Dynamic Instability Analysis of Thin Shell Structures subjected to Follower Forces
(2nd Report) Numerical Solutions and some Theoretical Concepts

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Summary

Dynamic instability behaviour of thin shell structures subjected to follower forces are governed by several geometric and physical factors like curvature, boundary conditions, loading, disturbances and so on, contributing to the overall possibility of a dynamic failure at intervals of its deformation history. However, an actual threat to the ultimate stability or the static critical points may not be posed, unless and until when the pertinent damping forces are overcome. Disturbance forces of various types, which are omnipresent in a marine environment, may be the single most serious factor that could undermine all the other considerations and lead the structure to exhibit dynamic instabilities at a much earlier stage than that could be predicted by a static stability criterion.

In the previous paper, we proposed a general governing equation in monoclinically convected coordinates to deal with shell deformations and validated the applicability of the method of small disturbances to unravel the problems of shell instabilities, in general, through analyzing the natural frequencies of deflected shells at subsequent disturbed equilibrium states. Here, the proposed method is used extensively to investigate the same problem in more detail and reached at some generalisations for singly and doubly curved shells. Also, a discretized approximate equation is proposed to get an insight into the total problem in a much simplified way.

1. Introduction

Marine structures, whether they are stationary or mobile, are persistently subjected to the action of numerous stray forces in the form of external disturbances, in addition to the internal static or dynamic functional loads and their external reactionary loads. This situation is clearly one that warrants an extremely careful investigation of the structural stability of such systems, since the foregoing situation could lead to some catastrophic situations in which both human lives and property are at stake. The stability of shells that form an integral part of most marine structures, undergoing the action of internal or external hydrostatic or dynamic loads which are mostly follower type, becomes an extremely sensitive subject in the presence of disturbances which again could be of the follower type.

In technical terms, a form of equilibrium is said to be unstable if a disturbance, however small, causes a finite deviation of the system from the considered form of equilibrium. In a broader sense, this could be interpreted as some gradual or sudden deviations in the equilibrium paths, geometric or structural configurations, energy states or even some localised phenomena, finally leading to the failure of the shell to function as a contributor to the overall structural strength in the intended way. The load-deformation history of a shell in static equilibrium gives a fairly accurate picture of some of the characteristic instability points which are mainly static in nature. However, it is contended here that the transformations of the shell structure and its behaviour in the deformed configurations are more elaborately described by the pattern of changes of its frequencies of natural vibrations. The underlying logic behind this can be easily understood if one considers that the shell undergoes changes in curvatures and accompanying changes in both its bending stiffness and form resistance, in a local as well as an overall perspective, as it deforms from an equilibrium state. Thus, some of the modes of natural vibrations are rendered not sustainable anymore without altering their frequencies, gradually or drastically, to suit the new configuration.

The characteristic values of the stability problem of shell structures are analysed in this paper to determine the particular cases of local or transient dynamic instability phenomena and correlate it to the question of overall stability. Thus, it is shown here that any particular disturbance values over some equilibrium state can give rise to various instability behaviours, depending on

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Received 10th January 1992
Read at the Spring Meeting, 12th, 13th MAY 1992
the particular stability state and the disturbance. The possibilities of instable behaviour mainly considered in this paper are the phenomena of mode shifts, flutters, mode divergences, bifurcation and finally the uniform mode variations characteristic of 'cable effects'. It can be seen that the shell can have several possible points of unstable behaviour leading to intermittent stages of stability and instability, finally leading to dynamic flutters, ultimate failures or cable effects, depending on the shell type.

The governing equations dealing with this stability problem is expressed here in a partial differential form, which can be transformed suitably into the required eigenvalue problem. This equation by itself is formulated in the monoclinically convected coordinates, which enable one to directly deal with follower type loads in terms of the deflection components. The basic formulation of these equations and their explanations with sufficient details were already presented in the previous paper (1st Report40) and elsewhere1223). Also, the corresponding validations were given, wherever found necessary to support the basic formulations.

Numerical examples are attempted for two principal shell types representing the singly and doubly curved shells and the results are presented in the forms of load-deflection curves, natural frequency curves, mode shape graphs and so on. These results are summarised for a range of curvatures from shallow to deep shells in each of the singly and doubly curved cases, and the comparison between the two types are presented, as well. The Galerkin methods is used for the numerical calculations and the boundaries are assumed to be simply supported with inplane deflections arrested along the boundaries. The loading is assumed to be a uniformly distributed hydrostatic pressure, in all cases, acting on a shell with a projected square base of unit area.

To obtain an approximate idea of the character of the proposed governing equations, a first approximation is obtained from a truncated form and discretized using the Galerkin method. It is shown here that this approximate form can be used to predict static stability with a fairly high degree of accuracy and a correlation to stability ranges predictable through the Mathieu equation can be obtained by making certain rough assumptions.


The constituent relationships and the fundamental tensor geometry of the finite deformation process of thin shells were already expounded in the previous paper (1st Report40) and all the underlying principles with their details can be found in the references1-310-7). For the sake of brevity, this paper intends to give only the most essential ones of the relationships which are found necessary to maintain continuity and integrity.

2.1 Definition of the Shell Geometry

The tensor geometry of the deformed shell is expressed here in terms of the pre-deformation geometry of the shell middle surface and the deformation values. The fundamental assumptions of the Kirchhoff-Love hypotheses are assumed to be true through the entire finite deformation process of a thin shell.

The geometry of two particular shell types, each representing the singly and doubly curved shells, along with the definition of monoclinically convected coordinates over the middle surface of a general shell are given in Fig. 1.

![Fig. 1 The geometry of two particular shell types and the definition of monoclinically convected coordinates](image)

2.2 Equilibrium Equations for Finite Deformations of Thin Shells

The equilibrium equation for the finite deformation of a thin shell in the inplane and normal directions can be expressed as follows:

\[ N^a \| = M^a \| B^a = - p^a - m^a B^a \]  \[ \text{(1)} \]

\[ N^a B^a - M^a \| = - p^a + m^a \| \]  \[ \text{(2)} \]

Here, the membrane force tensor \( N^a \), the moment tensor \( M^a \) and the curvature tensor \( B^a \) of the post-deformation shell middle surface can be expressed using the elasticity tensor \( a^{a\beta} \), the initial curvature tensor \( b_{a\beta} \), stain tensor \( e_{a\beta} \) and the change of curvature tensor \( K_{a\beta} \), which are quantities representing the pre-deformation middle surface geometry, and the deflection components \( u', u \), as follows:

\[ N^a = D a^{a\beta} e_{a\beta} + K a^{a\beta}(b_{a\beta} b_{a\beta} - b_{a\beta} b_{a\beta}) \]

\[ M^a = K a^{a\beta} e_{a\beta} + K a^{a\beta}(b_{a\beta} b_{a\beta} - b_{a\beta} b_{a\beta}) e_{a\beta} \]  \[ \text{(3)} \]

Range convention: Greek indices \( a, \beta, \gamma, \ldots = 1, 2 \) Latin indices \( i, j, k, \ldots = 1, 2, 3 \)
\[ B_{as} = b_{as} + u^i_{as} - u^i_{ai}u^i_{as} + \frac{1}{2} (u^i_{ai}u^i_{as} - u^i_{ai}u^i_{as} - a^i_{ai}u^i_{as}) \]  

The elasticity tensor can be expressed using the metric tensors \((g^{as})\) of the pre-deformation middle surface, as follows:

\[ a^{as} = \left(1 - \frac{\nu}{2}\right)(a^{as}g^{ai} + \alpha^{as}a^{ai}) + \nu a^{as}a^{ai} \]

In the above equation, \(p^a\) and \(p^i\) are respectively the tangential component and the normal component in the positive \(z\) direction of the applied loads. \(m^e\) is the sum of the surface tractions and body forces contributed moment load. Also, the character symbols \(D\) (Extensional stiffness parameter), \(K\) (Bending stiffness parameter), \(E\) (Young’s modulus of elasticity) and \(\nu\) (Poisson’s ratio) are adopted here in line with the 1st Report.

The symbols \(\delta\) and \(\beta\) represent respectively the 2 and 3-dimensional covariant differentiation of the preceding middle surface quantities with respect to the corresponding post-scripts, and the underscript \(a\) denotes that the differentiation is to be done over the post-deformation quantities.

It is to be noted that the particular type of loading determines the proper representation of all the load components, and they along with the rest of the tensor quantities are to be appropriately interpreted in terms of the corresponding physical components, wherever the necessity arises.

To obtain some easily interpretable and numerically manageable forms of the general governing equations, Eq. (1) and Eq. (2) have to undergo many drastic operations of neglecting the small order terms by a critical and objective evaluation of the order of each component term, before and after the covariant differentiation. This process and resulting equations in partial differential forms were given in the 1st Report.

The complete range of the component terms for the general shell types having all but the principal components of the initial metric and curvature tensors being zero were also given thereby.

2.3 Stability Equations for Thin Shells subjected to Small Disturbances

The equilibrium of a shell undergoing deformations is always prone to exhibit certain unstable behaviour depending on the various disturbances that accompany the state change or loading by virtue of either internal or external excitations those are omnipresent in any physical situation, as has already been stated. This feature can be brought out by examining the pattern of variations of the frequency of disturbed small vibrations at consecutive static equilibrium states of the shell. Equating the variations in the unloaded equilibrium state to the inertial forces introduced by such variations, the following equations can be formulated directly from the governing equations [Eq. (1) and Eq. (2)] and the d’Alembert’s principle, to represent the stability at the disturbed state of equilibrium.

\[ \delta N^{as}B_{as} + M^{as}B_{as} + M^{as}B_{as} + m^sB_{as} = -\rho\ddot{u}^{as} \]  

\[ \delta \dot{N}^{as}B_{as} = N^{as}\delta B_{as} - M^{as}\delta u = -\rho\ddot{u}^{as} \]

where, the (\(\delta\)) and (*) marks show whether the corresponding quantities consist of the static equilibrium state values or the incremental quantities at a momentarily disturbed state, respectively. Evidently, the righthand sides of the above equations give the inertia forces due to the disturbances, where \(\rho\) is the resultant mass density of the shell material per unit surface area at the disturbed state and \(t\) is the uniform shell thickness.

These stability equations can also be formulated independently from the fundamental tensor geometrical relationships by considering the variations in the position vector of any middle surface point, between the static and disturbed equilibrium states. The incremental property from a preceding static equilibrium state is shown in Fig. 2 and contained in the following equation:

\[ R + \delta R = r + \delta u \]

where, \(r\) and \(R\) are the position vectors at the initial equilibrium and an intermediate static equilibrium states, respectively. \(u(=u_{ai})\) is the deformation vector at the static equilibrium state.

Now, the stability equations can be rewritten in a simpler form by substituting for all the component terms and performing the covariant differentiations, as required. It can be seen that a considerable number of terms in this expanded equation are constituted of different combinations of quantities, neglecting which would not much affect the overall structure of the equations. Some of these terms which could be dropped are the different multiples of contravariant tensor combinations like inplane deflections, initial curvatures and curvature changes, strains, Christoffel symbols and finally the multiples of such smaller terms with the bending stiffness parameter.

A process of reevaluation of the expanded equation as given above, in a similar manner as was done in the case of the equilibrium equations [Eq. (23), Eq. (24) 1st Report] give us the following partial differential form.
of the stability equations.

\[
\begin{aligned}
D_{o}^{\alpha \beta \gamma} & \left[ u_{x_{\alpha}} + u_{x_{\beta}} + u_{x_{\gamma}} \right] - (u_{x_{\alpha}} + u_{x_{\beta}} + u_{x_{\gamma}}) = - \rho \iota \left( u_{x_{\alpha}} + u_{x_{\beta}} + u_{x_{\gamma}} \right) \\
& \left[ u_{x_{\alpha}} + u_{x_{\beta}} + u_{x_{\gamma}} \right] - \left. \rho \iota \left( u_{x_{\alpha}} + u_{x_{\beta}} + u_{x_{\gamma}} \right) \right|_{a} = - \rho \iota \left( u_{x_{\alpha}} + u_{x_{\beta}} + u_{x_{\gamma}} \right)
\end{aligned}
\]

Equation (10)

\[
D_{o}^{\alpha \beta \gamma} \left[ u_{x_{\alpha}} + u_{x_{\beta}} + u_{x_{\gamma}} \right] - \left. \rho \iota \left( u_{x_{\alpha}} + u_{x_{\beta}} + u_{x_{\gamma}} \right) \right|_{a} = - \rho \iota \left( u_{x_{\alpha}} + u_{x_{\beta}} + u_{x_{\gamma}} \right)
\]

Equations (10) can be treated as a truncated approximation of the discretized general governing equations of equilibrium for shells undergoing finite deformations. It can be shown that the above equation is sufficiently accurate to obtain a fair estimate of finite deformations of shells in general, and quite dependable to deal with shallow shells undergoing small deformations in particular.

The stability equations that can be derived from Eq. (14) using the method of small disturbances can be written as follows:

\[
m v^2 + k v + r (u^2) + c (u^2) = 0
\]

where, \( m = \rho t \) is the mass per unit area of the shell middle surface and the double dots above the deflection term denotes differentiation with respect to the time variable \( t \). The coefficients of other \( u \) terms are as follows:

\[
\begin{align*}
\kappa & = \psi_1 + 2 \psi_2 v + 2 \psi_3 (u^2) \\
r & = \psi_4 + 2 \psi_5 \chi \\
c & = \psi_3
\end{align*}
\]

Equation (15)

The above set of simultaneous equations can be solved algebraically in terms of the \( u \) terms and the resultant equation can be written as given below:

\[
\left. \psi_1 + \psi_2 (u^2) + \psi_3 (u^2) \right|_{a} = - \rho \iota
\]

Equation (14)

The coefficients of \( u \) terms in Eq. (13) and Eq. (14) are listed in Appendix A in their fully expanded explicit forms. Here, the load elements \( p \) are purely hydrostatic uniform pressure loads.

The eigenvalue problem and associated considerations for the stability analysis presented in the 1st Report remain the same for the numerical formulation and calculations in this paper, as well.

2.4 A Truncated First Approximation of the Governing Equations and a Stability Criterion

The equilibrium equations and the stability equations derived here can be treated as not easily visualizable in any simple analytical form, unless a very rigorous numerical approach is employed. As a result, no direct conclusions regarding their nature can be assumed at the first glance. However, certain simplifications in the form of a truncated deflection series and subsequent discretization using suitable variational methods could bring out an approximate form of the total equation to the scaled down level of a simple algebraic equation.

Here, we adopted the following truncated Fourier series to represent the deflection components, \( u_i \) (i=1, 2, 3).

\[
u_i = \sum_{m} \phi_1 (\theta_1, \theta_2)
\]

Equation (12)

\[
u_1 = \sum_{n} \phi_2 (\theta_1, \theta_2)
\]

Equation (13)

The above set of simultaneous equations can be solved algebraically in terms of the \( u \) terms and the resultant equation can be written as given below:

\[
\left. \psi_1 + \psi_2 (u^2) + \psi_3 (u^2) \right|_{a} = - \rho \iota
\]

Equation (14)

Considering the shell as a structural element encompasses the requirement that the initial continuity and strength are retained to a sufficient minimum level even at the maximum expected performance level of the total structure. This underlines the fact that the incidence of any unexpected factor, such as an external disturbance or a random excitation force, should not endanger the integrity of any element, within the normal performance ranges. Static failure levels, which are
predictable with considerable accuracy, may not pose a threat since that aspect would normally be taken care of in the design stage. However, various dynamic aspects which may induce some instability behaviour are not fully accountable with sufficient level of accuracy, since the unknown factors involved are too many.

To deal with the problem of shell stability, here we consider the cylindrical and spherical shells which are representative of singly and doubly curved shells, respectively. Results of numerical calculations performed using the stability equations derived from the general governing equations are presented in this section. All examples given here are for shells of projected square bases of unit area and simply supported with all inplane deflections at the boundaries arrested. A uniform hydraulic pressure loading acting on the entire shell surface in the anti-radial direction is adopted throughout. The shell thickness $t$ is taken to be of a sufficiently small order in comparison with the radius $R$ and the principal chord lengths $[a=l_1=l_2, t/a=0.01]$, to represent the fundamental assumptions properly.

Galerkin's method and numerical integrations using the Gaussian quadrature are used for numerical solutions and the following physical constants are adopted throughout.

- Young's modulus, $E = 2.0 \times 10^9$ kg/m$^2$ (1.96 $\times 10^{11}$ N/m$^2$)
- Poisson's ratio, $\nu = 0.3$
- Material density, $\rho = 7.85 \times 10^3$ kg/m$^3$
- Acceleration due to gravity, $g = 9.81$ m/s$^2$

The differential geometrical values of the shell middle surface are calculated from the basic relationships, a detailed account of which can be found in the references $^1$-$^4$, $^5$-$^8$.

3.1 Stability Ranges of Singly and Doubly Curved Shells

Numerical calculations performed on cylindrical and spherical shells of various curvatures, from shallow shells to considerably deep shells, have brought out some interesting characteristics of their stability behaviour. These results are presented in a condensed form in Fig. 3 and Fig. 4, where the types and ranges of instability in each case of curvature are shown. The figures are mostly self explanatory, whereas the method of each evaluation and their justifications are presented in subsequent sections.

Fig. 3 shows the case of the spherical shell, which by
virtue of its double curvature has a complicated division of stability ranges and instability characteristics. Fig. 4 shows the case of the cylindrical shell which is rendered less complicated by the absence of one principal curvature, in spite of the fact that the actual evaluation of individual characteristics is eventually made more problematic for the same reason, than a doubly curved shell.

The most notable feature of these figures are the lower bounds of dynamic instability, which are considerably below the curves of dynamic or static failures. The region of transient dynamic instability between these curves are prone to introduce instabilities of different kinds, mainly of the mode shift type, at several points of the loading history, depending on the type and amount of disturbances or other excitations. As a result, the shell as a structure can not be considered fully dependable during the range of transient instabilities.

Also, it can be noted that shells of large curvatures which are considerably 'membrane rigid' due to their deeper forms, are subject to dynamic flutters or local reverse bucklings, usually leading to ultimate failures. However, shallow shells may overcome some of the initial instabilities using their 'membrane flexibility' and are more likely to continue carrying further loads in some assumed new equilibrium state, mostly like an elastic cable. Evidently, the region of extremely shallow shells can be found to be almost completely stable, except for some transient initial instabilities which would be overcome in most cases, to continue deforming until an elastic failure occurs.

3.2 Particular Examples of Instability

The analysis of stability ranges for shells of various curvatures described in the previous section contains a number of specific instability characteristics. Many of these characteristics are common knowledge, but some examples for each particular class of instability are presented with relevant details in this section, for the sake of clarity. An overall perspective of the situation is supposed to be obtained from these examples, where the load-frequency curves, load-deflection curves, mode shape graphs and some other specific graphical explanations form the basis of all the conclusions reached thereby about the type of instability in each class.

3.2.1 Transient Unstable behaviour due to Mode Shifts

One of the most prominent instability behaviours exhibited by shells of generally any curvature is the transient phenomenon of shifting the mode frequencies at some particular loading stage abruptly from uniform or steady variation patterns to a different frequency level altogether. This could be either the case of an individual mode only or often an intermodal transfer, sometimes involving even multiple modes at the vicinity of the same loading stage. This phenomenon could be easily explained, if one considers that the geometrical process of deformations at some stage renders some of the natural modes of vibrations not sustainable anymore without altering their frequencies, to suit the newly attained stability configuration. In some other instances this could be due to transient vibrations introduced by some mode resonance too, which deposes the overall stability momentarily.

Fig. 5 shows an example of mode shifts occurred in the case of a spherical shell (R/a=8,0). Here, an intermodal shift occurs between modes (1, 3) and (3, 2) as shown. The corresponding mode shapes before and after the shift occurs are also shown alongside in the figure. It should be noted that this shift was a transient phenomenon that temporarily disrupted an otherwise smooth pattern of stable behaviour. It was observed that the occurrence of mode shifts is possible at any loading stage depending on the disturbances that are present.

![Fig. 5 Illustration of an example for a mode shift (spherical shell, R/a=8,0)](image)

3.2.2 Local Reverse Buckling and Average Deflection

It is generally understood that a load acting symmetrically to the axes of symmetry of a shell with uniform symmetric sections should produce symmetrical deformations with respect to the original axes of shell symmetry. However, this axiom fails when it comes to nonconservative loading and large deflections, especially in the case of deep shells. In such cases, the successive deformed shapes could produce some complicated equilibrium configurations giving rise to localised instability behaviours, such as 'local reverse bucklings' and 'local snap-backs'. This could lead one to lose an overall perspective of the deformation level and inhibits a proper evaluation of the total stability situation. In order to obtain an overall idea of the deflection level, one can resort to the concept of average deflections. This phenomenon is a common feature of deep shells, which are presented in the load-deflection part of Fig. 6 and Fig. 7, where a considerably deep spherical shell and a cylindrical shell (R/a=1,0) are shown, respectively. It can be seen that the centres of both the shells undergo reverse bucklings. In the case of Fig. 7, it can be seen that the two points in the longitudinal quarter line (points C and D) exhibit monotonically increasing.
deflection patterns. However, the average deflection curves provided alongside the other curves have the same monotonic pattern in both figures, which shows that the overall trend of deformation is monotonically increasing.

3.2.3 Dynamic Flutter and Instability

Transformations in natural frequencies of vibrations during the deformation process of a shell could ultimately bring it to a particular situation where a couple of geometrically compatible modes attain the same frequency of vibration. This will evidently lead to resonant vibrations and accompanying mode shifts which sometimes could be fatal to the stability of the structure to the extent of undermining the ultimate stability. The case of flutter being the cause of ultimate failure would be more predominant for shells acted upon by follower forces. Also, deep shells are more prone to fail this way than shallower ones, which is evident from the stability range curves, Fig. 3 and Fig. 4.

The case of the spherical shell and cylindrical shell ($R/a = 1.0$) in Fig. 6 and Fig. 7 are examples of this phenomenon. Both shells undergo a complicated deformation pattern involving local reverse bucklings and mode shifts at several points, finally leading to flutter type dynamic instability and ultimate failures.

3.2.4 Bifurcation and Unsymmetrical Buckling

Bifurcation of equilibrium paths could occur at points where the shell undergoes local reverse bucklings, mode shifts or transient flutters. These are evidently points where the equilibrium path has stationary points for monotonically increasing loads. Bifurcations will lead to unsymmetrical buckling when the preceding activity is an unsymmetrical movement like a local reverse buckling, among other possible reasons. A shell that has undergone bifurcations may still be statically stable until the ultimate failure at some higher state of loading, except for the frequent threat of other dynamic instabilities like mode shifts.

The case of a spherical shell ($R/a = 4.0$) undergoing bifurcations is shown in Fig. 8. It can be noted that this shell undergoes local reverse bucklings and bifurcations in an overall scale, as can be seen from the average deflection curve. However, the shell may still continue to bear more loads depending on the level of disturbances present. It was observed that the deformation process of shells in the bifurcation category [Fig. 3, Fig. 4] to be extremely complicated, as compared to shells of other categories.

For the sake of presenting a more visual idea of the actual process, the deformed shapes through the complete loading history at two transverse cross sections of the same shell as in Fig. 8, is given in Fig. 9. Different positions through which the shell deforms and bifurcates to attain a final unsymmetric configurations are marked in the figure.

3.2.5 Snapping or Symmetrical Buckling

Snapping or symmetrical bucklings are found to occur mainly for comparatively shallow doubly curved
shells, as is evident from the stability range curves [Fig. 3, Fig. 4]. Obviously, there is a range of loading stages for such shells at which the threat of such energy level shifts are the highest.

Fig. 10 shows the case of a spherical shell \((R/a=6.0)\) which undergoes snapping is shown. The natural frequency graphs show a severe pattern of mode shifts which indicate the equilibrium readjustments that take place much before the snapping occurs. It can be seen that the monotonic pattern of deflection graphs is not the decisive indication of a completely stable dynamic behaviour. Thus, the disturbance forces may endanger the stability of such shells at an earlier stage and enforce a dynamic failure.

### 3.2.6 Instability due to Divergence

Divergence occurs when any one of the modal frequencies ceases to be real, whereby rendering the shell equilibrium to ‘slide’ for some small increment in the load, which is found to occur mainly for shallow shells. [Fig. 3, Fig. 4]. Evidently, a diverged equilibrium may still regain its static stability at a different energy level when the new configuration allows to reestablish the vibration modes at a new frequency level, beyond which the cable effect predominates, except for some minor dynamic instabilities.

The case of a shallow spherical shell in the divergence category \((R/a=10.0)\) is shown in Fig. 11. It can be seen that the \((1,1)\) mode causes the possibility of a divergence at which all other modes also haphazardly shift to higher frequency levels. It can be observed that
the vanishing fundamental mode may render the shell 'weak' for a while, forcing it to diverge to some higher level of stability. Evidently, this could be very fatal depending on the amount of disturbance present at the time.

3.2.7 Cable Effect beyond the Critical Points of Dynamic Instability

Cable effect represents the stage of tranquil elastic deformation characteristic of extremely shallow shells. This could also occur in deeper shells, provided the dynamic instabilities in such cases are successfully overcome and the shell retained its static equilibrium configuration. Usually, the incidence of cable effect makes the shell able to withstand higher levels of loads until an elastic failure occurs.

Fig. 12 shows the case of an extremely shallow spherical shell (R/\(\alpha\) = 24.0) which has a pattern of comparatively smooth natural frequency transitions and load-deflection history, except for some mode shifts, which were evidently not fatal. It was observed that shallow shells that successfully overcome the dynamic and static critical points of instability would eventually attain the cable effect.

3.3 A Comparative Analysis of Stability Ranges between Singly and Doubly Curved Shells

The principal geometrical difference between singly and doubly curved shells is evidently that the former has a zero Gaussian curvature. This could be easily shown to have great implications when we compare the deformation or stability characteristics of these shells. The influence of curvature on bending deflections of shells can be considered to be a property of its 'form' and thus be called the 'form resistance'.

A study of the comparative absolute influence ranges of the linear terms of bending stiffness and form resistance on bending deflections of singly and doubly curved shells is shown in Fig. 13. The different cases into which the curvature ranges of shells are divided here, is supported by the explanations given in Appendix-B. The absolute influences of bending deflection and form resistance are denoted respectively by \(R_b\) and \(R_f\) in the figure, where a value of unity shows complete influence, which are defined as follows:

\[
R_b = \frac{L_{3\omega_1}}{L_{3\omega_1} + L_{3\omega_2}}, \quad R_f = \frac{L_{3\omega_2}}{L_{3\omega_1} + L_{3\omega_2}}
\]

The increasing absolute influence of form resistance and the correspondingly decreasing absolute influence of bending stiffness on bending deflections with increasing curvatures can be seen from the figure. Also, the total shift of the curves for singly curved shells towards the large curvature end explains the persisting influence of bending stiffness much longer than in the case of doubly curved shells.

Here, the ranges of various instability regions plotted on Fig. 13 enables one to correlate them to the influence of curvatures. It can be seen that clear subdivisions of instability regions of cylindrical shells are fewer and generally shifted towards the large curvature end, which
can be attributed to the prolonged influence of bending stiffness and smaller relative share of form resistance in the case of singly curved shells, when compared to doubly curved shells. Thus, it can be said that the extra curvature generally introduces more instability regions. A similar effect can be deduced from the fact that the regions of flutter instability in both cases begin in the vicinity of the same curvature range, which shows that the still remaining presence of bending stiffness in the case of singly curved shells does not contribute much to postpone the incidence of dynamic flutter.

The results of the critical evaluation of the dynamic instability behaviour of shells conducted in this paper are compared with some of the other results for doubly curved shells. Fig. 14 shows this comparison where it can be found that the numerical results obtained through the present analysis using the complete range of terms of the governing equations [Eq. (10), Eq. (11)] and the approximate criterion formulated thereafter [Eq. (16)] are compared well with a classical equation and with a well established numerical result. Obviously, the classical method, which gives an approximate formula for the critical loads of complete spherical shells, has predicted higher values of critical points when compared to the present method. However, it can be seen that the results of the present method can predict the lower bounds of dynamic instability also, and shells of much deeper curvatures too can be analyzed, than would be possible through other methods. Also, it should be noted that the approximate criterion of the present method gave fairly accurate predictions for the complete range of shallow shells. The present method has the added advantage of defining the shells in monoclincally convected coordinates and also the assurance of using fully follower type loads to represent uniform pressure loads.
4. Conclusions

The applicability of the general governing equations for finite deformations of shells to determine the stability characteristics of singly and doubly curved shallow and deep shells, using the method of small disturbances, is presented here with corresponding theoretical formulations, several numerical examples and detailed explanations. The stability characteristics determined here are justified in comparison with the result of a classical shell formula and a well-established numerical result, in addition to providing the results of a simplified formula derived from the present governing equation itself.

It is believed that the method of small vibrations applied to the proposed governing equations can provide a very good estimation of the stability characteristics and give a deeper insight into the intricacies of the problem of shell stability.

References


Appendix A

Coefficients of the Truncated Discretized Form of the Simplified Governing Equations

A-1: Coefficients of Eq. (13)

\[ l_{11} = -D \left\{ (a^{11})^2 \frac{\pi^2 \Omega^4}{4 \Omega^4} - \frac{1}{\pi} \right\} \]
\[ + a^{11} a^{22} \left\{ \frac{1 - \nu}{2} \frac{\pi^2 \Omega^4}{16 \Omega^4} - \frac{3 - \nu}{2} \frac{\Omega^4}{8 \Omega^4} \right\} \]
\[ l_{12} = D \left\{ (a^{11})^2 \frac{12 \Omega^4}{9 \pi^2} \Gamma_{11}^2 + \frac{1 + \nu}{2} a^{11} a^{22} \left\{ \frac{8 \Omega^4}{9 \pi^2} \Gamma_{11}^2 - \frac{4}{9} \right\} \right\} \]
\[ l_{13} = D \left\{ (a^{11})^2 b_{11} + \nu a^{11} a^{22} b_{22} \frac{\Omega^4}{4 \Omega^4} \right\} \]
\[ l_{14} = D \left\{ (a^{11})^2 \left\{ \frac{1 + \nu}{2} \frac{\Omega^4}{9 \pi^2} \Gamma_{11}^2 - \frac{4}{9} \right\} \right\} \]
\[ l_{22} = D \left\{ (a^{22})^2 \frac{\Omega^4}{4 \Omega^4} \frac{\pi^2}{4} \right\} \]
\[ l_{23} = D \left\{ (a^{22})^2 b_{22} + \nu a^{11} a^{22} b_{11} \frac{\Omega^4}{4 \Omega^4} \right\} \]
\[ l_{33} = D \left\{ (a^{11})^2 b_{11} + \nu a^{11} a^{22} b_{22} \frac{\Omega^4}{4 \Omega^4} \right\} \]
\[ n_1 = D \left\{ (a^{11})^2 \frac{\Omega^4}{4 \Omega^4} + (a^{22})^2 (n_1)^2 \frac{\pi^2}{256 (\Omega^4)^2} \right\} \]
\[ n_2 = D \left\{ (a^{22})^2 \frac{\Omega^4}{4 \Omega^4} + (a^{11})^2 (n_2)^2 \frac{\pi^2}{256 (\Omega^4)^2} \right\} \]
\[ n_3 = -D \left\{ (a^{11})^2 \left\{ \frac{2 \Omega^4}{\Omega^4} \right\} + \left( 1 + 3 \nu \right) a^{11} a^{22} \frac{\pi^2}{192 \Omega^4} \right\} \]
\[ n_{33} = D \left\{ 1 \right\} \left\{ (a^{11})^2 b_{11} (\Omega^4)^2 + \nu a^{11} a^{22} (b_{22} (\Omega^4))^2 \right\} \]
\[ n_{33} = -D \left\{ (a^{11})^2 (\Omega^4)^2 \left( \frac{5 \pi^2}{36 (\Omega^4)^2} \right) \right\} \]
\[ n_{33} = -D \left\{ (a^{22})^2 (\Omega^4)^2 \left( \frac{3 \pi^2}{812 (\Omega^4)^2} \right) \right\} \]

Here, \( \Omega \) and \( \Omega \) are the subtended half angles in the principal directions.

A-2: Coefficients of Eq. (14)

\[ \psi_1 = \frac{1}{\psi_1} \left\{ l_{11} \psi_1 - l_{12} \psi_2 + l_{13} \right\} \]
\[ \psi_2 = \frac{1}{\psi_2} \left\{ l_{11} \psi_2 - l_{12} \psi_1 + n_{331} \psi_2 - n_{332} \psi_1 + n_{333} \right\} \]
\[ \psi_3 = \frac{1}{\psi_3} \left\{ l_{11} \psi_3 - l_{12} \psi_1 + n_{331} \psi_3 - n_{332} \psi_2 + n_{333} \right\} \]

where the \( \psi \) terms are as follows:

\( \psi_1 = l_{11} l_{22} - l_{12} l_{12} \)
\( \psi_2 = l_{11} l_{33} - l_{13} l_{13} \)
\( \psi_3 = l_{11} l_{33} - l_{13} l_{13} \)
\( \psi_4 = l_{11} n_{22} - l_{12} n_{12} \)
\( \psi_5 = l_{11} n_{22} - l_{12} n_{12} \)
Appendix B

A Categorization of Doubly Curved Shells according to the Absolute Influences of various Terms in the Linear Stiffness Matrix

The absolute influence of different terms in the stiffness matrix can be numerically analyzed to determine the contribution of each towards the inplane or bending deflections. An approximate idea of such contributions can be obtained by analyzing only the linear terms from the general governing equations\(^{1-3}\). The complete range of terms of the general governing equations are given in the 1st Report\(^4\). The same notations as was used thereby are reproduced in the details of Table 1 below, which demonstrates a categorization of doubly curved shells with projected square bases of unit area, according to their subtended angles which are directly related to curvatures. The deterministic terms of the stiffness matrix in each category are shown in the Table.

It can be seen that the coefficient matrix \( C_i \) of Case 1 is the ordinary plate equation. Cases 2 and 3 can be analyzed to find that they correspond to the usual governing equations for shallow shells. The presence of terms with Christoffel symbols \( (L_{1,2}, L_{1,2}, L_{2,2}, L_{2,2}) \) in the range of Case 4 shows the differential influence of inplane elements. Finally, Case 5 shows that for shells of still higher curvatures the term due to bending stiffness \( (L_{3,2}) \) has vanished, retaining the form resistance term \( (L_{3,2}) \).

<table>
<thead>
<tr>
<th>Case</th>
<th>Range of Subtended half angles ( \Omega ) (rad.)</th>
<th>Coefficient Matrix ( C_i )</th>
</tr>
</thead>
</table>
| 1    | \( \Omega \leq 1 \times 10^{-4} \) | \[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\] |
| 2    | \( 1 \times 10^{-4} < \Omega \leq 5 \times 10^{-4} \) | \[
\begin{bmatrix}
l_{1,1} & l_{1,1} & l_{1,2} \\
l_{2,1} & l_{2,1} & l_{2,2} \\
l_{3,1} & l_{3,1} & l_{3,2}
\end{bmatrix}
\] |
| 3    | \( 5 \times 10^{-4} < \Omega \leq 5 \times 10^{-2} \) | \[
\begin{bmatrix}
l_{1,1} & l_{1,1} & l_{1,2} \\
l_{2,1} & l_{2,1} & l_{2,2} \\
l_{3,1} & l_{3,1} & l_{3,2}
\end{bmatrix}
\] |
| 4    | \( 5 \times 10^{-2} < \Omega \leq 1 \times 10^{-1} \) | \[
\begin{bmatrix}
[l_{1,1} + l_{1,2}] & [l_{1,1} + l_{1,2}] & l_{1,2} \\
[l_{2,1} + l_{2,2}] & [l_{2,1} + l_{2,2}] & l_{2,2} \\
[l_{3,1} + l_{3,2}] & [l_{3,1} + l_{3,2}] & l_{3,2}
\end{bmatrix}
\] |
| 5    | \( 1 \times 10^{-1} < \Omega \leq 5 \times 10^{-1} \) | \[
\begin{bmatrix}
[l_{1,1} + l_{1,2}] & [l_{1,1} + l_{1,2}] & l_{1,2} \\
[l_{2,1} + l_{2,2}] & [l_{2,1} + l_{2,2}] & l_{2,2} \\
[l_{3,1} + l_{3,2}] & [l_{3,1} + l_{3,2}] & l_{3,2}
\end{bmatrix}
\] |

Appendix C

Mathieu Equation and a Correlation to the Stability Ranges of Shells - A Conceptual Approach

Mathieu's differential equation, which is stated below, is bounded only for certain combinations of the parameters \( \delta \) and \( \epsilon \).

\[
\ddot{\theta} + (\delta + \epsilon \cos \tau)\theta = 0
\]  

(C.1)

The diagram giving the stable and unstable domains, which is due to Strutt\(^5\), in the plane of \( (\delta, \epsilon) \) is given in Fig. 15. The unstable regions are shown shaded and the complete diagram is symmetric with respect to the \( \delta \)-axis.

The stability criterion proposed in Section 2.4 can be extended further by introducing up to the second order terms of disturbances in Eq. (15) into consideration, neglecting the third order terms for being small, and a correlation can be brought out with the Mathieu equation and the Strutt's diagram using the following procedures. Eq. (15) can be rewritten in the following form by partly substituting its third term using the linear solution given by Eq. (17).

\[
u \ddot{\xi} + (\Omega^2 + \gamma \cos \Omega \tau) \nu \xi = 0
\]

(C.2)

where, \( \gamma = (rA_i/m) \)

The form of Eq. (C.2) can be seen to be identical to that of Eq. (C.1) and in such a case the stability characteristics of the shell would become dependent on the size of \( \Omega \) and \( \gamma \) in Eq. (C.2).

Now, in line with the linear solution Eq. (17), the following equation is assumed to be the solution to the nonlinear form of Eq. (15).

\[
u \ddot{\xi} = A_n \cos (\Omega + \Delta \Omega \tau)
\]

(C.3)
Galerkin’s method is employed to determine the unknown coefficient $A_n$ of Eq. (C.3) from Eq. (C.2), which gives the following result.

$$A_n = \frac{3k}{4\gamma} \left[ 1 + \frac{\Delta O}{\Omega} \right] \left[ \frac{\pi}{2} \left( \frac{\Delta O}{\Omega} \right)^2 \right]$$

$$+ \left( \frac{\Delta O}{\Omega} \right) \sin \left( \frac{\pi O}{\Omega + \Delta O} \right) + \sin \left( \frac{\pi \Delta O}{\Omega + \Delta O} \right)$$

(C.4)

Using Eq. (C.4) in Eq. (C.3) after making the following substitutions, we can rewrite the stability equations and obtain Eq. (C.8) below in the form of Eq. (C.1):

$$\tau' = (\Omega + \Delta O) \tau$$

(C.5)

$$\epsilon = \frac{\gamma m A_n}{(\Omega + \Delta O) \tau}$$

(C.6)

$$\delta = \frac{1}{(1 + \frac{\Delta O}{\Omega})^2}$$

(C.7)

Now, the boundary of the Principal Region of Instability\(^{(12)}\) can be considered to be reached at the point where the term $(\Omega + \Delta O)$ coincides with $2\Omega \sqrt{1 - \epsilon/2}$, as the nonlinear terms of disturbance become prominent in the stability equation. It has been numerically found that this happens approximately at $\delta = 0.44, \epsilon = 0.65$, where there may be a strong possibility that the shell may become unstable as shown in Fig. 15, where the progressive advance of the stability characteristic $(\delta, \epsilon)$ through the stable domain to the boundary of the principal region of instability is shown by arrows starting at $\Omega = 0$. Thus, it can be concluded that there exist possibilities of a dynamically unstable behaviour at several points other than the axis of zero stability coefficients.