ANALOGY BETWEEN GENERALIZED PALINDROMIC SYMMETRY
AND QUASI-ODDS SYMMETRY MODELS FOR
SQUARE CONTINGENCY TABLES WITH ORDERED CATEGORIES

Sadao Tomizawa

Department of Information Sciences, Faculty of Science
and Technology, Science University of Tokyo,
Noda City, Chiba 278, JAPAN

SUMMARY

For the analysis of square contingency tables having ordered categories, this note points out that the generalized palindromic symmetry (GPS) model introduced by McCullagh (1978) is very analogous (but nonequivalent) to the quasi-odds symmetry (QOS) model considered in Tomizawa (1985b). Some resemblances between two models are also found when these models are applied to two sets of unaided vision data of (i) British women earlier analysed by Stuart (1955), and (ii) students in a university in Japan earlier analysed by Tomizawa (1984). Moreover it is noted that the GPS model fits the data of (i) very poorly though the palindromic symmetry model introduced in McCullagh (1978) fitted the same data well. Further a suitable interpretation is shown when the GPS (QOS) model is applied to the data of (ii).

1. Introduction

For the analysis of square contingency tables having ordered categories, McCullagh (1978) introduced the palindromic symmetry (PS) model and the generalized PS (GPS) model. Tomizawa (1985b) considered the odds symmetry (OS) model and the quasi-odds symmetry (QOS) model being one of the decomposed models of the OS model. We note that the QOS model is also equivalent to an adjusted quasi-symmetry model in Bishop et al. (1975, p.292).

The purpose of this note is to point out that the GPS model is very analogous (but nonequivalent) to the QOS model and also to note that the GPS model fits the unaided vision data of British women (Table 1) very poorly though this fact was not

Key words: Analogy; Generalized palindromic symmetry; Quasi-odds symmetry; Square table; Unaided vision data.

described in McCullagh (1978) and though the PS model fitted the same data well.

2. Analogy between two models

For the $R \times R$ square contingency table, let p_{ij} denote the probability that an observation will fall in the cell in row i and column j ($i=1, \dots, R; j=1, \dots, R$). The GPS model defined in McCullagh (1978) is given by

$$\begin{cases} F_{ij} = \exp\{\Delta_i/2\} \cdot \frac{\alpha_i}{\alpha_{j-1}} \phi_{ij} & (1 \leq i < j \leq R), \\ G_{ji} = \exp\{-\Delta_i/2\} \cdot \frac{\alpha_{j-1}}{\alpha_i} \phi_{ji} & (1 \leq i < j \leq R), \\ p_{ii} = \phi_{ii} & (1 \leq i \leq R), \end{cases} \quad (1)$$

where $\phi_{ij} = \phi_{ji}$, $\alpha_1 = 1$ and where

$$F_{ij} = \sum_{s=1}^i \sum_{t=j}^R p_{st} \quad \text{and} \quad G_{ji} = \sum_{s=j}^R \sum_{t=1}^i p_{st}.$$

A special case of the GPS model obtained by putting $\Delta_1 = \dots = \Delta_{R-1}$ is the PS model. The QOS model defined in Tomizawa (1985b) is also expressed as

$$p_{ij} = \begin{cases} \gamma_i \mu_j \psi_{ij} & (1 \leq i < j \leq R), \\ \mu_j \psi_{ij} & (1 \leq j < i \leq R), \\ \psi_{ii} & (i=j), \end{cases} \quad (2)$$

where $\psi_{ij} = \psi_{ji}$, $\mu_1 = 1$ and $\gamma_1 = 1$ without loss of generality. (Note that γ_i in (2) may be replaced by γ_j with $\gamma_R = 1$.) Let $\theta_{ij, st}$ denote the odds-ratio for the 2×2 subtable obtained from the i th and j th rows and the s th and t th columns, i.e., $\theta_{ij, st} = p_{is}p_{jt} / (p_{js}p_{it})$. Then the QOS model is further expressed as

$$\theta_{ij, st} = \theta_{st, ij} \quad (1 \leq i < j < s < t \leq R). \quad (3)$$

This means that for $1 \leq i < j < s < t \leq R$, if the odds that an observation is in row j rather than in row i is θ times higher when the observation is in column t rather than when it is in column s , then the odds that the observation is in column j rather than in column i is identically θ times higher when the observation is in row t rather than when it is in row s .

By the way, the GPS model can be also expressed as

$$\begin{cases} F_{ij} = \delta_i \nu_j \Phi_{ij} & (1 \leq i < j \leq R), \\ G_{ij} = \nu_j \Phi_{ij} & (1 \leq j < i \leq R), \\ p_{ii} = \Phi_{ii} & (1 \leq i \leq R), \end{cases} \quad (4)$$

where $\Phi_{ij} = \Phi_{ji}$, $\nu_1 = 1$ and $\delta_1 = 1$ without loss of generality. (Note that δ_i in (4) may be replaced by δ_j with $\delta_R = 1$.) By comparing (4) with (2), we see that the GPS model is very analogous (but nonequivalent) to the QOS model. Namely, the QOS model has a multiplicative form for the cell probabilities $\{p_{ij}\}$ and the GPS model has the same multiplicative form for the cumulative probabilities $\{F_{ij}\}$ and $\{G_{ji}\}$.

By the way, the GPS model can be further expressed as follows (Tomizawa, 1987d):

$$\theta_{ij, st}^U = \theta_{st, ij}^L \quad (1 \leq i < j < s < t \leq R), \quad (5)$$

where $\theta_{ij, st}^U = F_{is} F_{jt} / (F_{js} F_{it})$ and $\theta_{st, ij}^L = G_{st} G_{tj} / (G_{sj} G_{ti})$.

This means that for $1 \leq i < j < s < t \leq R$, if the odds (>1) that an observation is in row j or below rather than in row i or below is θ times higher when the observation is in column t or above rather than when it is in column s or above, then the odds (>1) that the observation is in column j or below rather than in column i or below is identically θ times higher when the observation is in row t or above rather than when it is in row s or above.

Now by comparing (5) with (3), we see again that the GPS model is very analogous (but nonequivalent) to the QOS model. Of course, the GPS and QOS models have the same $(R-2)(R-3)/2$ degrees of freedom (df) for testing goodness of fit by a chi-squared statistic. The analogy between both models is furthermore found through the analysis of data in the next section.

3. Some resemblances between two models applied to unaided vision data

For two sets of data in this section, we note that the maximum likelihood estimates (MLEs) of expected frequencies under the GPS and QOS models are obtained by using the Newton-Raphson method to the log-likelihood equations.

Table 1 is constructed from the data on unaided distance vision of 7477 women aged 30-39 employed in Royal Ordnance

Table 1

Unaided distance vision of British women; from Stuart (1955).
(The upper and lower parenthesized values are MLEs of expected frequencies under the GPS and QOS models, respectively.)

Right eye grade	Left eye grade				Total
	Best (1)	Second (2)	Third (3)	Worst (4)	
Best (1)	1520 (1520) (1520)	266 (266) (266)	124 (133.1) (132.8)	66 (59.3) (57.2)	1976
Second (2)	234 (234) (234)	1512 (1512) (1512)	432 (421.1) (423.2)	78 (86.5) (86.8)	2256
Third (3)	117 (107.3) (108.2)	362 (373.4) (370.8)	1772 (1772) (1772)	205 (205) (205)	2456
Worst (4)	36 (43.5) (44.8)	82 (72.8) (73.2)	179 (179) (179)	492 (492) (492)	789
Total	1907	2222	2507	841	7477

Table 2

Unaided distance vision of students in a university in Japan; from Tomizawa (1984). (The upper and lower parenthesized values are MLEs of expected frequencies under the GPS and QOS models, respectively.)

Right eye grade	Left eye grade				Total
	Best (1)	Second (2)	Third (3)	Worst (4)	
Best (1)	1291 (1291) (1291)	130 (130) (130)	40 (42.8) (42.4)	22 (19.9) (19.6)	1483
Second (2)	149 (149) (149)	221 (221) (221)	114 (110.6) (111.6)	23 (25.6) (25.4)	507
Third (3)	64 (61.1) (61.6)	124 (127.3) (126.4)	660 (660) (660)	185 (185) (185)	1033
Worst (4)	20 (22.0) (22.4)	25 (22.5) (22.6)	249 (249) (249)	1429 (1429) (1429)	1723
Total	1524	500	1063	1659	4746

factories in Britain from 1943 to 1946. This data have been analysed by many statisticians using various statistical models and methods; see, for example, Stuart (1955), Bishop et al. (1975, p.284), McCullagh (1978), Goodman (1979, 1985), Tomizawa (1985a, 1987a, b, c, d). Table 2 is constructed from the data on unaided distance vision of 4746 students aged 18 to about 25 including women of about 10 % in Faculty of Science and Technology, Science University of Tokyo in Japan examined in April 1982. Note that Table 2 is obtained by reversing the orders among row categories and the orders among column categories respectively of the original table analysed by Tomizawa (1984, 1985a, b, 1986, 1987b, etc.). We now consider Table 2 instead of the original table because it is easier to compare Table 2 with Table 1 than to do the original table with Table 1, and the value of likelihood ratio chi-squared statistic (denoted by G^2) for each of the GPS and QOS models applied to Table 2 is theoretically equal to that applied to the original table.

When the GPS model is applied to the data in Table 1, the value of G^2 is 6.18 with 1 df, yielding a P -value of $P < 0.025$. Therefore the GPS model fits the data in Table 1 very poorly though this fact was not described in McCullagh (1978). However note that the PS model fits the data in Table 1 well having $G^2 = 6.2$ ($P > 0.05$) with 3 df (McCullagh). On the other hand, when the QOS model is applied to the data in Table 1, the value of G^2 is 6.79 with 1 df, yielding a P -value of $P < 0.01$. Therefore the QOS model also fits the data in Table 1 very poorly. Next, when the GPS model is applied to the data in Table 2, the value of G^2 is 1.47 ($P > 0.2$) with 1 df. Therefore the GPS model fits the data in Table 2 well in contrast with the case of Table 1. The QOS model also fits the data in Table 2 well since the value of G^2 is 1.35 ($P > 0.2$) with 1 df (Tomizawa, 1985b). We now find that for each of the data in Tables 1 and 2, the value of G^2 for the GPS model is close to that for the QOS model and also the MLEs of expected frequencies under the GPS model are very similar to those under the QOS model (see Tables 1 and 2).

Now under the GPS model applied to the data in Table 1, the MLEs of ν_2 , ν_3 , ν_4 , δ_2 and δ_3 are $\hat{\nu}_2 = 1.19$, $\hat{\nu}_3 = 1.28$, $\hat{\nu}_4 = 1.36$, $\hat{\delta}_2 = 1.09$ and $\hat{\delta}_3 = 1.11$. On the other hand, under the QOS model ap-

plied to the data in Table 1, the MLEs of $\mu_2, \mu_3, \mu_4, \gamma_2$ and γ_3 are $\hat{\mu}_2=1.14, \hat{\mu}_3=1.23, \hat{\mu}_4=1.28, \hat{\gamma}_2=1.06$ and $\hat{\gamma}_3=1.10$. Similarly, under the GPS model applied to the data in Table 2, we get $\hat{\psi}_2=0.83, \hat{\psi}_3=0.75, \hat{\psi}_4=0.90, \hat{\delta}_2=0.94$ and $\hat{\delta}_3=0.66$; and under the QOS model applied to the data in Table 2, $\hat{\mu}_2=0.87, \hat{\mu}_3=0.69, \hat{\mu}_4=0.88, \hat{\gamma}_2=1.12$ and $\hat{\gamma}_3=0.58$. Therefore in addition to some resemblances between both models described earlier, we also find that for each of the data in Tables 1 and 2, the $\{\hat{\psi}_l\}$ and $\{\hat{\delta}_k\}$ under the GPS model are very close to the corresponding $\{\hat{\mu}_l\}$ and $\{\hat{\gamma}_k\}$ under the QOS model.

In closing this section, we shall describe the interpretations for the QOS and GPS models applied to the data in Table 2 for the comparison. Now the MLE of $1/\theta_{12,34}$ ($1/\theta_{12,34}^U$) under the QOS (GPS) model applied to the data in Table 2 is 2.03 (1.39). Thus when the QOS model is applied to the data in Table 2, the odds that a student's right eye grade is 2 (Second) instead of 1 (Best) is estimated to be 2.03 times higher when the student's left eye grade is 3 (Third) rather than when it is 4 (Worst); and also the odds that the student's left eye grade is 2 (Second) instead of 1 (Best) is estimated to be identically 2.03 times higher when the student's right eye grade is 3 (Third) rather than when it is 4 (Worst). On the other hand, when the GPS model is applied to the data in Table 2, the odds (>1) that a student's right eye grade is 1 (Best) or 2 (Second) rather than 1 (Best) is estimated to be 1.39 times higher when the student's left eye grade is 3 (Third) or 4 (Worst) rather than when it is 4 (Worst); and also the odds (>1) that the student's left eye grade is 1 (Best) or 2 (Second) rather than 1 (Best) is estimated to be identically 1.39 times higher when the student's right eye grade is 3 (Third) or 4 (Worst) rather than when it is 4 (Worst).

4. Note

Consider the 4×4 table. Then $1 > \theta_{12,34}^U > \theta_{12,34}$ (or $1 \leq \theta_{12,34}^U \leq \theta_{12,34}$) holds and similarly $1 > \theta_{34,12}^L > \theta_{34,12}$ (or $1 \leq \theta_{34,12}^L \leq \theta_{34,12}$) holds without the assumption that a model holds. (These inequalities are easily obtained by considering the differences.) Thus

$|\log \theta_{12,34}^U|$ and $|\log \theta_{34,12}^L|$ are smaller than $|\log \theta_{12,34}|$ and $|\log \theta_{34,12}|$, respectively. In other words, the association based on $\theta_{12,34}^U$ and $\theta_{34,12}^L$ are weaker than the association based on $\theta_{12,34}$ and $\theta_{34,12}$, respectively. Thus when both of the GPS and QOS models fit the data fairly well (as with the data in Table 2), the estimated value of $|\log \theta_{12,34}^U|$ ($=|\log \theta_{34,12}^L|$) for the GPS model tends to be typically smaller than the estimated value of $|\log \theta_{12,34}|$ ($=|\log \theta_{34,12}|$) for the QOS model. In fact, for the data in Table 2, the MLE of $|\log \theta_{12,34}^U|$ under the GPS model is 0.33 and the MLE of $|\log \theta_{12,34}|$ under the QOS model is 0.71 (see Section 3).

By the way, for the $R \times R$ table, an extended QS (denoted by EQS) model considered by Tomizawa (1984, 1985a) is defined by (2) with $\gamma_2 = \dots = \gamma_{R-1}$. Thus the QOS model is a generalization of the EQS model. On the other hand, the GPS model is, of course, a generalization of the PS model. We already saw that the GPS model was very analogous to the QOS model. However, the PS model is not analogous to the EQS model because the PS model cannot be expressed as (4) with $\delta_2 = \dots = \delta_{R-1}$ except the case that the parameters $\{\alpha_t\}$ in (1) for $t=1, \dots, R-1$, are exponential monotone, i.e., $\alpha_t = \alpha^{t-1}$ where $t=1, \dots, R-1$ and α is unspecified.

Finally, we note that the analogy between the QOS and GPS models seems to be like the analogy between the log-linear uniform association model and the logit uniform association model compared in Agresti (1984, p.219).

ACKNOWLEDGEMENTS

The author would like to express his sincere thanks to Professors K. Kunisawa, T. Makino and Dr. K. Shimizu, Science University of Tokyo, for their encouragements. He is also grateful to the referees for their useful comments.

REFERENCES

- Agresti, A. (1984). Analysis of ordinal categorical data.
John Wiley & Sons, New York.
- Bishop, Y. M. M., Fienberg, S. E. and Holland, P. W. (1975).
Discrete Multivariate Analysis: Theory and Practice.

- Cambridge, Mass: MIT Press.
- Goodman, L. A. (1979). Multiplicative models for square contingency tables with ordered categories. Biometrika, 66, 413-418.
- Goodman, L. A. (1985). The analysis of cross-classified data having ordered and/or unordered categories: association models, correlation models, and asymmetry models for contingency tables with or without missing entries. The Annals of Statistics, 13, 10-69.
- McCullagh, P. (1978). A class of parametric models for the analysis of square contingency tables with ordered categories. Biometrika, 65, 413-418.
- Stuart, A. (1955). A test for homogeneity of the marginal distributions in a two-way classification. Biometrika, 42, 412-416.
- Tomizawa, S. (1984). Three kinds of decompositions for the conditional symmetry model in a square contingency table. Journal of the Japan Statistical Society, 14, 35-42.
- Tomizawa, S. (1985a). Analysis of data in square contingency tables with ordered categories using the conditional symmetry model and its decomposed models. Environmental Health Perspectives, 63, 235-239.
- Tomizawa, S. (1985b). Decompositions for odds-symmetry models in a square contingency table with ordered categories. Journal of the Japan Statistical Society, 15, 151-159.
- Tomizawa, S. (1986). A new method of applying an extended marginal homogeneity model for square contingency tables with ordered categories. Journal of the Japan Statistical Society, 16, 187-190.
- Tomizawa, S. (1987a). Diagonal weighted marginal homogeneity models and decompositions for linear diagonals-parameter symmetry model. Communications in Statistics-Theory and Methods, 16, 477-488.
- Tomizawa, S. (1987b). Analysis of square contingency tables with ordered categories by a new method of applying the 1-weight modified marginal homogeneity models. Biometrical Journal, 29, 667-674.

- Tomizawa, S. (1987c). Analysis of data of contingency tables using the decompositions for models. Japanese Journal of Applied Statistics, 16, 131-139, (in Japanese).
- Tomizawa, S. (1987d): Decompositions for conditional symmetry model into palindromic symmetry and modified marginal homogeneity models. Manuscript.

[Received December, 1988]