment of load and shear deflection will be different from that presented in Appendix 2. The new relationship is developed and presented in Appendix 4.

Then

$$\delta \bar{F} = \bar{K}_{B1} \delta \bar{U} \quad (3)$$

This equation will hold till one side or both sides of the element satisfy the yield condition, point 3 in Fig. 2.

5.3 Behavior of girder element after yielding

Ueda & Others³⁾ presented the relation between the increment of load and the increment of deformation after one side of the element or both sides satisfy the yield condition as follows

$$\delta \bar{F} = \bar{K}_p \delta \bar{U} \quad (4)$$

The form of $\bar{K}_p$ changes according to the plastic sides and its different forms are given in the same reference.

This load-deformation equation, Eq. (4) ensures that the condition of loading at the plastic sides will follow the interaction curves between the different load components, until the structure collapses, line 3-4 in Fig. 2.

The analysis, as presented in 5.1, 5.2 and 5.3 takes account of web buckling (ultimate strength analysis). On the other hand if the web does not buckle, or buckling is disregarded (plastic strength analysis), an element will behave according to Eq. (1) till

Fig. 5 Methods of analysis

yielding, line O-I in Fig. 2. After yielding it will follow Eq. (4) till collapse, line I-II in Fig. 2.

A flow chart showing the different stages which an element may undergo according to the different ways of analysis, accompanied with the appropriate load-deformation relationships, is shown in Fig. 5.

5.4 Solution for the complete structure

A structure of deep girders is regarded as an assemblage of girder elements. The assembly of these girder elements into the whole structure is made according to the usual way, that is an assemblage of the stiffness matrices. Imposing the boundary and loading conditions, the deformation of the structure will be computed. The load is applied proportionally in increments. Each increment of load will be the minimum increment which will cause one element or more to reach its next local failure. The course of loading will continue until the structure collapses.

6 Example of analysis

In the previous chapter, the three different methods of analysis and their procedures are described. In order to examine the nature of these methods, several simple structures are analysed with the aid of the three methods and their results are compared.

6.1 Analytical models

Several deep girders are used for the analysis. Their dimensions are chosen like those of oil tankers' structures. These dimensions are shown in Table 1. The external loads are uniformly distributed normal to one flange, and applied proportionally. The girders are fixed at both ends.

6.2 Results of the analysis

Table 3 shows the values of ultimate load and the associated deflection at mid-span for several girders, analysed by the different methods. Fig. 6 shows the relation between the applied load and the corresponding deflection at mid-span for two of these girders. Fig. 7 shows the load condition at each section for different loading stages on the $M-V$ diagram for girder No. 7.

7 Concluding remarks

From the results of analysis, the following information is obtained.

1. Comparing the results of the elastic design method

Table 1 Girders' scantlings

Model No.	h	t_w	b	t_f	L	n
G 1	1200	3	200	10	600	6
G 2	1200	3	200	10	1200	6
G 3	1200	6	200	10	600	6
G 4	1200	6	200	20	600	6
G 5	2800	11	630	30.5	1500	6
G 6	3000	17	475	30.5	1200	6
G 7	1800	12	335	31	2000	6

Dimensions are in mm.

$E=2.1 \times 10^4 \text{ kg/mm}^2$, $\sigma_y=30 \text{ kg/mm}^2$,

$\nu=0.3$

Table 2 Plastic, ultimate and critical loads

Model No.	M_p	V_p	M_{fp}	M_u	V_u	M_c	M_u'	V_u'	M_{cr}	V_{cr}	M_y
G 1	0.1050 $\times 10^9$	0.5400 $\times 10^5$	0.7260 $\times 10^8$	0.7948 $\times 10^8$	0.4667 $\times 10^5$	0.3536 $\times 10^8$	0.7484 $\times 10^8$	0.1174 $\times 10^5$	0.9580 $\times 10^7$	0.1174 $\times 10^5$	0.9481 $\times 10^8$
G 2	0.1050 $\times 10^9$	0.5400 $\times 10^5$	0.7260 $\times 10^8$	0.7937 $\times 10^8$	0.3017 $\times 10^5$	0.3362 $\times 10^8$	0.7470 $\times 10^8$	0.5256 $\times 10^4$	0.8979 $\times 10^7$	0.5256 $\times 10^4$	0.9481 $\times 10^8$
G 3	0.1374 $\times 10^9$	0.1080 $\times 10^6$	0.7260 $\times 10^8$	0.1051 $\times 10^9$	0.1068 $\times 10^6$	0.4368 $\times 10^8$	0.9029 $\times 10^8$	0.7694 $\times 10^5$	0.4705 $\times 10^8$	0.7694 $\times 10^5$	0.1164 $\times 10^9$
G 4	0.2112 $\times 10^9$	0.1080 $\times 10^6$	0.1464 $\times 10^9$	0.1779 $\times 10^9$	0.1080 $\times 10^6$	0.1275 $\times 10^9$	0.1649 $\times 10^9$	0.7694 $\times 10^5$	0.7763 $\times 10^8$	0.7694 $\times 10^5$	0.1921 $\times 10^9$
G 5	0.2278 $\times 10^{10}$	0.4620 $\times 10^6$	0.1632 $\times 10^{10}$	0.1872 $\times 10^{10}$	0.4572 $\times 10^6$	0.1196 $\times 10^{10}$	0.1741 $\times 10^{10}$	0.2230 $\times 10^6$	0.5061 $\times 10^9$	0.2230 $\times 10^6$	0.2081 $\times 10^{10}$
G 6	0.2465 $\times 10^{10}$	0.7650 $\times 10^6$	0.1317 $\times 10^{10}$	0.1975 $\times 10^{10}$	0.7650 $\times 10^6$	0.1133 $\times 10^{10}$	0.1776 $\times 10^{10}$	0.6433 $\times 10^6$	0.1234 $\times 10^{10}$	0.6433 $\times 10^6$	0.2096 $\times 10^{10}$
G 7	0.8620 $\times 10^9$	0.3240 $\times 10^6$	0.5704 $\times 10^9$	0.7503 $\times 10^9$	0.3099 $\times 10^6$	0.3656 $\times 10^9$	0.7092 $\times 10^9$	0.2036 $\times 10^6$	0.5218 $\times 10^9$	0.2036 $\times 10^6$	0.7747 $\times 10^9$

M in kg/mm., V in kg.

Table 3 Maximum strength and maximum deflection

Model No.	Plastic design		Ultimate strength design'		Ultimate strength design''		Elastic design	
	P	v	P	v	P	v	P	v
G1	0.2160×10^5	2.40	0.1867×10^5	7.50	0.4694×10^4	0.53	0.4215×10^4	0.47
G2	0.2160×10^5	11.23	0.1207×10^5	28.02	0.2102×10^4	0.72	0.1904×10^4	0.63
G3	0.4320×10^5	3.58	0.4273×10^5	6.65	0.3078×10^5	1.95	0.2576×10^5	1.58
G4	0.4320×10^5	2.40	0.4320×10^5	3.31	0.3078×10^5	1.72	0.2861×10^5	1.58
G5	0.1848×10^6	6.03	0.1829×10^6	11.49	0.8919×10^5	2.94	0.8162×10^5	2.66
G6	0.3060×10^6	4.80	0.3060×10^6	6.15	0.2573×10^6	4.13	0.2375×10^6	3.73
G7	0.1296×10^6	21.42	0.1240×10^6	37.73	0.8143×10^5	9.06	0.7224×10^5	7.71

P in kg. v in mm.

' with consideration of post buckling strength of the web.

'' without consideration of post buckling strength of the web.

to those of the ultimate strength design method, it is clear that the elastic strength cannot represent the real strength of the structure; especially with the wide variety of scantlings and dimensions possible in such structures, buckling may occur very early or very little before collapse.

2. Comparing the results of the plastic design method to those of the ultimate strength design method, although the values of the ultimate loads are very near in many cases, they are very far in some others. Further consideration of axial force, unsymmetrical structures, and unsymmetrical loading, should be done, before having any limits within which the plastic design method can be used, neglecting buckling of the web, to determine the ultimate load with acceptable error.

3. Comparing the results of the ultimate design method to those of the ultimate design method which neglects the post buckling strength of the web, it can be concluded that the second gives ultimate loads much less than those which can be actually carried by the structure. This method is not recommended to represent the strength of the structure.

Fig. 6 Load-deformation relationship

i : Plastic strength curve ii : Ultimate strength curve iii : Buckling strength curve

Fig. 7 Internal force combinations at different sections at different load stages

4. The developed analytical method is considered to be a successful method for evaluation of the ultimate strength of structures with consideration of the effects of local failures,

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The numerical calculations are performed by the computer, FACOM 230-60, installed at Kyoto University.

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Appendices

1. Critical, ultimate and plastic strength of a girder element

(1) Critical loads

$$V_{cr} = V_p \frac{k_s \pi^2}{12(1-\nu^2)} \left(\frac{E}{\tau_{wy}} \right) \left(\frac{t_w}{h} \right)$$

$$V_{cr} \leq 0.5 V_p$$

$$V_{cr} = V_p \left[1 - \frac{3(1-\nu^2)}{k_s \pi^2} \left(\frac{\tau_{wy}}{E} \right) \left(\frac{h}{t_w} \right)^2 \right] \quad V_{cr} > 0.5 V_p$$

$$k_s = -38.86 + \frac{27.14}{\lambda^2} + 24.0 \lambda \quad \lambda \leq 1.0$$

$$k_s = 8.99 + \frac{6.73}{\lambda^2} - \frac{3.44}{\lambda^3} \quad \lambda > 1.0$$

$$M_{crw} = M_{wy} \frac{k \pi^2}{12(1-\nu^2)} \left(\frac{E}{\sigma_y} \right) \left(\frac{t_w}{h} \right)^2 \quad M_{crw} \leq M_{wy}$$

$$M_{crw} = M_{wy} \frac{[0.952\sqrt{k} - (h/t_w) \sqrt{\sigma_y/E}] f_s + [(h/t_w) \sqrt{\sigma_y/E} - 2.42]}{0.952\sqrt{k} - 2.42} \quad M_{crw} > M_{wy}$$

$$k = 15.87 + \frac{1.87}{\lambda^2} + 8.6 \lambda^2 \quad \lambda \leq 2/3$$

$$k = 23.9 \quad \lambda > 2/3$$

$$M_{cr} = M_{crw} \frac{I}{I_w}$$

(2) Buckling condition

$$(V/V_{cr})^2 + (M_w/M_{crw})^2 - 1 = 0$$

$$(V/V_{cr})^2 + (M/M_{cr})^2 - 1 = 0$$

(3) Plastic loads

$$V_p = h t_w \tau_{wy}$$

$$M_p = M_y [6(A_f/A_w) + (3/2)] / [6(A_f/A_w) + 1]$$

$$M_{fp} = A_f \sigma_y (h + t_f)$$

(4) Plastic yield condition

$$(V/V_p)^2 + [(M - M_{fp}) / (M_p - M_{fp})]^2 - 1 = 0$$

$$M > M_{fp}$$

$$|V/V_p| - 1 = 0$$

$$M \leq M_{fp}$$

(5) Ultimate loads

$$\left. \begin{aligned} V_u &= V_p \\ c^* &= 1 - v_{cr} \end{aligned} \right\} (1 - v_{cr}) < \varepsilon$$

$$\left. \begin{aligned} V_u &= V_p [(1 - \varepsilon) v_{cr} + \sqrt{1 + v_{cr}^2 - (1 - \varepsilon)^2}] / (1 + v_{cr})^2 \\ c^* &= [2 - 2 v_{cr} \sqrt{1 + v_{cr}^2 - (1 - \varepsilon)^2} - \varepsilon (1 - v_{cr})^2] / (1 + v_{cr})^2 \end{aligned} \right\} 1 - \frac{\lambda + v_{cr} (1 + \lambda v_{cr})}{\sqrt{\lambda^2 + (1 + \lambda v_{cr})^2}} < \varepsilon \leq (1 - v_{cr})$$

$$\left. \begin{aligned} V_u &= \sqrt{\lambda^2 + (1 + \lambda v_{cr})^2} - (1 - \varepsilon) \lambda \\ c^* &= \varepsilon [\lambda (\sqrt{\lambda^2 + (1 + \lambda v_{cr})^2} + \lambda) / (1 + \lambda v_{cr}) \\ &\quad + (1 + \lambda v_{cr}) (\sqrt{\lambda^2 + (1 + \lambda v_{cr})^2} - v_{cr}) / \sqrt{\lambda^2 + (1 + \lambda v_{cr})^2}] \end{aligned} \right\} \varepsilon \leq 1 - \frac{\lambda + v_{cr} (1 + \lambda v_{cr})}{\sqrt{\lambda^2 + (1 + \lambda v_{cr})^2}}$$

where

$$v_{cr} = V_{cr} / V_p, \quad \varepsilon = 8(h t_f / L^2) (A_f / A_w)$$

$$M_u = M_{cr} + \beta (M_y - M_{cr}) \quad M_{cr} \leq M_y$$

$$M_u = M_{cr} \quad M_{cr} > M_y$$

where

$$\beta = [1 / (1 + 2e)], \quad (I_{eBB} / I) \quad \text{for } e \text{ and } I_{eBB} \text{ see appendix 3}$$

(6) Ultimate strength yield condition

$$(V/V_u)^2 + [(M - M_c) / (M_p - M_c)]^2 - 1 = 0 \quad M > M_c$$

$$|V/V_u| - 1 = 0 \quad M \leq M_c$$

$$M_c = M_{fp} - c^* A_w \sigma_y / 4$$

(7) Ultimate loads without consideration of postbuckling strength of the web

$$V_u' = V_{cr}$$

$$M_u' = M_{fp} + M_{cr} (I_w / I) \quad M_{cr} \leq M_y$$

$$M_u' = M_{cr} \quad M_{cr} > M_y$$

(8) Ultimate strength yield condition without consideration of postbuckling strength of the web

$$(V/V_u')^2 + [(M - M_{fp}) / (M_u' - M_{fp})]^2 - 1 = 0 \quad M > M_{fp}$$

$$|V/V_u'| - 1 = 0 \quad M \leq M_{fp}$$

2. Girder element stiffness matrix accounting for shear deformation

2.1 Shear deflection

Timoshenko⁵⁾ presented the relation between the shearing force V and the corresponding shear deflection v_s as follows.

$$v_s = \alpha VL / A_w G$$

For sufficiently large flanges $\alpha = 1$ and

$$v_s = VL / A_w G = \gamma VL^3 / EI \quad \text{where} \quad \gamma = EI / A_w GL^2$$

2.2 Stiffness matrix

$$K = \frac{EI}{1 + 12\gamma} \begin{bmatrix} 12/L^3 & 6/L^2 & -12/L^3 & 6/L^2 \\ 6/L^2 & 4(1+3\gamma)/L & -6/L^2 & 2(1-6\gamma)/L \\ -12/L^3 & -6/L^2 & 12/L^3 & -6/L^2 \\ 6/L^2 & 2(1-6\gamma)/L & -6/L^2 & 4(1+3\gamma)/L \end{bmatrix}$$

3. Post-buckling stiffness matrix and effective section

Increasing the load from zero, the web will buckle at certain values of the shearing force and bending moment at the ends i, j of an element.

These values are such that their mean shearing force and mean bending moment will satisfy the buckling condition. For further loading, the web will follow its buckling shear-bending condition, Fig. 5. As the web should support the increase in the shearing force, some of the bending moment which was carried before by the web at the moment of buckling will be transferred to the effective section after buckling. This operation will continue until the value of the shearing force reaches V_{cr} , and the web will be in the same state as that of buckling under shearing force only.

To obtain the load-deformation relationship in this range, it is necessary at first to determine the amount of bending moment transferred to the effective section after buckling, for increments of load, ∂V_i , ∂V_j , ∂M_i and ∂M_j at the ends i, j of an element. Fig. 8, shows points i, j the condition of loading at any stage after the web has buckled. Points i', j' , are the condition of loading after the increments mentioned above. Points i_B, j_B, i_B' and j_B' on the buckling interaction curve of the whole section correspond to points i, j, i' and j' respectively, and points i_{BW}, j_{BW}, i_{BW}' and j_{BW}' correspond to them

Fig. 8 Rejected bending moment after the web has buckled

on the web interaction curve. The values δM_{V_i} and δM_{V_j} are the moments transferred from the web to the effective section after buckling and are to be evaluated here.

For small values of ∂V_i , ∂V_j the curves iA and jB can be considered straight lines.

$$\text{Then } \delta M_{V_i} = \partial V_i \tan \phi_i,$$

$$\delta M_{V_j} = \partial V_j \tan \phi_j$$

in magnitude, and follow the sign of M when ∂V follows the sign of V . When V_i , ∂V_i , M_i , δM_i are positive, $\tan \phi_i$ is negative.

Then

$$\delta M_{V_i} = -\delta V_i \tan \phi_i$$

To get $\tan \phi_i$ and $\tan \phi_j$, let us first find $\tan \phi_m$ at the mean point on the interaction curve.

Equation (2) gives

$$(M_{wm}/M_{crw})^2 + (V_m/V_{cr})^2 - 1 = 0$$

where

$$V_m = (V_i - V_j)/2, \quad M_{wm} = (M_{wi} - M_{wj})/2$$

differentiating with respect to V_m and substituting the value of M_{wm} from the same equation, we get

$$\tan \phi_m = -(M_{crw}/V_{cr}^2) (V_m/\sqrt{1 - (V_m/V_{cr})^2})$$

but

$$M_{crw} = M_{cr} (I_w/I)$$

then

$$\tan \phi_m = -(M_{cr}/V_{cr}^2) (V_m/\sqrt{1 - (V_m/V_{cr})^2}) (I_w/I)$$

$$\text{and } \tan \phi_i = (M_{wi}' - M_{wi}) / (V_i' - V_i)$$

$$= [\tan \phi_m + (1/2) (I_w/I) (\delta M_i + \delta M_j) / \delta V_m] / [1 + (1/2) (\delta V_i + \delta V_j) / \delta V_m]$$

$$\text{and } \tan \phi_j = [\tan \phi_m - (1/2) (I_w/I) (\delta M_i + \delta M_j) / \delta V_m] / [1 - (1/2) (\delta V_i + \delta V_j) / \delta V_m]$$

in the case when there is no load acting between the nodal points, Fig. 8., we have

$$\Sigma Y = 0 \quad \text{i.e. } \delta V_i + \delta V_j = 0$$

$$\text{and } \Sigma M = 0 \quad \text{i.e. } \delta M_i + \delta M_j = 0$$

Then

$$\tan \phi_i = \tan \phi_j = \tan \phi_m$$

defining factor C_{mv} as

$$C_{mv} = \tan \phi / L$$

then

$$\delta M_{V_i} = -C_{mv} L \delta V_i \quad \text{and} \quad \delta M_{V_j} = -C_{mv} L \delta V_j$$

Then stiffness matrix will be

$$K_B = \frac{EI_e}{1+12\gamma} \begin{bmatrix} 12/L^3 & 6/L^2 & -12/L^3 & 6/L^2 \\ 6(1+2C_{mv})/L^2 & 4(1+3\gamma+3C_{mv}/2)/L & -6(1+2C_{mv})/L^2 & 2(1-6\gamma+3C_{mv})/L \\ -12/L^3 & -6/L^2 & 12/L^3 & -6/L^2 \\ 6(1-2C_{mv})/L^2 & 2(1-6\gamma-3C_{mv})/L & -6(1-2C_{mv})/L^2 & 4(1+3\gamma-3C_{mv}/2)/L \end{bmatrix}$$

This load-deformation relationship is valid only after web buckling and V less than V_{cr} .

Effective sectional moment of inertia after web buckling

a) Effective section with consideration of post buckling strength of the web

After pure shear buckling the web will not be able to support any bending moment, and only the flanges are expected to do so, then

$$I_{eSB} = I_f$$

On the other hand with pure bending buckling, Akita and Fujii¹⁾ gave the value

$$I_{eBB} = ((1/2) + 2e^2) h^2 A_f + h^2 A_w [(1/4) + 3e^2 - ((1/2) + e - c)^3] / 3$$

$$e = [(1/2) + c + 2A_f/A_w] - \sqrt{2(1+2A_f/A_w)(c + A_f/A_w)}$$

$$c = 1.21 (t_w/h) \sqrt{E/\sigma_y}$$

This value corresponds to the flanges and the effective portion of the web (the hatched portion in Fig. 9. a).

For combined shearing force and bending moment, the effective portion of the web will decrease with the increase of the ratio V/V_{cr} . In the beginning the part near by the neutral axis will turn ineffective, with a small reduction in the effective moment of inertia, then, the part further from the neutral axis, with a bigger reduction. This will result in a convex curve for the effective moment of inertia, as shown in Fig. 9. b., which can be assumed in the form :

$$I_e = I_{eBB} - (I_{eBB} - I_{eSB}) (V/V_{cr})^2$$

b) Effective section neglecting post buckling strength of the web.

In this case, as there is no contribution by the web once it buckles, then,

$$I_e = I_f$$

Fig. 9 Effective sectional moment of inertia

4. Shear rigidity of web after the shearing force has been increased over the critical shearing force

At the moment when the shearing force V in the web reaches the value V_{cr} ,

$$\sigma_{ta} = \tau_{cr} \sin 2\alpha$$

At the shear plastic condition, $V = V_u$

$$\sigma_1 = \sigma_y - \tau_{cr} \sin 2\alpha$$

Therefore the increment of the shear force δV will be

$$\delta V = V_u - V_{cr}$$

and the increase in stress $\delta\sigma$ in the direction α will be

$$\delta\sigma_\alpha = \sigma_y - 2\tau_{cr} \sin 2\alpha$$

Then the increase in strain δe in the direction α will be

$$\delta e_\alpha = (\sigma_y - 2\tau_{cr} \sin 2\alpha)/E$$

The flanges will be subjected to compressive force ΔF due to the shear buckling tension field. Its value differs according to the scantlings of the girder and is given in ref. (1).

Then the flanges will undergo a compressive strain $\delta e_f = \Delta F/E$

Regarding Fig. 10., the increase in shear deflection can be expressed as follows

$$\delta v_V = L \sin \phi$$

The angle ϕ can be determined from the hatched triangles, as follows

$$\sin \phi = [L^2(1 + \delta e_\alpha)^2 \cos^2 \alpha - L^2(1 - \delta e_f)^2 - L^2 \tan^2 \alpha] / 2L^2(1 - \delta e_f) \tan \alpha \quad \alpha \leq \tan^{-1} h/L$$

$$\sin \phi = [(h/\sin \alpha)^2(1 + \delta e_\alpha)^2 - h^2 \cotan^2 \alpha(1 - \delta e_f)^2 - h^2] / 2h^2(1 - \delta e_f) \cotan \alpha \quad \alpha > \tan^{-1} h/L$$

defining a shear deflection coefficient γ' so that

$$\delta v_V = \gamma' \delta VL^3/EI_e \quad \text{then,} \quad \gamma' = \delta v_V EI_e / \delta VL^3$$

This coefficient is to be used in the matrix presented in Appendix 2 instead of the shear coefficient γ .

Fig. 10 Shear deflection after web has buckled