Optimal Elastoplastic Structural Design Using Sensitivities by Incremental Procedure  
(1st Report) Plane Truss Structure

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Summary

This paper describes a method for structural optimization considering the elastoplastic structural behaviour. The proposed method consists of three parts: elastoplastic incremental structural analysis, elastoplastic sensitivity analysis, and optimization. The structural analysis and sensitivity analysis are performed using the incremental finite element method. For structural optimization, the Sequential Unconstrained Minimization Technique (SUMT) is used.

In the optimization process, reanalysis is performed using the elastoplastic sensitivities. To get rational reanalysis solutions, some limitations should be imposed on the magnitude of changes in design variables. The maximum allowable changes in design variables are automatically evaluated by applying repeated sensitivity analysis during the optimization process.

After verifying the accuracy of the reanalysis solution based on repeated sensitivity analysis, a three-bar and a ten-bar plane trusses are analyzed taking the cross-sectional areas as design variables. Comparison of the results shows that the proposed method gives very good results.

1. Introduction

The aim of optimization of a structural system is to find out an optimum combination of design parameters that minimizes an objective function (cost, weight, etc.) without sacrificing the functional and performance requirements. Extensive research has been carried out on structural optimization in the last few decades. A review of various mathematical programming methods for structural optimization was made by Belegundu and Arora.1,2) They discussed the applicability of various optimization methods to structural design and presented a unified viewpoint of mathematical programming methods.

Almost all of the practical structural optimization techniques are iterative and require a number of design cycles before an optimum structure is obtained. The structural response is to be recomputed at each design cycle after design changes have been incorporated. Direct structural analysis at each design cycle will result in very high computing cost, especially for large structures. Various rapid reanalysis techniques have been developed in the past years in an effort to reduce the computing time. A review of these techniques were given by Arora3) and Abu Kassim.4)

One of the efficient methods for reanalysis is to use the sensitivities of structural responses with respect to design variables, and in this sense, design sensitivity analysis is an integral part of structural optimization. Various methods of design sensitivity analysis in structural optimization were reviewed by Arora and Haug.5) However, the available literatures mainly cover only linear elastic sensitivity analysis. Research on nonlinear sensitivity analysis is in the initial stage.6,7) The authors have developed a method for the elastoplastic sensitivity analysis and reanalysis based on the incremental finite element method applying the Plastic Flow Theory.8,9)

Optimal design of nonlinear structural systems is also in the initial stage, and only a few papers have been presented on this subject.10-14) In the present paper, a new method for the optimal elastoplastic structural design is presented using the elastoplastic sensitivities formulated in Ref. 9). According to Ref. 9), the elastoplastic reanalysis solution is reasonable only when the yielding order of elements in the reanalysis solution is the same as that in the exact solution. Therefore, the magnitude of changes in design variables should be limited so that the yielding orders become the same. For further changes in design variables, the sensitivities newly evaluated at the limit point have to be used. This procedure of repeated sensitivity analysis is incorporated in the present optimization, and its usefulness is examined on test problems.

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Received 10th January 1992.
Read at the Spring Meeting, 12th and 13th May 1992.
A computer code ‘EPOPT’ has been developed for the elastoplastic optimal design of plane truss structures taking cross-sectional areas of the members as design variables. Total weight of the truss is taken as the objective function, and constraints are imposed on element stresses, nodal displacements and plastic collapse load. Gauge constraints are also introduced. A three-bar truss and a ten-bar truss have been optimized as test problems, and the results are compared with those in the available literatures.

2. Elastoplastic Structural Analysis and Sensitivity Analysis

2.1 Structural Analysis

Incremental elastoplastic analysis based on the finite element method is employed for structural analysis.

The equilibrium equation for the jth element can be expressed as

\[
\{\delta R\}_j = [K]_j \{\delta U\}_j,
\]

where \{\delta R\}_j and \{\delta U\}_j are the increments of nodal forces and displacements respectively, and \([K]\)_j is the tangential stiffness matrix. The stress increments of the jth element are related to the nodal displacement increments by the relation

\[
\{\delta \sigma\}_j = [D]_j [B]_j \{\delta U\}_j,
\]

where \([D]\)_j and \([B]\)_j represent the stress-strain and strain-displacement matrices, respectively.

At the Mth step, the nodal displacements and nodal forces can be expressed as follows.

\[
[U] = \sum_{k=1}^{N} (a \{\delta U\})_k,
\]

\[
[R] = \sum_{k=1}^{N} (a \{\delta R\})_k,
\]

where \(a\) is the load magnification factor which controls the magnitude of load increment at each incremental step so that the yielding of individual finite elements takes place exactly. The stress of the jth element at the Mth step is expressed as:

\[
\{\sigma\}_j = \sum_{k=1}^{N} (a \{\delta \sigma\})_j.
\]

2.2 Derivation of Elastoplastic Sensitivities

Partially differentiating Eqs. (3), (4) and (5) with respect to the ith design variable \(b_i\), the first-order sensitivities of nodal displacements, nodal forces and element stresses at the Mth step are derived as:

\[
\frac{\partial [U]}{\partial b_i} = \sum_{k=1}^{N} \left( a \frac{\partial \{\delta U\}}{\partial b_i} + \frac{\partial a}{\partial b_i} \{\delta U\}_k \right),
\]

\[
\frac{\partial [R]}{\partial b_i} = \sum_{k=1}^{N} \left( a \frac{\partial \{\delta R\}}{\partial b_i} + \frac{\partial a}{\partial b_i} \{\delta R\}_k \right),
\]

\[
\frac{\partial \{\sigma\}_j}{\partial b_i} = \sum_{k=1}^{N} \left( a \frac{\partial \{\delta \sigma\}_j}{\partial b_i} + \frac{\partial a}{\partial b_i} \{\delta \sigma\}_j \right).
\]

General forms of \(\partial \{\delta U\}/\partial b_i\) and \(\partial \{\delta \sigma\}_j/\partial b_i\), in these expressions are given in Refs. 8 and 9. \(\partial \{\delta \sigma\}_j/\partial b_i\) becomes zero when the loads are independent of the ith design variable.

For the truss elements used in this paper, the load magnification factor \(a\) is expressed as:

\[
a = \text{sgn}(\delta \sigma) \sigma_R - \sigma_i/\delta \sigma_i,
\]

where

\[
\text{sgn}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}
\]

and the subscript \(f\) represents the \(f\)th yielded element.

The partial derivative of \(a\) with respect to \(b_i\) is given as:

\[
\frac{\partial a}{\partial b_i} = \frac{1}{\delta \sigma_i} \left( \frac{\partial \delta \sigma}{\partial b_i} + a \frac{\partial \sigma}{\partial b_i} \right)
\]

3. Optimization

3.1 Sequential Unconstrained Minimization Technique

A structural optimization problem can be expressed as:

\[
\text{minimize } f((B))
\]

subject to \(g_i((B)) \geq 0, \quad i = 1, \ldots, m\)

where \(f((B))\) is the objective function, \(g_i((B))\) are the constraints and \((B) = \begin{bmatrix} b_1, & \ldots, & b_n \end{bmatrix}^T\) is the design variable vector. \(m\) and \(n\) are the numbers of constraints and design variables, respectively.

Numerous numerical techniques are available to obtain a solution to Eq.(11). One popular technique among them is the Sequential Unconstrained Minimization Technique (SUMT).\(^{10}\) In this technique, a constrained optimization problem is transformed into a series of unconstrained minimization problems by replacing the constraints with penalty terms. Here, the inverse barrier function method is used, and based on it, an unconstrained transformation function \(\Phi\) is defined as

\[
\Phi((B), r) = f((B)) + r \sum_{i=1}^{m} \left( \frac{1}{g_i((B))} \right)
\]

where \(r\) is a penalty multiplier.

As a starting point of optimization, the initial value of \((B)\) should be such that \(g_i((B)) \geq 0\) for all \(i\). With an initial value of \(r = r_0\), the transformation function \(\Phi\) given by Eq.(12) is minimized. Then, \(r\) is reduced by dividing it by a factor \(r_c\), and the next response surface is optimized. This process is continued, and when \(r\) is fully reduced to zero, or convergence to a desired accuracy is achieved, the optimal value of \((B)\) is obtained.

3.2 Objective Function

In the structural optimization, the objective function which is minimized is often taken to be the weight of the structure. For the trusses in the test problems in Chap. 4, it is given by

\[
f((B)) = \sum_{i=1}^{N} \rho_i A_i L_i,
\]

where the subscript \(i\) denotes the group number (a group is a collection of elements having the same cross section and material properties), \(\rho_i\) is the weight density of the material, \(A_i\) is the cross-sectional area, \(L_i\) is the sum of the lengths of all elements belonging to group \(i\), and \(NG\) is the total number of groups in the structure.

3.3 Constraint Functions

The constraints that are imposed on the structure are
as follows:

**Displacement Constraints**: Allowable limits are imposed on the displacements of the nodes:

\[ |u_i| \leq u_i^* \]  

where \( u_i \) is the displacement of the \( i \)th degree-of-freedom and \( u_i^* > 0 \) is its upper bound.

**Stress Constraints**: Stress constraints are of the form:

\[ |\sigma_i| \leq \sigma_i^* \]  

where \( \sigma_i \) is the stress of the \( i \)th structural element and \( \sigma_i^* > 0 \) is its allowable stress.

**Gauge Constraints**: Lower bound constraints are imposed on all design variables:

\[ b_i \geq b_i^l \]  

where \( b_i \) is the value of the \( i \)th design variable and \( b_i^l \) is its lower bound.

**Load Constraint**: The plastic collapse load should be greater than the design load. Denoting the plastic collapse load and design load as \( P_k \) and \( P_d \), respectively, this constraint is expressed as:

\[ P_k > P_d \]  

The plastic collapse load depends on material properties and structural geometry. When strain hardening of the material is low, the load at which the plastic flow of the whole structure begins is regarded as the plastic collapse load.

### 3.4 Elasticplastic Reanalysis

Structural reanalysis is performed based on the Taylor Series Expansion Method. Only the first-order sensitivities are evaluated in this paper and based on it, the elasticplastic reanalysis solution can be expressed as follows:

\[ \{U\} = \{U^0\} + \sum \frac{\partial \{U\}}{\partial b_i} \cdot \Delta b_i \]  

\[ \{R\} = \{R^0\} + \sum \frac{\partial \{R\}}{\partial b_i} \cdot \Delta b_i \]  

\[ \{\sigma\} = \{\sigma^0\} + \sum \frac{\partial \{|\sigma|\}}{\partial u_i} \cdot \Delta u_i \]  

In the above expressions, \( \{U\} \), \( \{R\} \) and \( \{|\sigma|\} \) are the nodal displacements, nodal forces and element stresses obtained for the initial design variables at each design cycle.

In the optimization, the design load is specified in general. Reanalysis is performed to derive the change in displacements or stresses at the specified load. Here, the displacement is considered. If the design load \( P_k \) is in between the loads at the \( k \)th and \((k+1)\)th steps, that is, \( P_k \) and \( P_{k+1} \) of the reanalysis solution, the corresponding reanalysis displacement \( u \) is expressed as:

\[ u = u_k + (u_{k+1} - u_k) \frac{P_k - P_k}{P_{k+1} - P_k} \]  

where \( u_k \) and \( u_{k+1} \) are the reanalysis results at the \( k \)th and \((k+1)\)th steps, respectively. The reanalysis stress is also similarly calculated.

### 3.5 Repeated Sensitivity Analysis

It has been shown in Ref. 9) that in elastoplastic reanalysis using sensitivities, the predicted yielding order of elements sometimes differs from that of actual behaviour when the change in design variable is large.

Is such a case, a rational reanalysis solution cannot be obtained. Therefore, the change in design variable must be within an allowable limit such that the yielding order of elements for predicted and actual behaviours are the same. When the design change is beyond this allowable limit, sensitivity analysis is to be done again at this allowable limit, and based on it further reanalysis is to be done. This procedure of repeating sensitivity analysis at the allowable limit is hereafter termed as 'repeated sensitivity analysis'.

At a repeated sensitivity analysis step \( j \), the original design variables are updated to

\[ b_i = b_i^{-1} + \Delta b_i^{j-1} \]  

where \( b_i \) and \( b_i^{-1} \) are the original design variables at steps \( j \) and \( j-1 \), respectively, and \( \Delta b_i^{j-1} \) is the allowable change in design variables at step \( j-1 \).

In the optimization process, initially, design sensitivity analysis is performed and the sensitivities are obtained. The allowable design changes are also estimated based on the procedure given in Sec. 3.6. However, these sensitivities can be used for reanalysis at a design cycle only if the design changes at the cycle are within the respective allowable limits. If this condition is violated, repeated sensitivity analysis is performed and further reanalysis is done using the sensitivities obtained by repeated sensitivity analysis. A flow chart of this procedure is given in Appendix I.

### 3.6 Maximum Allowable Changes in Design Variable

As described in Sec. 3.5, the magnitude of change in design variables has to be constrained to get a rational reanalysis solution. The procedure to find out such a constraint - the allowable change in design variable - is detailed here.

When only the first-order sensitivity is considered, the locus of reanalysis points representing yielding forms a straight line for each structural element. For each adjacent pair of loci, there may exist an intersecting point beyond which the yielding order of the two related elements reverses compared to the original yielding order. So, the change in design variable should be such that the reanalysis solution falls within the intersecting point.

For each adjacent pair of loci of yielding points, the intersecting point is found out which represents either the upper or lower limit point depending upon the orientation of the loci. For a structure of \( N \) elements, there will be a maximum of \( N-1 \) intersecting points, whether upper or lower limit points.

In actual calculation, for a design variable \( b_i \) and for a virtual change in design variable of \( \Delta b_i \), the original and reanalysis solutions are obtained as shown in Fig. 1. The points \( P(x_i, y_i) \) and \( Q(x_{i+1}, y_{i+1}) \) represent the yielding points obtained from reanalysis for two succes-
sively yielded elements \( j \) and \( j + 1 \). Similarly, \( R(x^j, y^j) \) and \( S(x^{j+1}, y^{j+1}) \) represent the respective yielding points of original analysis. The loci of yielding points for the two successively yielded elements \( j \) and \( j + 1 \) can be represented as

\[
y_j = a_jx_j + b_j
\]

\[
y_{j+1} = a_{j+1}x_{j+1} + b_{j+1}
\]  
(23)

Their intersecting point \( O(x, y) \) is obtained as

\[
x = \frac{b_j - b_{j+1}}{a_j - a_{j+1}}, \quad y = \frac{a_jy_j - b_j}{a_j - a_{j+1}}
\]  
(24)

where

\[
a_j = \frac{y_j^0 - y_j^0}{x_j^0 - x_j^0}, \quad b_j = \frac{x_j^0y_j^0 - x_j^0}{x_j^0 - x_j^0}
\]  
(25)

\( a_{j+1} \) and \( b_{j+1} \) are obtained by changing \( j \) to \( j + 1 \) in Eq. (25).

Using these, the distances \( d_1 \) and \( d_2 \) are evaluated. Then the allowable change for the design variable \( b_i \) in relation to the yielding order of elements \( j \) and \( j + 1 \) is given as

\[
\Delta b_i = \frac{d_i \Delta b_i}{d_a}
\]  
(26)

Whether the point \( O(x, y) \) represents an upper or a lower limit point for a change of \( +\Delta b_i \) in design variable is determined from the relative positions of \( O(x, y) \), the original solution and the reanalysis solution. The procedure for determining whether \( O(x, y) \) represents an upper or a lower limit point is given as follows.

Let \( V_a \) and \( V_c \) represent the vectors \( RS \) and \( RP \) in Fig. 1, respectively. The value of the vector product \( V = V_a \times V_c \) gives the relative position of point \( P \) with respect to line \( RS \). \( P \) comes below and above \( RS \) for positive and negative values of \( V \), respectively. From this and the relative values of \( d_1 \), \( d_2 \) and \( d_a \), the status of the point \( O(x, y) \) is determined as to whether it represents an upper or a lower limit point.

If the allowable limit described above is considered, the reanalysis solution may be rational since the yielding order is the same between the original and modified structures. However, when higher order sensitivities cannot be ignored, the reanalysis solution based on Eqs. (18), (19) and (20) are not accurate, especially for a large change in design variable. Therefore, some limit should be imposed on the change in design variable from this aspect also.

### 3.7 Grouping of Structural Elements

In structural optimization, in order to reduce the number of design variables and in turn the computing time, it is a general practice to group the structural elements based on basic parameters such as thickness, cross-sectional area etc. Such a grouping is considered in the present method.

When the elements are grouped, in a group, there will be \( n \) allowable changes corresponding to the \( n \) elements of the group. However, the allowable change for this group must be the minimum of these \( n \) elemental allowable changes. This underestimates the allowable changes of the other elements in the group during optimization, and results in additional repeated sensitivity analyses and in turn increases the computing time. However, efficient structural grouping can reduce the need for additional repeated sensitivity analyses.

### 4. Example Problems and Discussion

#### 4.1 Accuracy of Repeated Sensitivity Analysis

Before performing optimal elastoplastic design, the accuracy of repeated sensitivity analysis is examined on a five-bar truss shown in Fig. 2(a) analyzed in Ref. 9. The load-load point displacement relationship for this truss is indicated by a broken line in Figs. 2(b) and (c). The yielding starts initially in member 1, and follows in the order of members 5, 4, 2 and 3.

In Fig. 2(b), the cross-sectional area of member 1 is chosen as the design variable, and it is reduced to 0.05 % of its initial value. The first limit point is obtained from the condition that the third yielding takes place in elements 2 and 4 simultaneously for a reduction of 61.9 % in sectional area. This point is indicated in Fig. 2(b) as the intersecting point of the loci of yielding points of members 2 and 4 obtained by reanalysis shown by fine dashed lines. At the first limit point, the first repeated sensitivity analysis is performed with the sectional area of member 1 reduced by 61.9 %. The newly derived load-load point displacement relationship is shown by a double-dotted chain line. Considering this relationship as a new initial state, the second limit point is obtained with the newly evaluated sensitivities. At this point, the second yielding occurs in members 2 and 3 simultaneously for a reduction of 94.9 % in sectional area. Exact solution after the sectional area has been reduced by 94.9 \% is indicated by a single-dotted chain line. No more limit point exists during a further reduction of sectional area to 0.05 % of its initial value, and the corresponding exact solution is shown by a solid line. It should be noted that the slope of the dashed line representing the locus of yielding points changes much when the yielding order of the corresponding member changes; that is, when the...
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(a) Dimensions of original truss

(b) Relationship between load and displacement for decrease of sectional area

(c) Relationship between load and displacement for increase of sectional area

(d) Comparison of reanalysis solutions with and without repeated sensitivity analysis

Fig. 2 Repeated sensitivity analysis and reanalysis of five-bar truss

locus intersects with another locus.

Similar analysis is performed taking the cross-sectional area of member 5 as a design variable. The sectional area is increased up to 359.0% of its initial value. The first and second limit points are obtained for increases of 124.3% and 191.7% of the original sectional area, respectively. At the limit points, the corresponding elements which yielded simultaneously are (2, 4) and (2, 5). Similar features are observed as in the case shown in Fig. 2(b).

The reanalysis solutions using the results of repeated sensitivity analysis are plotted by ○ in Figs. 2(b) and (c). They are in very good agreement with the exact solutions when the sectional area is reduced. When it is increased, some differences are observed in some cases. This may be because only the first-order sensitivity is used in reanalysis. However, when cross-sectional area is chosen as the design variable, it is usually reduced
from a large initial value in the optimization process. Therefore, the results of optimization may not be affected much even if the higher order sensitivities are neglected. Nevertheless, it will be better to check if the higher order sensitivities affect the reanalysis solution for individual structural geometry and types of loads.

Figure 2(d) shows the effectiveness of the repeated sensitivity analysis. The broken line represents the load-load point displacement relationship for the original truss. The upper solid line is the exact solution after the cross-sectional area of member 5 is increased up to 359.0% of its original value. The lower solid line corresponds to the case when the sectional area of member 1 is reduced to 0.05% of its original value. The reanalysis solutions with and without repeated sensitivity analysis are indicated by ○ and △, respectively. Solutions with repeated sensitivity analysis coincide with the exact solutions quite well, whereas those without repeated sensitivity analysis differ much from the exact solutions. It can therefore be concluded that the repeated sensitivity analysis proposed in this paper gives accurate reanalysis solution.

4.2 Optimal Elastoplastic Design of Plane Truss

Two test problems, a three-bar truss and a ten-bar truss, are considered for optimization. In both the problems, the objective function is taken to be the total structural weight, and the design variables are the cross-sectional areas. In the analysis, the strain hardening of the material is considered. Therefore, the plastic collapse load does not necessarily coincide with the collapse load obtained by plastic mechanism analysis assuming perfectly rigid plastic material.

4.2.1 Three-bar truss

The geometry of the 3-bar truss which is widely used for benchmark test of optimization is shown in Fig. 3(a). The cross-sectional areas \( b_1, b_2, \) and \( b_3 \) are the design variables. The other parameters are

\[
E = 21000 \text{ kgf/mm}^2, \quad H = E/50, \\
\rho = 0.78 \times 10^{-6} \text{ kgf/mm}^3, \\
\sigma_0 = 30 \text{ kgf/mm}^2, \quad P_a = 100 \text{ tonf}, \\
(\sigma(B)) = (6450.0, 3220.0, 3220.0) \text{ mm}^2
\]

and the initial weight \( f(o^B) \) is calculated as 32.97 kgf. The value of penalty multiplier \( \rho \) in Eq. (12) has been changed in a wide range to obtain the local minima. The reduction factor \( r \) of penalty multiplier is taken as 10. The minimum among the local minima is taken as the global minimum. For this case, the load at the instant when all the members have been yielded is regarded as the plastic collapse load.

Two series of optimizations are carried out. Firstly, the displacement constraint is imposed changing the allowable horizontal displacement \( u_1^A \) at the loading point from 0.32 mm to 10.0 mm. The gauge constraint on sectional area is 10% of its initial value. No constraint is imposed on member stresses.

The optimized weights of the whole system and member 1 are plotted in Fig. 3(b) against allowable displacement. The stress in member 1 at the optimum is also plotted in the same figure. Table 1 shows how the various constraint conditions are satisfied in the optimized truss.
To examine whether the optimized truss obtained by the present method is really optimum, elastoplastic stress analyses are performed on the optimized truss. Figure 3(c) shows the relationships between the load and load-point displacement in the horizontal direction. When the allowable displacement is 0.32 mm, no member has yielded at the design load, whereas member 1 is yielded when it is 0.45 mm and 0.6 mm. In other cases, members 1 and 2 are yielded at the design load. It may be said that elastic optimization is performed when allowable displacement is 0.32 mm, and elastoplastic optimization has been performed in other cases of larger allowable displacement.

As indicated in Fig. 3(c) and Table 1, the displacement of each optimized truss at the design load is very close to the respective allowable displacement. This indicates that constraint on displacement is active in these cases. Table 1 also indicates that the displacement constraint is not active when the allowable displacement is 10.0 mm. On the other hand, the constraint on plastic collapse load is not active when the displacement constraint is below 8.0 mm. With an increase in allowable displacement, the total weight of the optimized truss decreases, and hence its rigidity. This causes a reduction in plastic collapse load, and thus the load constraint becomes active. Gauge constraints on cross-sectional areas of members 2 and 3 are very active in the elastoplastic optimization.

Figure 3(b) indicates that the rates of change of stress and weights become lower with increase in allowable displacement. This is because an elastic element has much higher rigidity than a plastic element, and the change in load carrying capacity is more sensitive to the change in sectional area, in other words the weight.

Similar optimization is performed considering the stress constraints. The allowable stress is changed between 0.8 and 2.0 times the yield stress of the material. No constraint is imposed on displacement in this case. Other constraints are the same as those in the former series. The obtained optimal values are summarized in Fig. 4 and Table 2. $u_1$ in Fig. 4 indicates the horizontal displacement of node 1 at the optimum.

In this case, the stress constraints of members 1 and 3...
are active as well as the gauge constraint of member 3. The constraint on plastic collapse load tends to be active as the allowable stress is increased.

4.2.2 Ten-bar truss

The ten-bar truss used for optimization is similar to that in Refs. 10) and 11), and is shown in Fig. 5. The structural elements are grouped into 6 groups based on cross-sectional area as shown in the figure, and thus there are 6 design variables $b_1, \ldots, b_6$. The other parameters are:

$$E = 21000 \text{ kgf/mm}^2, \quad H = E/1000$$
$$\rho = 0.78 \times 10^{-5} \text{ kgf/mm}^3, \quad \sigma_0 = 24 \text{ kgf/mm}^2$$
$$\sigma_1 = 24.24 \text{ kgf/mm}^2, \quad \rho = \frac{\sigma_0}{1000}$$
$$b(i = 1, 2) = 2000 \text{ mm}^2$$
$$b(i = 4, 5, 6) = 3000 \text{ mm}^2, \quad P = 34.0 \text{ tonf}$$
$$b(i = 1, \ldots, 6) = 4500 \text{ mm}^2$$
$$f(B) = 722.86 \text{ kgf}.$$

Except the strain hardening rate, all other parameters are kept the same as those for the truss in Ref. 11) for comparison purposes.

The initial value of the penalty multiplier $r$ in Eq. (12) has been varied, and the minimum from among the evaluated local minima is taken as the global minimum. The reduction factor $r_p$ of the penalty multiplier is chosen as 10. The same definition as that in the case of three-bar truss is used for the plastic collapse load.

The optimization result is shown in Fig. 6 (a) together with the result from Ref. 11) for comparison. In Table 3, the optimized cross-sectional areas and structural weight are compared with those in Refs. 10) and 11). In these references, an elastic perfectly-plastic material is assumed and the deformation theory of plasticity is applied for structural analysis. The optimization technique is based on the dual approach.

As indicated in Fig. 6 (a), the process of convergence is different at the earlier stages of design cycle. This may be attributed to the difference in optimization techniques and the initial value of the penalty multiplier. The converged optimal weight of 502.05 kgf is a
little smaller than the weights of 502.32 kgf in Ref. 10) and 503.88 kgf in Ref. 11). This may be because the strain hardening is considered in the present analysis, and nearly the same load carrying capacity may be attained with less sectional area.

The active constraints at the optimum are the displacement at node 1 in x-direction, the stress constraint in member 2, and the gauge constraints corresponding to design variables \( b_1, b_4, \) and \( b_6 \). To reduce the error due to the absence of higher order sensitivities, the maximum allowable change in design variable is set as 10% of its initial value, if the evaluated allowable change is greater than 10%, at every design cycle.

For the optimized structure, the relationship between the load and the horizontal displacement at node 1 is plotted in Fig. 6(b). At the design load, the displacement is nearly the same as its allowable value. Judging from Fig. 6 and Table 3, it may be concluded that the present method gives very good result and that it is very useful in the optimal elastoplastic structural design.

5. Conclusions

A method of optimal elastoplastic design of structural systems is presented. The proposed method has the following characteristics.

1. It consists of three parts: elastoplastic structural analysis, elastoplastic sensitivity analysis and optimization.
2. The structural analysis and sensitivity analysis are performed in an incremental manner within the framework of the finite element method.
3. The Sequential Unconstrained Minimization Technique (SUMT) is used for the structural optimization.

In elastoplastic reanalysis, when the design changes are large, the yielding order of elements of reanalysis solution may differ from that of exact solution, and if it differs, the reanalysis solution is not reasonable. To avoid this, maximum allowable changes are to be imposed on the design variables, and when the design change is beyond this imposed limit, repeated sensitivity analysis is to be performed at the limit point and the new sensitivities are to be used for further reanalysis. The accuracy of reanalysis solution based on the results of repeated sensitivity analysis has been verified on a five-bar truss.

In the proposed method, the allowable changes are automatically evaluated and incorporated into the optimization process. A computer code 'EPOPT' has been developed for the elastoplastic optimal design of plane truss structures. Total weight of truss and member cross-sectional areas are taken as the objective function and design variables, respectively. Constraints are imposed on member stresses, nodal displacements, plastic collapse load and gauge limits. A three-bar truss and a five-bar truss have been optimized as test problems. It has been found that:

1. Optimal elastoplastic design results in less weight compared to optimal elastic design.
2. For larger allowable values of displacements or stresses, constraint on plastic collapse load becomes active.
3. The results of optimal elastoplastic design are less sensitive to changes in allowable values of constraints.
4. The results of optimal elastoplastic design by the present method show good agreement with those in the available literature.

References

Appendix 1: Optimization Flow Chart

START

INITIAL DESIGN $b_0^k$
set $k=1$, $j=1$

STRUCTURAL ANALYSIS
SENSITIVITY ANALYSIS
ALLOWABLE DESIGN CHANGE $\Delta h_j^{k,j}$

Yes

$k = k + 1$

No

$|\Delta h_j^{k,j} - \Delta h_j^{k-1,j}| \geq \Delta h_j^{k-1,j}$

Yes

update $\Delta h_j^{k,j} = \Delta h_j^{k-1,j} + \Delta h_j^{k-1,j}$

STOP

OPTIMUM

Yes

REANALYSIS

OBJECTIVE FUNCTION

No

NEXT DESIGN CYCLE
set $k=k+1$
NEW DESIGN $b_0^{k+1}$