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Application of the Vortex Lattice Method to the Three-Dimensional Theory of a Cavitating Propeller

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Summary

A new theory on a cavitating propeller is proposed, treating the cavitation three-dimensionally, and using a lifting surface theory. A numerical method has also been developed based on this theory, applying the vortex lattice method and replacing the cavitation and the blade thickness by source elements, for a cavitating propeller operating in a uniform flow.

To evaluate the present method, the calculations are carried out for a model propeller and compared with other prediction methods and the model test results.

It is confirmed that the present method gives considerable improvements, in comparison with the conventional methods, on the evaluation of the cavity thickness due to the three-dimensional flow, and on the elimination of the instability of the solution caused by the two-dimensional theory.

1. Introduction

Several theoretical and numerical calculations have been carried out by many authors on cavitating propellers; however, the agreements between the calculated and observed results are not fully satisfactory. Most of the theoretical contributions to this problem are based on the combination of two-dimensional cavitation theory and the angle of attack concept.

For a steady, two-dimensional, thin or slender body, the linearized cavitation problem has been formulated and solved by Tulin¹⁾ and Geurst²⁾. These previous methods gave the solutions by the analytic function theory in the mapped plane or by the mode function method expressing the singularity distributions in the physical plane. The discrete function method has been developed by Golden³⁾ to solve a cavitating two-dimensional hydrofoil.

It was clarified that either the mode function method or the discrete function method has the same characteristics of the solutions, which for instance the cavity becomes very unstable in the transition region between partial and full cavitation⁴⁾.

Uhlman and Persson⁵⁾ tried to improve the instability of the two-dimensional solution by applying the concentrated vortex ahead of the leading edge of a flat plate, and found the stabiliz-

ing effect on the cavity length increased beyond 75% of chord length.

However, at the outer radial sections of the blade for a cavitating propeller, the traditional method could not yield the solution with a thicker cavity due to the three-dimensional effect.

The natural approach at this point was attempted by Frydenlund and Persson⁶⁾ describing cavitation on a propeller blade based on the three-dimensional equations but solving them stripwise including three-dimensional induced velocities. For the unsteady cavitating propeller problem, Lee⁷⁾ introduced numerical solutions by using a three-dimensional cavity concept, but the cavity length and source distribution were iteratively confined to one chordwise strip by assuming that the cavity extent and source and vortex distributions on the other chordwise strips be known either by an initial approximation or by the most recent solutions on those strips.

These two previous works seem rationally to introduce the three-dimensional effect on a propeller blade, however the process for cavity solution is still limited to the stripwise manner.

In this regard pure three-dimensional cavitation theory and numerical processes are desired to describe cavitation on a propeller blade.

The author attempts to formulate more exactly three-dimensional cavitation theory combined with lifting surface for the steady cavitating propeller problem using the vortex lattice method, and to evaluate the numerical solutions in com-

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parison with experiments, two-dimensional theory and lift equivalent method.

2. Fundamental Assumption

The propeller is assumed to operate in an inviscid, unbounded and incompressible fluid, rotating with a constant angular velocity and entailing partial sheet cavitation to a certain extent from the leading edge on the back surface of the propeller blade in a uniform flow.

Fig. 1 shows the co-ordinate system for propeller geometry. The propeller blades, cavity thickness and loading are denoted by the distribution of sources $q(\xi, \eta, \zeta)$ and vortices $\gamma(\xi, \eta, \zeta)$ lying on the mean camber surface of each blade and free vortex sheet.

These sources and vortices will be determined by solving the boundary value problem on the back and face surface of the propeller blade simultaneously.

3. Boundary Value Problem

The fluid in which the propeller operates is assumed to be an unbounded, incompressible and inviscid. Therefore, Laplace's equation governs throughout the fluid or the solution must satisfy the boundary conditions specified, which are the normal velocity being zero on the wetted surface of the blade, a constant pressure in the cavity and the closure of the cavity as follows:

3.1 Wetted Surface Condition

On a wetted portion of the back and whole face surface of the blade, the kinetic boundary condition that is the normal component of the total fluid velocity should be zero is considered.

Namely, on the portion of the back surface,

Fig. 1 Co-ordinate system for propeller three-dimensional cavitation theory

$$\mathbf{V}^{GB} \cdot \mathbf{n}^B + \mathbf{V}^{QB} \cdot \mathbf{n}^B = -\mathbf{W}^B \cdot \mathbf{n}^B, \quad (1)$$

$$l_r^c < l_r \leq 1$$

and on the whole face surface,

$$\mathbf{V}^{GF} \cdot \mathbf{n}^F + \mathbf{V}^{QF} \cdot \mathbf{n}^F = -\mathbf{W}^F \cdot \mathbf{n}^F, \quad (2)$$

$$0 \leq l_r \leq 1$$

where $\mathbf{V}^G$ and $\mathbf{V}^Q$ denote respectively the vectors of induced velocities from the vortex and source distribution with unknown strengths, $\mathbf{W}$ is the vector of known inflow velocity, and $\mathbf{n}$ is the unit normal vector on the surface specified by the superscripts B and F which indicate Back and Face. l_r is the normalized ordinate along the camber line on a blade radial position r , where $l_r = 0$ at the leading edge, $l_r = l_r^c$ at the cavity end and $l_r = 1$ at the trailing edge.

3.2 Cavity Surface Condition

On the cavity surface the pressure coefficient $-C_p$ expressed by Bernoulli's equation is considered to be the same as the cavitation number σ defined by the prescribed vapor pressure inside the cavity;

$$\sigma = -C_p \equiv -1 + \frac{\{(\mathbf{W} + \mathbf{V}^G + \mathbf{V}^Q) \cdot \mathbf{t}\}^2}{|\mathbf{W}|^2}. \quad (3)$$

σ is constant on the cavity surface. Hence rewriting expression (3) the following equation is derived,

$$|\mathbf{W}| \sqrt{\sigma + 1} = \mathbf{W} \cdot \mathbf{t} + \mathbf{V}^G \cdot \mathbf{t} + \mathbf{V}^Q \cdot \mathbf{t}, \quad (4)$$

where $\mathbf{t}$ denotes the unit tangential vector on the portion of the back surface (see Fig. 2).

3.3 Cavity End Condition

It is assumed that the cavity thickness be zero at the leading edge and be l_r^c including half the blade thickness at the cavity end l_r^c which is geometrically shown in Fig. 2. Therefore l_r^c must satisfy the following equation which is obtained by integrating the gradient of streamline along the camber line;

$$l_r^c = \int_0^{l_r^c} \{(\mathbf{W}^B + \mathbf{V}^{GB} + \mathbf{V}^{QB}) \cdot \mathbf{n} / (\mathbf{W}^B + \mathbf{V}^{GB} + \mathbf{V}^{QB}) \cdot \mathbf{t}\} dt$$

$$= \int_0^{l_r^c} \{(\mathbf{W}^B + \mathbf{V}^{GB} + \mathbf{V}^{QB}) \cdot \mathbf{n} / \mathbf{W}^B \cdot \mathbf{t}\} dt, \quad (5)$$

where $\mathbf{n}$ and $\mathbf{t}$ denote the unit normal and tangen-

Fig. 2 Definition of cavity end

Fig. 3 Velocity diagram

tial vectors on the camber surface in this case.

4. General Expression of Simultaneous Integral Equation

As shown in Fig. 3 the vector of known inflow velocity $\mathbf{W}$ at a point $P(x, r, \theta)$ on the r_G radial section of the propeller blade can be expressed by using the velocity of uniform flow V_a and the propeller turning speed n as follows;

$$\mathbf{W}(x, r, \theta) = (V_a, 0, \pi n r_G D), \quad (6)$$

where D denotes the diameter of the propeller, and the relation between the polar and the Cartesian co-ordinate system is $y = r \cos \theta$, $z = r \sin \theta$.

The vectors of vortex and source induced velocities are expressed at the same point $P(x, y, z)$,

$$\mathbf{V}^G(x, y, z) = -\frac{1}{4\pi} \iint_{S+W} \frac{\gamma(\xi, \eta, \zeta) \mathbf{l} \times \mathbf{R}}{|\mathbf{R}|^3} d(S+W) \quad (7)$$

$$\mathbf{V}^Q(x, y, z) = -\frac{1}{4\pi} \iint_S \frac{q(\xi, \eta, \zeta) d\mathbf{l} \cdot \mathbf{R}}{|\mathbf{R}|^3} dS \quad (8)$$

Here, the definition of integration is the same as ref. 4. S denotes the camber surface, W the shed wake vortex sheet, $\mathbf{R}$ the vector of length between $P(x, y, z)$ and the singularity point $Q(\xi, \eta, \zeta)$, $\mathbf{l}$ the unit vector of the vortices with the arbitrary direction along the camber surface and the shed wake vortex sheet.

Co-ordinate (ξ, η, ζ) are defined as corresponding to the Cartesian co-ordinate system (x, y, z) for the singularity distributions.

Substituting Eqs. (6) to (8) into Eqs. (1) to (5) the following expression of a set of simultaneous integral equations which satisfy the boundary conditions are obtained. These equations for the point $P(x, y, z)$, include the known values on the right hand side and the unknown γ , q and σ on the left.

Fig. 4 Definition of cavity thickness

$$\begin{aligned} & \iint_{S+W} \gamma(\xi, \eta, \zeta) \sum_{p=1}^Z \mathbf{t}^B \cdot \mathbf{K}^{GB}(x, y, z; \xi, \eta, \zeta; p) d(S+W) \\ & + \iint_S q(\xi, \eta, \zeta) \sum_{p=1}^Z \mathbf{t}^B \cdot \mathbf{K}^{QB}(x, y, z; \xi, \eta, \zeta; p) dS \\ & - |\mathbf{W}^B| \sqrt{\sigma + 1} = \mathbf{W}^B \cdot \mathbf{t}^B, \quad (9) \\ & 0 \leq l_r \leq l_r^c \end{aligned}$$

$$\begin{aligned} & \iint_{S+W} \gamma(\xi, \eta, \zeta) \sum_{p=1}^Z \mathbf{n}^B \cdot \mathbf{K}^{GB}(x, y, z; \xi, \eta, \zeta; p) d(S+W) \\ & + \iint_S q(\xi, \eta, \zeta) \sum_{p=1}^Z \mathbf{n}^B \cdot \mathbf{K}^{QB}(x, y, z; \xi, \eta, \zeta; p) dS \\ & = \mathbf{W}^B \cdot \mathbf{n}^B, \quad (10) \\ & l_r^c < l_r \leq 1 \end{aligned}$$

$$\begin{aligned} & \iint_{S+W} \gamma(\xi, \eta, \zeta) \sum_{p=1}^Z \mathbf{n}^F \cdot \mathbf{K}^{GF}(x, y, z; \xi, \eta, \zeta; p) d(S+W) \\ & + \iint_S q(\xi, \eta, \zeta) \sum_{p=1}^Z \mathbf{n}^F \cdot \mathbf{K}^{QF}(x, y, z; \xi, \eta, \zeta; p) dS \\ & = \mathbf{W}^F \cdot \mathbf{n}^F \quad (11) \end{aligned}$$

$$\begin{aligned} & \iint_{S+W} \gamma(\xi, \eta, \zeta) \sum_{p=1}^Z \int_0^{l_r^c} \{ \mathbf{n} \cdot \mathbf{K}^{GB}(x, y, z; \xi, \eta, \zeta; p) / \\ & |\mathbf{W}^B| \} dt \cdot d(S+W) \\ & + \iint_S q(\xi, \eta, \zeta) \sum_{p=1}^Z \int_0^{l_r^c} \{ \mathbf{n} \cdot \mathbf{K}^{QB}(x, y, z; \xi, \eta, \zeta; p) / \\ & |\mathbf{W}^B| \} dt \cdot dS \\ & = -l_r^c + \int_0^{l_r^c} (n/t) dt \quad (12) \end{aligned}$$

Here, $\mathbf{K}^G$ and $\mathbf{K}^Q$ are the kernel functions as follows;

$$\mathbf{K}^G(x, y, z; \xi, \eta, \zeta; p) = \frac{1}{4\pi} \cdot \frac{\mathbf{l} \times \mathbf{R}}{|\mathbf{R}|^3} \quad (13)$$

$$\mathbf{K}^Q(x, y, z; \xi, \eta, \zeta; p) = \frac{1}{4\pi} \cdot \frac{\mathbf{R}}{|\mathbf{R}|^3} \quad (14)$$

These Eqs. indicate the difficulties of numerical analysis, due to the doubled singularities in the integral equations.

As shown in Eqs. (9), (10) the cavity length l_r^c is considered as known and the cavitation number $\sigma (= -C_p \cdot \text{kinetic pressure coefficient})$ is computed. Iterations with respect to the cavity

length are carried out until σ is computed to reach the static pressure coefficient σ_s given in advance with the prescribed accuracy at each radial section r . Z is the number of blades.

5. Numerical Analysis

Reducing these equations by a mode function method to a set of simultaneous algebraic equations to solve the problem, one faces the difficulties due to the complex treatments required for the singular integrals. The discrete function method has become popular with respect to the lifting surface problems in recent practical applications^{(4), (8), (9), (10)}.

In this chapter the author tries to discretize the problem by using the so called vortex lattice method following the same steps of formulation of the general expression.

5.1 Arrangement of Singularities

The camber surface of propeller blades is divided into small rectangular panels the same as the propeller lifting surface.

Dividing the radial (spanwise) direction of the blade into M elements with constant spacing, the radial position of the vortex grid r_G is expressed as follows;

$$r_G = r_h + (1 + r_h) \frac{4m-3}{4(M-1)+2}, \quad (15)$$

$$m = 1, 2, \dots, M$$

where r_h is a hub radius, r_G and r_h are normalized by the propeller radius. The chordwise position of the vortex grid S_G is defined by applying N constant spacing expressed by the normalized ordinate S along the chord;

$$S_G = \frac{n-1}{N} + \frac{1}{4} \cdot \frac{1}{N}, \quad (16)$$

$$n = 1, 2, \dots, N.$$

Including the whole geometrical factors (pitch, rake, offset etc. of the propeller), the vector of specified vortex grid point P_{ij}^G on the camber surface of a reference blade is described as follows with respect to the origine of the co-ordinate system;

$$P_{ij}^G = (x_{ij}^G, y_{ij}^G, z_{ij}^G), \quad (17)$$

where the subscripts i and j denote the indices of spanwise and chordwise directions as shown in Fig. 5. The control points P_{ij}^{CB} and P_{ij}^{CF} on the back and face surface of the blade are denoted by using the point P_{ij}^G on the camber surface with the unit normal vector n_{ij}^G and half the blade thickness t_{ij} at the same point.

$$P_{ij}^{CB} = P_{ij}^G + n_{ij}^G t_{ij}, \quad (18)$$

Fig. 5 Schematic view of co-ordinate system and the arrangement of horse-shoe vortex system for a propeller

$$P_{ij}^{CF} = P_{ij}^G - n_{ij}^G t_{ij}, \quad (19)$$

where

$$P_{ij}^G = \frac{1}{4} (P_{ij}^G + P_{i+1,j}^G + P_{i,j+1}^G + P_{i+1,j+1}^G). \quad (20)$$

Regarding the arrangements of singularities, the bound vortex and source elements are placed on the straight line segment between P_{ij}^G and $P_{i+1,j}^G$ and the trailing vortex element on that between P_{ij}^G and $P_{i,j+1}^G$. A bound vortex and a couple of trailing elements constitute a horse-shoe vortex. The latter forms a set of helicoidal free vortices toward the infinitive downstream satisfying Kelvin's theorem at each vortex grid point on the blade.

5.2 Inflow and Induced Velocities

First, let us discretize the expression (1) to (3). The vector of inflow velocity Eq. (6) can be denoted in the following discrete form;

$$W_{ij}^B = W_{ij}^F = (V_a, 0, \pi n r_G D). \quad (21)$$

The total velocity induced by a vortex distribution is the sum of the influence of all the spanwise and chordwise vortices. Thus the velocity V_{ij}^{GB} at the control point P_{ij}^{CB} on the back surface of the reference blade can be expressed as follows;

$$\frac{V_{ij}^{GB}}{W} = \sum_{m=1}^M \sum_{n=1}^N G_{mn} \cdot \left\{ K_{ijmn}^{GBS} + \sum_{k=N+1}^{N_T} (K_{ijmk}^{GBW} - K_{ijm+1,k}^{GBW}) \right\}, \quad (22)$$

where $N_T - N$ denotes the number of free vortex elements. Here, W is the reference velocity at 0.7 radial position of the blade; G_{mn} is the unknown strength of vortex elements, and detailed algebraic work is shown in Appendix 1. Moreover, by replacing the superscript B by F , V_{ij}^{GF}/W can also be obtained for the face surface.

Referring to the **Appendix 2**, the source induced velocity can be described for the unknown strength of line source elements for the same control point.

$$\left. \begin{aligned} \frac{V_{ij}^{QB}}{W} &= \sum_{m=1}^M \sum_{n=1}^N Q_{mn} \mathbf{K}_{ijmn}^{QB}, \\ \frac{V_{ij}^{QF}}{W} &= \sum_{m=1}^M \sum_{n=1}^N Q_{mn} \mathbf{K}_{ijmn}^{QF}. \end{aligned} \right\} \quad (23)$$

Concerning the cavity surface condition of Eq. (4), the self-induced velocity from the vortex distributions is required for the control point. The following can be denoted from **Appendix III**.

$$\left. \begin{aligned} \frac{bV_{ij}^{GB}}{W} + \frac{tV_{ij}^{GB}}{W} &= G_{ij} \mathbf{H}_{ij}^{GB} + \sum_{k=1}^j (G_{i-1,k} - G_{i+1,k}) \cdot \mathbf{H}_{ik}^{GB}, \\ \frac{bV_{ij}^{GF}}{W} + \frac{tV_{ij}^{GF}}{W} &= -G_{ij} \mathbf{H}_{ij}^{GF} - \sum_{k=1}^j (G_{i-1,k} - G_{i+1,k}) \cdot \mathbf{H}_{ik}^{GF}, \end{aligned} \right\} \quad (24)$$

where the superscripts *b* and *t* mean the contribution from bound and trailing vortex distribution respectively, and

$$\left. \begin{aligned} \mathbf{H}_{ij}^{GB} &= -\pi \mathbf{t}_{ij} \left/ \left(\frac{\Delta c}{R} \right) \right., \\ \mathbf{H}_{ij}^{GF} &= -\pi \mathbf{t}_{ij} \left/ \left(\frac{\Delta c}{R} \right) \right., \end{aligned} \right\} \quad (25)$$

$$\left. \begin{aligned} \mathbf{H}_{ik}^{GB} &= -\pi \mathbf{t}_{ik} \left/ \left(\frac{\Delta s}{R} \right) \right., \\ \mathbf{H}_{ik}^{GF} &= -\pi \mathbf{t}_{ik} \left/ \left(\frac{\Delta s}{R} \right) \right.. \end{aligned} \right\} \quad (26)$$

While for the wetted surface condition of Eqs. (1), (2) the self-induced velocity from the source distribution is required. From **Appendix 4** the following can be expressed.

$$\left. \begin{aligned} \frac{V_{ij}^{QB}}{W} &= Q_{ij} \mathbf{H}_{ij}^{QB}, \\ \frac{V_{ij}^{QF}}{W} &= -Q_{ij} \mathbf{H}_{ij}^{QF}. \end{aligned} \right\} \quad (27)$$

Where,

$$\left. \begin{aligned} \mathbf{H}_{ij}^{QB} &= -\pi \mathbf{n}_{ij} \left/ \left(\frac{\Delta c}{R} \right) \right., \\ \mathbf{H}_{ij}^{QF} &= -\pi \mathbf{n}_{ij} \left/ \left(\frac{\Delta c}{R} \right) \right., \end{aligned} \right\} \quad (28)$$

$\mathbf{t}_{ij}$, $\mathbf{n}_{ij}$ being the unit vectors of tangential and normal directions at the point $\mathbf{P}_{ij}^c$ on the camber surface on which the singularities are placed.

5.3 Simultaneous Algebraic Equations

As the simultaneous integral equations which satisfy the boundary conditions have been derived

in the former chapter, the resultant expressions of simultaneous algebraic equation can be obtained to follow the same procedure. Substituting Eqs. (21) to (28) into Eqs. (1) to (5) the discretized simultaneous equations are derived as follows;

$$\begin{aligned} G_{ij} \mathbf{H}_{ij}^{GB} \mathbf{t}_{ij}^B + \sum_{k=1}^j (G_{i-1,k} - G_{i+1,k}) \mathbf{H}_{ik}^{GB} \mathbf{t}_{ij}^B \\ + \sum_{m=1}^M \sum_{n=1}^N G_{mn} \sum_{p=1}^Z \left\{ \mathbf{K}_{ijmn}^{GBS}(\theta_p) + \sum_{k=N+1}^{NT} (\mathbf{K}_{ijmk}^{GBW}(\theta_p) \right. \\ \left. - \mathbf{K}_{ijm+1,k}^{GBW}(\theta_p)) \right\} \mathbf{t}_{ij}^B + Q_{ij} \mathbf{H}_{ij}^{GB} \mathbf{t}_{ij}^B \\ + \sum_{m=1}^M \sum_{n=1}^N Q_{mn} \sum_{p=1}^Z \mathbf{K}_{ijmn}^{QB}(\theta_p) \mathbf{t}_{ij}^B - \frac{|\mathbf{W}_{ij}^B|}{W} \sqrt{\sigma_i + 1} \\ = -\frac{\mathbf{W}_{ij}^B}{W} \mathbf{t}_{ij}^B \quad (0 \leq l_i \leq l_i^c) \end{aligned} \quad (29)$$

$$\begin{aligned} \sum_{m=1}^M \sum_{n=1}^N G_{mn} \sum_{p=1}^Z \left\{ \mathbf{K}_{ijmn}^{GBS}(\theta_p) + \sum_{k=N+1}^{NT} (\mathbf{K}_{ijmk}^{GBW}(\theta_p) \right. \\ \left. - \mathbf{K}_{ijm+1,k}^{GBW}(\theta_p)) \right\} \mathbf{n}_{ij}^B + Q_{ij} \mathbf{H}_{ij}^{QB} \mathbf{n}_{ij}^B \\ + \sum_{m=1}^M \sum_{n=1}^N Q_{mn} \sum_{p=1}^Z \mathbf{K}_{ijmn}^{QF}(\theta_p) \mathbf{n}_{ij}^B + G_{ij} \mathbf{H}_{ij}^{GF} \mathbf{n}_{ij}^B \\ = -\frac{\mathbf{W}_{ij}^B}{W} \mathbf{n}_{ij}^B \quad (l_i^c < l_i \leq 1) \end{aligned} \quad (30)$$

$$\begin{aligned} \sum_{m=1}^M \sum_{n=1}^N G_{mn} \sum_{p=1}^Z \left\{ \mathbf{K}_{ijmn}^{GF}(\theta_p) + \sum_{k=N+1}^{NT} (\mathbf{K}_{ijmk}^{GBW}(\theta_p) \right. \\ \left. - \mathbf{K}_{ijm+1,k}^{GBW}(\theta_p)) \right\} \mathbf{n}_{ij}^F + Q_{ij} \mathbf{H}_{ij}^{QF} \mathbf{n}_{ij}^F \\ + \sum_{m=1}^M \sum_{n=1}^N Q_{mn} \sum_{p=1}^Z \mathbf{K}_{ijmn}^{QF}(\theta_p) \mathbf{n}_{ij}^F + G_{ij} \mathbf{H}_{ij}^{GF} \mathbf{n}_{ij}^F \\ = -\frac{\mathbf{W}_{ij}^F}{W} \mathbf{n}_{ij}^F \end{aligned} \quad (31)$$

$$\begin{aligned} \sum_{j=1}^J G_{ij} \Delta l_{ij} \mathbf{H}_{ij}^{GB} \mathbf{n}_{ij} / \mathbf{W}_{ij}^* \mathbf{t}_{ij} \\ + \sum_{m=1}^M \sum_{n=1}^N G_{mn} \sum_{p=1}^Z \sum_{j=1}^J \Delta l_{ij} \left\{ \mathbf{K}_{ijmn}^{GBS}(\theta_p) \right. \\ \left. + \sum_{k=N+1}^{NT} (\mathbf{K}_{ijmk}^{GBW}(\theta_p) - \mathbf{K}_{ijm+1,k}^{GBW}(\theta_p)) \right\} \mathbf{n}_{ij} / \mathbf{W}_{ij}^* \mathbf{t}_{ij} \\ + \sum_{j=1}^J Q_{ij} \Delta l_{ij} \mathbf{H}_{ij}^{QB} \mathbf{n}_{ij} / \mathbf{W}_{ij}^* \mathbf{t}_{ij} \\ + \sum_{m=1}^M \sum_{n=1}^N Q_{mn} \sum_{p=1}^Z \sum_{j=1}^J \Delta l_{ij} \mathbf{K}_{ijmn}^{QF}(\theta_p) \mathbf{n}_{ij} / \mathbf{W}_{ij}^* \mathbf{t}_{ij} \\ = t_i^c - \sum_{j=1}^J \Delta l_{ij} \mathbf{W}_{ij} \mathbf{n}_{ij} / W \mathbf{W}_{ij}^* \mathbf{t}_{ij}. \end{aligned} \quad (32)$$

Where *J* is the corresponding *j* of the unknown ordinate l_i^c along the chord where the cavity terminates on the *i*-th radial panel of the reference blade. *Z* is the number of blades and θ_p is the angle position of each blade.

Therefore Eqs. (29) through (32) lead to a set of simultaneous linear equations for the unknown

vortices G_{mn} , sources Q_{mn} and cavitation numbers σ_i , if initially assuming the cavity length at spanwise section r_i .

Here $\mathbf{W}_{ij}^*$ in Eq. (32) is corresponding to $\mathbf{W}^*$ in Eq. (5) and $\mathbf{W}^B$ may be given as its value at the initial step of the computation.

5.4 Cavity Shape

Applying the solutions obtained from the simultaneous equations, the cavity shape can be calculated from Eq. (5). This can be rewritten as the following discretized form at l_{ij} of the arbitrary chordwise ordinate which yields the height from the camber surface summing up the cavity and half the blade thickness.

$$\begin{aligned} t_{ij}^c = & \sum_{j=1}^J \left\{ \Delta l_{ij} \mathbf{W}_{ij} \mathbf{n}_{ij} + \Delta l_{ij} G_{ij} \mathbf{H}_{ij}^{GB} \mathbf{n}_{ij} \right. \\ & + \sum_{m=1}^M \sum_{n=1}^N G_{mn} \sum_{p=1}^Z \Delta l_{ij} \left\{ \mathbf{K}_{ijmn}^{GBS}(\theta_p) \right. \\ & + \sum_{k=N+1}^{NT} \left(\mathbf{K}_{ijmk}^{GBW}(\theta_p) - \mathbf{K}_{ijm+1,k}^{GBW}(\theta_p) \right) \left. \right\} \mathbf{n}_{ij} \\ & + \Delta l_{ij} Q_{ij} \mathbf{H}_{ij}^{QB} \mathbf{n}_{ij} \\ & + \sum_{m=1}^M \sum_{n=1}^N Q_{mn} \sum_{p=1}^Z \Delta l_{ij} \left\{ \mathbf{K}_{ijmn}^{QB}(\theta_p) \mathbf{n}_{ij} \right\} / \mathbf{W}_{ij}^* t_{ij} \left. \right\}, \end{aligned} \quad (33)$$

where J is chordwise index corresponding to the ordinate l_i as shown in Fig. 4.

The cavity thickness t_{ij}^{cav} can be obtained at l_i as follows;

$$t_{ij}^{cav} = t_{ij}^c - t_{ij}. \quad (34)$$

Here t_{ij} is half the blade thickness.

6. Results of Computation

In order to confirm the validity of the present method, the computation has been carried out for a Japanese standard propeller model (Table 1) with partially cavitating condition operating in a uniform flow.

The number of panels on the blade surface has been applied as follows:

Table 1 Principal particulars of model propeller

Model Propeller	MAU 3-50
Dia. (M)	0.250
Pitch Ratio	1.000
Exp. Area Ratio	0.500
Number of Blades	3

11 elements along the chord

6 elements along the span (equal spacing),

where the chord length is equally divided into 10 and, moreover, the nearest panel to the leading edge is divided equally.

It is required to increase the number of panels if one intends to improve the accuracy of the computation. The number of panels described above has been determined by examining the balance between the accuracy and the cost of the computation for the present purposes.

Each element of bound vortex and line source has been placed on the quarter chord line and the control point on the midspan of three-quarter chord position on each panel according to James⁽²⁾.

For the propeller wake geometry, the deformed model has been adopted which was introduced by the author⁽¹⁾ combining the contraction and the classical wake models with the same pitch as the local blade pitch distributions. This is suggested to predict both the propeller performance and the slipstream over the wide range of propeller loading with sufficient accuracy.

Thus defining the arrangements of panel division and propeller wake geometry, the simultaneous Eqs. (29) to (32) can be solved by setting up an operating condition of the propeller.

6.1 Influence of Blade Loading

In order to confirm the tendency of the solu-

Fig. 6 Cavitation patterns given for the trial calculations (MAU 3-50, $H/D=1.0$)

Fig. 7 Solutions of cavitation number σ for various loading conditions in case of Trial-1

tion toward the influence of blade loading, a simple pattern of sheet cavitation has been assumed (Trial-1 in Fig. 6) with the cavity end being zero.

Fig. 7 shows the solutions of the cavitation number σ at three advance ratios. The results that the solution σ becomes higher in the case of fixed cavity length when the propeller loading increases, indicating the same characteristics of the actual phenomena. The description "Non-Linear Solution" in the figure means the results of iterative computation with respect to the denominator in Eq. (5). The Non-Linear Solution agrees with the Linear Solution in the region from the midspan to the tip of the blade.

Fig. 8 displays the cavity shape by putting the solutions of vortices and sources into Eq. (33).

For the cavity length larger than 75%C (of the chord length) the solutions are also obtained with no sign of instability as in the solution of the basic two-dimensional cavitation theory. The results of cavity thickness show an optimal tendency from the fact that it becomes larger due to the increment of propeller loading.

The author is especially impressed that the cavity thickness around the tip region or at $0.9R$ reaches large enough, i.e. 10 to 25%C as shown in Fig. 8, because he introduced a three-dimensional correction factor to the solution of the two-dimensional cavitation theory for that region of propeller blade cavitation in the previous paper⁴. The correction factor, determined by

Fig. 8 Cavity shapes obtained for Trial-1 with closed cavity ends (No. 1)

investigating the full-scale measurements of cavity thickness, was multiplying by "five" the two-dimensional solution. The cavity thickness obtained by the two-dimensional theory is 2 to 3%C at that region in general. Since the present solution gains "five to ten" times that, it might be said that the prescribed correction factor "five" is correct.

6.2 Iteration for Cavity Shape

The computation has to be repeated to determine the cavity shape until the solution $\sigma(r)$ reaches the value of given cavitation number $\sigma_s(r)$ at each radial section of a propeller blade. According to the comparison of both mentioned above;

$\sigma(r) > \sigma_s(r)$: the cavity extent being increased along the chordwise panel at radial position r , or the thickness of the cavity end being expanded in the case of full cavity;

$\sigma(r) < \sigma_s(r)$: reverse of that described above. Thus, the iterative computation is to be carried out.

Fig. 9 shows the results of a that calculation in which the cavity extent and its closure condition are varied at each radial section in order to let the solution $\sigma(r)$ approach $\sigma_s(r)$ of the model experiment. As denoted in the upper table in Fig. 9, one can investigate the solutions approaching the given values by varying the combination of the cavity extent, its closure condition and

Fig. 9 Solutions of cavitation number σ for the various cavitation patterns (Trial-1, 2, & 3) and cavity end conditions (No. 1~5)

Linear or Nonlinear regarding each case of No. 1 to 5 at the same loading condition.

In the case of No. 5 the cavitation pattern cor-

responds to the trial-3 in Fig. 6 which has little larger extent and wider thickness of the cavity end at the sections with full cavity than that of No. 4. The solution of No. 5 comes so close to $\sigma_s(r)$ that it might be regarded as the approximate converged solution.

Figs. 10 and 11 are the solutions of sources q and vortices γ in the case of No. 5. With respect to the blade section with partial cavity, the distributions of q and γ show the discontinuity at the cavity ends, which is the common characteristic of two-dimensional cavitation theory. Moreover, the distribution of γ on the blade section with full cavity can considerably be seen flat.

Fig. 12 displays the results of cavity shape in the case of No. 5. For the blade sections near the root with large thickness, for example $0.53R$, the maximum thickness of the partial cavity is 1 to 2% C even if the cavity is assumed to separate at the leading edge. The corresponding solution of two-dimensional theory is 3 to 4% C , but the actual measurement is about 1% C quoted from ref. 4. The maximum thickness of the partial cavity can be reduced in the case of two-dimensional theory by assuming that the cavity starts at the minimum pressure point which locates downstream near the leading edge, and its results

Fig. 10 Source strength obtained for Trial-3 with open cavity ends (No. 5)

Fig. 11 Vortex strength obtained for Trial-3 with open cavity ends (No. 5)

Fig. 12 Cavity shapes obtained for Trial-3 with open cavity ends (No. 5)

show good agreement with the measurement¹³⁾.

From these considerations, the solution of the present three-dimensional theory can yield good results for the maximum thickness of the partial

cavity even if the cavity is assumed to separate at the leading edge.

In the present method the thickness of the full cavity end is supposed to vary in order to obtain the converged solution. However it must actually become a super-cavity stretching its tail longer when the cavitation number decreases.

Accordingly, one should expand this method to simulate such an actual phenomenon as three-dimensional cavitation including the super-cavity, which requires the increase of the unknown values and computing time in proportion to the cavity length.

In any case it is considered that the present method is enough to discuss the characteristics of the three-dimensional solution for the sake of simplification and time saving. With respect to the convergence of the solution, the present method shows the following relations by inquiring into the cases of No. 1 to No. 5;

cavity length $l_r^c \rightarrow$ increase solution $\sigma(r) \rightarrow$ decrease
 $\rightarrow$ decrease $\rightarrow$ increase

thickness of the full cavity end
 $l_r^c \rightarrow$ increase $\sigma(r) \rightarrow$ decrease
 $\rightarrow$ decrease $\rightarrow$ increase

propeller loading
 $K_T \rightarrow$ increase $\sigma(r) \rightarrow$ decrease
 $\rightarrow$ decrease $\rightarrow$ increase.

These tendencies guarantee the convergence of the solution by the present method, since the

Fig. 13 Comparison of cavitation patterns between the experiment and Trial-3 corresponding to the converged solution by the present three-dimensional calculation (No. 5)

parameters described above vary to adapt themselves to the tendency of the actual phenomena. That is to say, one can let $\sigma(\nu)$ agree with $\sigma_s(\nu)$ by controlling the parameters l_r^c or l_r^c at $l_r^c=1$.

6.3 Comparison with Experiment

Fig. 13 shows the comparison between the model experiment and the present solution of the case of No. 5 in a uniform flow. The experiment discloses the narrow sheet cavitation along the leading edge, while the solution presents the larger extent of sheet cavitation with the full cavity in the region from $0.65R$ through the tip.

Though the convergence of the present solution has been confirmed by referring to the actual phenomena in the former section, the calculated extent of cavity by the present method could not agree with the model experiment. In order to investigate the disagreement of the present solution with the model experiment, this has been compared with the conventional lift-equivalent method and the modified two-dimensional cavitation theory quoted from ref. 4.

Fig. 14 shows the pressure distributions on the propeller blade at the wetted surface condition with three kinds of cavitation number $\sigma_N (=P/\frac{1}{2}\rho n^2 D^2)$ including $\sigma_N=1.50$ in the case of No. 5 for each radial section. For the three conditions of σ_N the calculated results by the conventional method are shown in Fig. 15 compared with the present method. Here, the two-dimensional calculation has been carried out by using the

Fig. 14 Pressure distributions for lift-equivalent method (MAU 3-50, $H/D=1.0$)

interpolation method referred to in ref. 4 for the case that the cavity length increases beyond 75%C.

Fig. 15 shows the good agreement between the present method and the two-dimensional method, but the result of the lift-equivalent method is rather close to the experiment. The reason for the difference between the present method and the experiment is thought to be caused by σ_N being similar to the pressure distribution downstream the hollowed pressure near the leading edge on the blade. Accordingly, it should be noted that the observed sheet cavitation in the negative pressure region at the leading edge terminates around the hollowed pressure point of 10 to 20%C while the present result extends the cavity length to the trailing edge because of the slight difference between σ_N and the pressure distribution.

Another reason could be that the sheet cavitation is shortened by the effect of the laminar boundary layer on the blade surface due to the

Fig. 15 Comparison of cavitation patterns between the experiment, lift-equivalent method, 2-D cavity conventional method and 3-D cavity present method (MAU 3-50, $H/D=1.0$)

model scale, because it was observed that the lower radial position of the sheet cavitation had no variation when σ_N decreased.

Therefore adopting the leading edge roughness as a turbulent stimulator, the experiment will be close to the present results. This is thought to be a proper problem with respect to the technique for model testing¹⁴⁾. On reflection, the full scale observation may be rather close to the present method.

7. Concluding Remarks

A new theory on a cavitating propeller is proposed, treating the cavitation three-dimensionally, and using a lifting surface theory. A numerical method has also been developed based on this theory, applying the vortex lattice method and replacing the cavitation and the blade thickness by source elements, for a cavitating propeller operating in a uniform flow.

Based on the proposed theory, the following approaches are attempted.

- Discrete panels, vortex and source elements are placed on the camber surface to denote the blade loading and thickness effect including three-dimensional cavitation, while the boundary conditions are to be satisfied at the control points on both back and face surface of propeller blades.
- With respect to the self-induced velocity due to vortex distributions, the contributions from the chordwise elements are exactly considered together with that of the spanwise elements.
- The induced velocities by the whole vortices and sources are taken into account on the cavity closure condition without simplification.
- To evaluate the present method, the calculations are carried out for a model propeller and compared with other prediction methods and the

model test results.

The advantages of this method could be summarized as follows:

- At a certain loading condition, the solution $\sigma(r)$ can reach the value of given cavitation number $\sigma_s(r)$ by the iterative computation controlling the cavity length and the thickness of cavity end.
- The present method indicates no sign of the 75% chord length break down which is well-known shortcoming in case of the two-dimensional cavitation theory.
- With respect to the cavity thickness around the tip region, three-dimensional effect can naturally be taken into account, and the results give the cavity shapes that are sufficiently close to the ones observed in full-scale measurements.
- The results show the possibilities to eliminate the overestimate for a thick profile around the blade root which is generally the case by the two-dimensional theory.
- As for the cavitation extent, the present method gives a good agreement with other prediction method by the modified two-dimensional theory, but shows disagreements with the model test results and also with the results by the lift-equivalent method. This insufficient agreement of cavity extent between the present method and the model test may be caused by the effect of laminar boundary layer on the blade surface in the model test.

It is confirmed that the present method gives considerable improvements, in comparison with the conventional method, on the cavity thickness due to the three-dimensional flow. By examining the detailed characteristics of the present solution, it is however confirmed that the numerical

computation is exactly performed. Thus, the discrete function method as vortex lattice concept is a powerful technique to solve such a complex problem which seems very difficult to solve by applying mode function method etc.

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Appendix 1 Vortex Induced Velocity

The induced velocity $V_{ij}^G(\bar{X}_k\bar{X}_{k+1})$ at a control point P_{ij} due to a vortex element $\bar{X}_k\bar{X}_{k+1}$ on a set of horse-shoe vortex which is constituted from mn -th bound vortex and a couple of trailing vortices can be expressed by the law of Biot-Savart as follows:

$$V_{ij}^G(\bar{X}_k\bar{X}_{k+1}) \equiv V_{ijmn}^G = \frac{\Gamma_{mn}}{4\pi} \int_{\bar{X}_k}^{\bar{X}_{k+1}} \frac{d\mathbf{l} \times \mathbf{D}}{|\mathbf{D}|^3}, \quad (\text{A1.1})$$

where the vortex strength per unit length Γ_{mn} is constant on the horse-shoe vortex, $d\mathbf{l}$ the differential vector of the element, $\mathbf{D}$ the vector of the distance between $d\mathbf{l}$ and the control point. The subscripts m and n demote the indices of spanwise and chordwise directions on the propeller blade.

Dividing Eq. (A1.1) by the reference velocity W the following expression is derived:

$$V_{ijmn}^G/W = G_{mn} \mathbf{K}_{ijmn}^G \quad (\text{A1.2})$$

where supposing R being the radius of the propeller,

$$G_{mn} = \Gamma_{mn}/2\pi WR, \quad (\text{A1.3})$$

$$\mathbf{K}_{ijmn}^G = \frac{1}{2}R \int_{\bar{X}_k}^{\bar{X}_{k+1}} \frac{d\mathbf{l} \times \mathbf{D}}{|\mathbf{D}|^3}. \quad (\text{A1.4})$$

Therefore, the induced velocity V_{ij}^G due to the whole horseshoe vortices denoting propeller blades and its free vortex sheet can be described as follows:

$$V_{ij}^G/W = \sum_{m=1}^M \sum_{n=1}^N G_{mn} \left\{ \mathbf{K}_{ijmn}^{GS} + \sum_{k=N+1}^{N_T} (\mathbf{K}_{ijmk}^{GW} - \mathbf{K}_{ijm+1,k}^{GW}) \right\} \quad (\text{A1.5})$$

where,

$$\mathbf{K}_{ijmn}^{GS} = \mathbf{K}_{ijmn}^{GS \text{ span}} + \sum_{k=n}^N (\mathbf{K}_{ijmk}^{GS \text{ chord}} - \mathbf{K}_{ijm+1,k}^{GS \text{ chord}})$$

Hence, the superscripts S and W mean blade surface and free wake vortex sheet, M , N and $N_T - N$ denote the numbers of spanwise, chordwise and free vortex elements respectively.

Appendix 2 Source Induced Velocity

The source strength per unit length σ_{mn} is defined to be constant along mn -th spanwise element. Supposing the co-ordinate position of its both ends $\mathbf{X}_{mn}$ and $\mathbf{X}_{m+1,n}$, the induced velocity $\mathbf{V}_{ijmn}^Q$ at the control point P_{ij} due to the source element $\overline{\mathbf{X}_{mn}\mathbf{X}_{m+1,n}}$ as follows:

$$\mathbf{V}_{ijmn}^Q = \frac{\sigma_{mn}}{4\pi} \int_{\mathbf{X}_{mn}}^{\mathbf{X}_{m+1,n}} \text{grad} \left(\frac{1}{|\mathbf{D}|} \right) d\mathbf{l} \quad (\text{A2.1})$$

Non-dimensionalizing Eq. (AII.1)

$$\mathbf{V}_{ijmn}^Q/W = Q_{mn} \mathbf{K}_{ijmn}^Q \quad (\text{A2.2})$$

is expressed where

$$Q_{mn} = \sigma_{mn}/2\pi WR, \quad (\text{A2.3})$$

$$\mathbf{K}_{ijmn}^Q = \frac{1}{2}R \int_{\mathbf{X}_{mn}}^{\mathbf{X}_{m+1,n}} \text{grad} \left(\frac{1}{|\mathbf{D}|} \right) d\mathbf{l}. \quad (\text{A2.4})$$

The contribution from the whole source elements can therefore be denoted as follows:

$$\mathbf{V}_{ij}^Q/W = \sum_{m=1}^M \sum_{n=1}^N Q_{mn} \mathbf{K}_{ijmn}^Q \quad (\text{A2.5})$$

Appendix 3 Self-Induced Velocity due to Vortex Distribution

Discrete vortex elements are regarded as the concentration of the vortex distribution density γ . The relation between the discretized vortex strength per unit length Γ_{ij} and γ on ij -th panel of a propeller blade can be derived as follows.

$${}^b\gamma_{ij} = \Gamma_{ij}/\Delta c \quad (\text{A3.1})$$

$${}^t\gamma_{ij} = \frac{1}{2} \sum_{k=1}^j (\Gamma_{i-1,k} - \Gamma_{i+1,k})/\Delta s \quad (\text{A3.2})$$

where the superscripts b and t denote bound and trailing, Δc and Δs the mean length along chord and span of the panel. The vortex distribution densities ${}^b\gamma_{ij}$ and ${}^t\gamma_{ij}$ are supposed to be constantly distributed on ij -th panel.

The self-induced velocities due to the vortex distribution can be expressed at the control point P_{ij} as follows:

$${}^bV_{ij}^Q/W = -\frac{1}{2}{}^b\gamma_{ij}/W \equiv -\pi G_{ij}/(\Delta c/R) \quad (\text{A3.3})$$

$$\begin{aligned} {}^tV_{ij}^Q/W &= -\frac{1}{2}{}^t\gamma_{ij}/W \\ &\equiv -\pi \sum_{k=1}^j (G_{i-1,k} - G_{i+1,k})/(\Delta s/R) \end{aligned} \quad (\text{A3.4})$$

Here, ${}^bV_{ij}^Q$ and ${}^tV_{ij}^Q$ denote the induced velocities due to bound and trailing vortex distribution densities, and the values G_{0k} and $G_{M+1,k}$ of the root and tip panels are defined as follows:

$$G_{0k} = G_{1k}, \quad G_{M+1,k} = G_{M,k} \quad (\text{A3.5})$$

The expressions (A3.3) and (A3.4) yield the tangential velocities to the camber surface.

Appendix 4 Self-Induced Velocity due to Source Distribution

Referring to the case of vortex the source distribution density q_{ij} is expressed by using the discretized source strength per unit length σ_{ij} as follows:

$$q_{ij} = \sigma_{ij}/\Delta c \quad (\text{A4.1})$$

The self-induced velocity due to the vortex distribution can be described as the following expression at the control point P_{ij} .

$$V_{ij}^Q/W = -\frac{1}{2}q_{ij}/W \equiv -\pi Q_{ij}/(\Delta c/R) \quad (\text{A4.2})$$

This yields the normal velocity to the camber surface.