Generation of the Irregular Wave Specified by the Spectrum of the Wave Envelope and Some Results of a Model Test

by Ben T. Nohara, Member and Masami Matsuura, Member

Summary

This paper describes the generation of the irregular wave specified by the spectrum of the wave envelope using a wave maker in a test basin. An effective generation algorithm for the irregular wave specified by the wave envelope spectrum is proposed.

The design for mooring systems of a vessel in a port as well as a floating offshore structure is important because the long period wave happens to cause the break of mooring lines in spite of relatively calm ocean conditions. In very large floating structures, such as the mega-float case, especially, this problem will become a severe one due to no past experience in real design. The long period wave can be induced by the irregular wave specified by the spectrum of the wave envelope.

The proposed algorithm is verified by the computer simulation. The experiment results using a pontoon model are also presented.

1. Introduction

The design for mooring systems of a vessel in a port is important because long period waves happen to cause the break of mooring lines in spite of relatively calm ocean conditions. In the mega-float case, especially, this problem will become a severe one due to no past experience in real design.

We can consider some types of long period waves and some observations in ports are reported. However, we can find few studies for generation of long period waves in a test basin in the literature. It need scarcely be said that model tests using a wave maker in a test basin is important to obtain the design data.

In this paper, the authors deal with the irregular wave specified by the spectrum of the wave envelope because the irregular wave with short period can form the long period wave according to its envelope spectrum. An effective generation algorithm for the irregular wave specified by the wave envelope spectrum is proposed.

The following section presents the theoretical background of the irregular wave specified by the spectrum of the wave envelope and also in this section, the authors propose an efficient algorithm in real time for generation of the defined envelope wave. Section 3 deals with some simulation results of generation of the discussed wave. Section 4 presents the results of some experiments in a test basin. Using a pontoon model, the experiments has been performed by the traditionally irregular wave and the grouping wave. The final section describes the concluding remarks.

2. Generation of the Irregular Wave Specified by the Spectrum of the Wave Envelope

2.1 Theoretical background

Using the concept of modulation in the signal processing field, the irregular ocean wave \( \eta(t) \) can be generally written as follows:

\[
\eta(t) = r(t) \cos(\phi(t))
\]

Here, \( r(t) \) and \( \phi(t) \) mean amplitude modulation and phase modulation as the function of time \( t \), respectively.

Conversely, the conventional method for generation of the irregular wave is based on the summation of the frequency component waves as follows:

\[
\eta(t) = \sum_{n=1}^{N} a_n \cos(\sigma_n t + \varepsilon_n)
\]

Here, \( a, \sigma, \varepsilon \) and \( N \) denote amplitude, angular frequency, random phase lag defined from 0 to \( 2\pi \) and the number of the frequency component wave, respectively. Let the spectrum of this wave be \( S(\sigma) \), then \( a_n \) is written by the following equation.

\[
a_n = \sqrt{2S(\sigma_n) \Delta \sigma_n}
\]

Here, \( \Delta \sigma \) represents the defined frequency bandwidth in the effectively spectral domain. That is, the given wave spectrum \( S(\sigma) \) can be realized by Equation(3).

However, this method can not define the spectrum of the wave envelope \( r(t) \) explicitly. The wave envelope
is determined by the course of events as the results of the mutual effects of the characteristics of random value $\varepsilon$ and amplitude $a$. In order to specify the spectrum of the wave envelope explicitly, the following method has been proposed, which is based on Equation (1). \(^{29}\)

1. Creation of phase modulation $\phi(t)$

Firstly, the following irregular wave $\eta(t)$ of which spectrum is given by $S_{\omega}(\omega)$ is obtained from Equation (2).

$$
\eta(t) = \sum_{n=1}^{\infty} a_n \cos(\omega_n t + \varepsilon_n)
$$

(4)

Here,

$$
a_n = \sqrt{2 S_{\omega}(\omega_n)} \Delta \omega_n
$$

(5)

The given spectrum $S_{\omega}(\omega)$ characterizes the carrier of the wave.

Secondly, the wave $\xi(t)$ which is rotated by $\pi/2$ in terms of phase in Equation (4) is earned.

$$
\xi(t) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \eta(t) dt
$$

(6)

The above integral is known as Hilbert transform. \(^{40}\)

Finally, phase modulation $\phi(t)$ is obtained by

$$
\phi(t) = \tan^{-1} \frac{\xi(t)}{\eta(t)}
$$

(7)

2. Creation of amplitude modulation $r(t)$

Let $S_{\sigma}(\sigma)$ be the spectrum of the wave envelope, then amplitude modulation $r(t)$ is performed as follows in the same manner of Equations (4) and (5).

$$
r(t) = \sum_{n=1}^{\infty} a_n^2 \cos(\omega_n t + \varepsilon_n)
$$

$$
a_n^2 = \sqrt{2 S_{\sigma}(\sigma_n) \Delta \sigma_n}
$$

(8)

3. Modification of amplitude modulation $r(t)$

To obtain the real envelope $\tilde{r}(t)$, the mean value of $r(t)$: $E(r(t))$ must be added to $r(t)$ as follows:

$$
\tilde{r}(t) = r(t) + E(r(t))
$$

Here, $E(r(t))$ is obtained as the following manner based on the assumption of Gaussian distribution of $\eta(t)$ and $\xi(t)$. Let the mean value and the standard deviation of the Gaussian distribution of $\eta(t)$ and $\xi(t)$ be $0$ and $m_{\sigma}$, respectively, then

$$
E(r(t)) = \int_{-\infty}^{\infty} r \cdot \frac{1}{\sqrt{2\pi m_{\sigma}}} \exp\left(-\frac{r^2}{2m_{\sigma}^2}\right) dr = -\sqrt{\frac{\pi}{2}} m_{\sigma}
$$

(9)

Here, $m_{\sigma}$ represents the 0th moment of the spectrum $S_{\sigma}(\sigma)$, that is,

$$
m_{\sigma} = \int_{-\infty}^{\infty} S_{\sigma}(\sigma) d\sigma.
$$

(10)

Moreover, the generation of $\eta(t)$ and $\xi(t)$ by Equations (4) and (6) can not create the Gaussian distribution completely. So, the error based on this assumption must be modified as follows:

The value $\Delta m_{\sigma}$ which means the difference between the 0th moment of the given spectrum $S_{\sigma}(\sigma)$ and the generated spectrum $S_{\omega}(\sigma)$ is

$$
\Delta m_{\sigma} = m_{\sigma} - m_{\sigma_0}
$$

(11)

The mean value of the envelope is modified as

$$
E'(r(t)) = \sqrt{\frac{\pi}{2}} (m_{\sigma} + \Delta m_{\sigma})
$$

(12)

Then the formulation of the final envelope $\tilde{r}(t)$ is

$$
\tilde{r}(t) = r(t) + E'(r(t))
$$

(13)

(4) Building up the irregular wave $\tilde{h}(t)$ specified by the spectrum of the wave envelope

The irregular wave $\tilde{h}(t)$ is written by the following equation using amplitude modulation $\tilde{r}(t)$ and phase modulation $\phi(t)$.

$$
\tilde{h}(t) = \tilde{r}(t) \cos \phi(t)
$$

(14)

2.2 An effective algorithm

However, this method is not suitable to the algorithm for a wave maker in a test basin because this procedure doesn't allow the algorithm to execute in real time. The desired wave can be generated after the complete calculation of $\phi(t)$ and $\tilde{r}(t)$. So, the authors propose the real time algorithm for the generation of the irregular wave specified by the envelope spectrum as follows.

First of all, the numerical calculation of phase modulation in Equation (7) is unstable because the time series $\eta(t)$ necessarily has zero-cross points. So, we need the stable procedure for calculation of phase modulation in stead of Equation (7). The following two equations are obtained by modifying Equation (4).

$$
\tilde{h}(t) = \sum_{n=1}^{\infty} a_n^2 \cos(\omega_n t - \sigma_n^2 t + \varepsilon_n)
$$

(15)

$$
\tilde{h}(t) = \sum_{n=1}^{\infty} a_n^2 \sin(\omega_n t - \sigma_n^2 t + \varepsilon_n)
$$

(16)

where $\sigma_n^0$ is the center angular frequency defined by

$$
\sigma_n^0 = \frac{m_1}{m_0}
$$

(17)

$$
m_1 = \int_{-\infty}^{\infty} \sigma S_{\sigma}(\sigma) d\sigma.
$$

(18)

(Note 1: As far as the formulation of Equations (15) and (16) in the numerical calculation is concerned, it doesn't matter whatever the value of $\sigma_n^0$ is. However, $\sigma_n^0$ of Equation (17) matches the physical meaning in the frequency domain of the spectrum.)

(Note 2: Each of two equations: Equation (15)and Equation (16)substantially has the relationship of Hilbert transform, however, these two equations must be calculated independently in exchange for using Hilbert transform in order to execute in real time.)

Using Equations (15) and (16), $\tilde{h}(t)$ is written by

$$
\tilde{h}(t) = \tilde{h}(t) \cos(\sigma \tilde{t}) - \tilde{h}(t) \sin(\sigma \tilde{t})
$$

$$
= R(t) \cos(\sigma \tilde{t} + \phi(t))
$$

(19)

where

$$
R(t) = \sqrt{(\tilde{h}(t))^2 + (\tilde{r}(t))^2}
$$

$$
\phi(t) = \tan^{-1} \frac{\tilde{r}(t)}{\tilde{h}(t)}
$$

(20)

(21)

In Equation (19) $R(t)$ represents the wave envelope which is implicitly defined by Equations (15) and (16) and $\cos(\sigma \tilde{t} + \phi(t))$ denotes the carrier of the wave $\tilde{h}(t)$. We need phase modulation only in this point. However $\tilde{h}(t)$ also has zero-cross points, so Equation (21) is unstable. Therefore the carrier of the wave can be obtained directly in stead of the calculation of phase modulation as follows:

$$
\cos(\sigma \tilde{t} + \phi(t)) = \cos(\sigma \tilde{t}) \frac{\tilde{h}(t)}{R(t)} - \sin(\sigma \tilde{t}) \frac{\tilde{r}(t)}{R(t)}
$$

(22)
Fig. 1 Flowchart of the Proposed Algorithm for Generation of the Grouping Wave.
(Note 3: Using the algorithm developed by Nohara\(^7\), the calculation of Equation (22) can be executed in real time because both \(a_0(t)\) and \(s_0(t)\) are written in an iterative formulation in stead of the summation of frequency. See Appendix.) Then the wave envelope \(R(t)\) is calculated in real time by the same manner. Here, in reality, \(\Delta m_0\) can be neglected if the stationary characteristics of random value is preserved and the energy equivalence divided frequency method for \(\sigma\) is used.\(^8\)

\[
\ddot{R}(t) + \omega^2 R(t) = \ddot{r}(t) + E(\ddot{r}(t)) = \dot{\ddot{r}}(t)
\]

Finally, the desired wave \(\ddot{r}(t)\) is obtained as the following formulation.

\[
\ddot{r}(t) = \dot{\ddot{r}}(t) \cos(\sigma t + \phi(t))
\]

= \(\{r(t) + E(r(t))\} \cdot \cos(\sigma t) \frac{s(t)}{R(t)} - \sin(\sigma t) \frac{a_0(t)}{R(t)}\)

(24)

Figure 1 summarizes the flowchart of this algorithm.

3. Simulation

This chapter shows some simulation results of generation of the irregular wave specified by the spectrum of the wave envelope. In the coastal engineering field, this wave is called the grouping wave\(^2\) of which envelope is a relatively simple form just like a single frequency. The swell which consists of two close frequencies is in the special case of the grouping wave. The grouping wave has been known as the wave which sometimes causes the break of mooring ropes of the vessels and/or terminates the cargo handling at ports.

The solid line of Figure 2 presents the time histories of the irregular wave with the parameters: \(N_0 = 200\), \(\Delta t = 4[\text{ms}]\), \(H_{1/3} = 0.3[\text{m}]\), \(T_{1/3} = 1.0[\text{s}]\) and the generated spectrum: Bretschneider - Mitsuyasu calculated by Equation (4). The dotted line of Figure 2 shows the envelope of this wave, calculated by Equation (20). This wave's carrier is shown in Figure 3 by Equation (22).

Figure 4 gives the spectra of this wave and its envelope. Conversely, the dotted line of Figure 5 indicates the re-calculated time histories which is reconstructed by the carrier: Equation (22) and the envelope: Equation (20). The solid line is the original one. To clarify the difference between the re-calculated and original time histories, the former is transferred to an upward movement of 0.1[\text{m}]. Figure 5 evidences the precise reorgani-
zation for the original by the proposed algorithm. Using Equation (24), the time histories of the grouping wave of which envelope spectrum has a single frequency: 0.1 [Hz] is demonstrated in Figure 6 (the dotted line). In order to compare with the original irregular wave, the original is shown by the solid line. However the wave spectrum of the grouping wave gives good agreement with the original one. This situation is indicated by Figure 7. Especially, the split frequencies due to the energy equal frequency division method in Equation (3) or Equation (5) are in beautiful agreement with the values which are obtained by the original irregular wave. The spectrum shape (the dotted line) induced from the grouping wave is shown by transferring to an upward movement of 10 [dB] in order to understand clearly compared to the original (the solid line). Figure 8 shows the envelope spectra of the grouping wave (the dotted line) and the original (the solid line).

4. Results of Some Experiments in a Test Basin

In this chapter, the results of some experiments in a test basin are presented. The experiments have been performed by both the traditionally irregular wave and the grouping wave described above. Using a scale model of a pontoon type platform, the authors have obtained the platform motions in the generated waves. Table 1 indicates major dimensions of a pontoon. The model was moored softly and maintained in the beam sea condition by linear coil springs. The natural periods

---

**Fig. 5** Re-constructed Time Histories from the Carrier and the Envelope. 
Solid Line: Original Time Histories 
Dotted Line: Re-constructed One

**Fig. 6** Time Histories of the Grouping Wave of which Envelope Spectrum Has a Single Frequency: 0.1 [Hz]. 
Solid Line: Original Time Histories 
Dotted Line: The Grouping Wave

**Fig. 7** Spectra of the Grouping Wave and the Original Irregular Wave. 
Solid Line: The Original Wave 
Dotted Line: The Grouping Wave

**Fig. 8** Spectra of the Envelope of the Grouping Wave and the Original Irregular Wave. 
Solid Line: The Original Wave 
Dotted Line: The Grouping Wave
Table 1 Major Dimensions of a Pontoon Model.

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>2.34 m</td>
</tr>
<tr>
<td>Breadth</td>
<td>1.0 m</td>
</tr>
<tr>
<td>Draft</td>
<td>0.2 m</td>
</tr>
<tr>
<td>Displacement</td>
<td>468 kg</td>
</tr>
<tr>
<td>Metacentic Height, GM</td>
<td>0.48 m</td>
</tr>
<tr>
<td>Natural Period in Roll</td>
<td>1.84 s</td>
</tr>
<tr>
<td>Natural Period in Sway</td>
<td>8.8 s</td>
</tr>
</tbody>
</table>

of a pontoon in roll and sway are 1.84[s] and 8.8[s], respectively.

Figure 9 shows the results generated by the traditional irregular wave with parameters: $H_{1/3}=0.04[m]$, $T_{1/3}=1.84[s]$ and Bretschneider-Mitsuyasu spectrum. The figure indicates the surface elevation [mm], the heave [mm], the sway [mm], the roll [deg] and the yaw [deg] in tune from top.

Figure 10 is the case of the grouping wave of which the envelope period is 8.8[s]. However the other parameters equals the wave of Figure 9. Figure 11 and Figure 12 similarly correspond to the envelope period of 17.6[s] and 4.4[s], respectively.

The surface elevation of Figure 10 to 12 demonstrates the generation of the defined grouping wave with its envelope spectrum by the proposed algorithm.

Comparing with these figures, the grouping wave produces the remarkable and low frequency sway

Fig. 9 Time Histories of Measured Data Generated by the Traditional Irregular Wave in a Test Basin.

Fig. 10 Time Histories of Measured Data Generated by the Grouping Wave in a Test Basin. The Envelope Period : 8.8[s]

Fig. 11 Time Histories of Measured Data Generated by the Grouping Wave in a Test Basin. The Envelope Period : 17.6[s]

Fig. 12 Time Histories of Measured Data Generated by the Grouping Wave in a Test Basin. The Envelope Period : 4.4[s]

Photo 1 Experimental View of a Pontoon Model in a Test Basin.
motion. The low frequency sway motion resonant with its natural period 8.8 [s] is appeared in Figure 12, especially. This situation has occurred when the envelope period is a half period of sway resonance. Namely, the sub-harmonic motion subject to the envelope period has occurred in the sway motion.

Photograph 1 presents the experimental view of a pontoon model in a test basin.

5. Concluding Remark

The authors proposed an effective algorithm in real time for generation of the irregular wave specified by the spectrum of the wave envelope. The validity of the proposed algorithm was verified through the experiments in a test basin as well as computer simulations.

The fact that the remarkable and low frequency sway motion of a model is produced by the grouping wave, especially, of which envelope period is a half of sway resonance of a model, is cleared by the experiments.

The study of generation of the irregular wave specified by the defined envelope spectrum is an important problem in the field of the design of the mooring systems, which is explored much more than it used to be.

Appendix

The real time algorithm for generation of several types of ocean waves, such as regular waves, irregular waves, multi-directional irregular waves and complex waves, etc., has been obtained in the unified methodology. Here, the case of the irregular wave is shown.

The discrete time equation for the irregular wave is as follows:

\[ \eta_j = \sum_{k=1}^{N} a_k \cos(j \sigma_n \Delta t + \epsilon_k) \]

(A 1)

where \( j \) and \( \Delta t \) denote the \( j \)-th time step and the sampling period, respectively.

Here,

\[ \xi = a_k \cos(\epsilon_k) \]

(A 2)

\[ \xi = a_k \sin(\epsilon_k) \]

(A 3)

then

\[ \eta_j = \sum_{k=1}^{N} [\xi_k \cos(j \sigma_n \Delta t) - \xi_k \sin(j \sigma_n \Delta t)]. \]

(A 4)

Now, let \( \varphi_n \) and \( \Phi_n \) be

\[ \begin{pmatrix} \varphi_n \\ \Phi_n \end{pmatrix} = \begin{pmatrix} a_k - \beta_k \\ \beta_k \end{pmatrix} \begin{pmatrix} \cos(j \sigma_n \Delta t) \\ \sin(j \sigma_n \Delta t) \end{pmatrix}, \]

(A 5)

then the following iterative equation is obtained. That is,

\[ \begin{pmatrix} \varphi_{n,j+1} \\ \Phi_{n,j+1} \end{pmatrix} = \begin{pmatrix} a_k - \beta_k \\ \beta_k \end{pmatrix} \begin{pmatrix} \varphi_n \\ \Phi_n \end{pmatrix}, \]

(A 6)

where

\[ a_k = \cos(\sigma_n \Delta t) \]

(A 7)

\[ \beta_k = \sin(\sigma_n \Delta t) \]

(A 8)

By calculating equations (A 2), (A 3), (A 7), and (A 8) in advance, \( \varphi_{n,j+1} \) is obtained by equation (A 6) without sinusoidal calculation.

Finally, \( \eta_{j+1} \) is obtained by equation (A 4) as follows:

\[ \eta_{j+1} = \sum_{n=1}^{N} \varphi_{n,j+1}. \]

(A 9)

References