18. On the Distribution of the Zero Points of Sections of a Power Series.

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1.

By *Jentzsch's* theorem we know that every point on the convergence circle is a limiting point of the zero point of sections of a power series and *Hurwitz's* theorem teaches us that inside the convergence circle the limiting point is the zero of the function defined by the power series. The purpose of this paper is to investigate the distribution of the limiting points outside the convergence circle under a certain broad assumption.

1. Let the power series be

$$F(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots \quad (1)$$

and any root of the equation

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n = 0 \quad (2)$$

be denoted by x_n .

Theorem I. *If for non-vanishing coefficients a_n of (1)*

$$\overline{\lim}_{n \rightarrow \infty} \left| \frac{a_n}{n^{\lambda} \log^{\lambda_0} n \log_1^{\lambda_1} n \dots \log_v^{\lambda_v} n} \right| = K < \infty$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_n}{n^{\lambda} \log^{\lambda_0} n \log_1^{\lambda_1} n \dots \log_v^{\lambda_v} n} \right| = L > 0$$

where $\log_1 n = \log(\log n)$ etc. and $\lambda, \lambda_0, \lambda_1, \dots, \lambda_v$ are any real numbers, then

$$\overline{\lim}_{n \rightarrow \infty} |x_n| \leq e^{\frac{K}{L}}$$

Remark. The zero points are bounded and the limiting points lie in the circle

$$|x| = e^{\frac{K}{L}}$$

2. We first prove two lemmas:

Lemma I. *When $r \geq r_0 > 1$, $\Phi(t) = r^t t^{\lambda} \log^{\lambda_0} t \log_1^{\lambda_1} t \dots \log_v^{\lambda_v} t$*

becomes increasing for $t \geq n_0$, where n_0 is independent of r .

Proof. Putting

$$f(t) = t^{\lambda} \log^{\lambda_0} t \dots \log_v^{\lambda_v} t,$$

We get

$$tf'(t) = f(t)\psi(t), \quad (3)$$

where
$$\psi(t) = \lambda + \frac{\lambda_0}{\log t} + \cdots + \frac{\lambda_\nu}{\log t \cdot \log_\nu t}$$

Since $\Phi(t) = r^t f(t)$, we get from (3)

$$\Phi'(t) = r^t f'(t) + r^t f(t) \log r = r^t f(t) \left[\log r + \frac{\psi(t)}{t} \right] \quad (4)$$

Now since $\lim_{t \rightarrow \infty} \frac{\psi(t)}{t} = 0$, we can find n_0 so that for $t \geq n_0$

$$\left| \frac{\psi(t)}{t} \right| < \varepsilon_1 \quad (5)$$

where ε_1 is a small quantity.

If we determine ε_1 so that $\log r_0 > \varepsilon_1$ or $r_0 > e^{\varepsilon_1}$ then we have $\Phi'(t) > 0$ uniformly for $r \geq r_0$. Hence the lemma is proved.
 Lemma. II. When μ is given, we can find n_0 and $\bar{r}_0$ such that for $n_2 \geq n_0$,

$$\int_{n_2}^n \Phi(t) dt < \frac{1}{\mu} \Phi(n)$$

uniformly for $r \geq \bar{r}_0 > 1$.

Proof.

By integrating

$$\Phi'(t) = \Phi(t) \left[\log r + \frac{\psi(t)}{t} \right]$$

we get

$$\int_{n_2}^n \Phi(t) \left[\log r + \frac{\psi(t)}{t} \right] dt = \left[\Phi(t) \right]_{n_2}^n$$

Since for a large value of t , $\Phi(t)$ is positive, we get

$$\int_{n_2}^n \Phi(t) \left[\log r + \frac{\psi(t)}{t} \right] dt < \Phi(n) \quad (6)$$

Now we determine $\bar{r}_0$ so that

$$\log \bar{r}_0 > \mu + \varepsilon_1 \quad \text{or} \quad \bar{r}_0 > e^{\mu + \varepsilon_1} \quad (7)$$

Then from (5), we see that

$$\log r + \frac{\psi(t)}{t} > \mu$$

for $t \geq n_0$ and $r \geq \bar{r}_0$.

Consequently from (6), we get,

$$\mu \int_{n_2}^n \Phi(t) dt < \int_{n_2}^n \Phi(t) \left[\log r + \frac{\psi(t)}{t} \right] dt < \Phi(n)$$

or
$$\int_{n_2}^n \Phi(t) dt < \frac{1}{\mu} \Phi(n)$$

for $n_2 \geq n_0$ and $r \geq \bar{r}_0$

3. Now from the hypothesis of the theorem I, we can put for $n \geq n_1$

$$L - \varepsilon \leq \left| \frac{a_n}{n^\lambda \log^{\lambda_0} n \dots \log^{\lambda_\nu} n} \right| \leq K + \varepsilon$$

or
$$L - \varepsilon \leq \left| \frac{a_n}{f(n)} \right| \leq K + \varepsilon$$

where ε is a small quantity.

Putting $|x| = r$, we get,

$$|a_{n_1} x^{n_1} + \dots + a_{n-1} x^{n-1}| \leq |a_{n_1} x^{n_1}| + \dots + |a_{n-1} x^{n-1}| \leq$$

$$(K + \varepsilon)[f(n_1)r^{n_1} + \dots + f(n-1)r^{n-1}] = (K + \varepsilon)[\Phi(n_1) + \dots + \Phi(n-1)] \quad (8)$$

Take n_1 greater than n_0 of Lemma I, and $r \geq \bar{r}_0$, then, since by Lemma I, $\Phi(t)$ is increasing hence we have,

$$\Phi(n_1) + \dots + \Phi(n-1) \leq \int_{n_1}^n \Phi(t) dt$$

Take n_1 greater than n_0 of Lemma II and $r \geq \bar{r}_0$, then,

$$\int_{n_1}^n \Phi(t) dt < \frac{1}{\mu} \Phi(n)$$

Hence we have

$$|a_{n_1} x^{n_1} + \dots + a_{n-1} x^{n-1}| < \frac{K + \varepsilon}{\mu} \Phi(n) \quad (9)$$

Putting

$$M = |a_0| + \dots + |a_{n_1-1}| \quad (10)$$

we get

$$|a_0 + a_1 x + \dots + a_{n_1-1} x^{n_1-1}| < M r^{n_1} \quad (11)$$

Therefore

$$\begin{aligned}
 |a_0 + \dots + a_{n-1} x^{n-1}| &< M r^{n_1} + \frac{K+\varepsilon}{\mu} \Phi(n) \\
 &= \frac{K+\varepsilon}{\mu} \Phi(n) \left[1 + \frac{M\mu}{(K+\varepsilon)} \frac{r^{n_1}}{\Phi(n)} \right]
 \end{aligned} \tag{12}$$

We will show that the second term in the bracket in (12) can be made very small.

For $r \geq \bar{r}_0$

$$\frac{r^{n_1}}{\Phi(n)} = \frac{1}{r^{n-n_1} n^\lambda \log_{\lambda_0} n \dots \log_{\lambda_v} n} \leq \frac{1}{\bar{r}^{n-n_1} n^\lambda \dots \log_{\lambda_v} n}$$

Since $\lim_{n \rightarrow \infty} \frac{1}{\bar{r}^{n-n_1} n^\lambda \dots \log_{\lambda_v} n} = 0$, we can determine n_3 so large, that for

$n \geq n_3$ and $r \geq \bar{r}_0$ we have,

$$\left| \frac{M r^{n_1}}{(K+\varepsilon)\Phi(n)} \right| < \varepsilon_2. \tag{13}$$

Hence

$$|a_0 + a_1 x + \dots + a_{n-1} x^{n-1}| < \frac{K+\varepsilon}{\mu} \Phi(n) [1 + \varepsilon_2 \mu]$$

If we determine μ , not specified till now, so that

$$L - \varepsilon \geq \frac{K+\varepsilon}{\mu} (1 + \varepsilon_2 \mu) \quad \text{or} \quad \mu \geq \frac{K+\varepsilon}{L - \varepsilon - \varepsilon_2 (K+\varepsilon)}, \tag{14}$$

Then

$$\begin{aligned}
 |a_n x^n| &\geq (L - \varepsilon) f(n) r^n \geq \frac{K+\varepsilon}{\mu} (1 + \varepsilon_2 \mu) \Phi(n) > \\
 &\quad |a_0 + a_1 x + \dots + a_{n-1} x^{n-1}|
 \end{aligned}$$

Hence for $r \geq \bar{r}_0$ and $n \geq n_3$

$$a_0 + a_1 x + \dots + a_n x^n \neq 0$$

consequently we get,

$$\overline{\lim}_{n \rightarrow \infty} |x_n| \leq \bar{r}_0$$

or from (7)

$$\overline{\lim}_{n \rightarrow \infty} |x_n| \leq e^{\mu + \varepsilon_1} \tag{15}$$

Since ε_1 , ε_2 and ε_3 can be taken very small, we get from (14) and (15)

$$\overline{\lim}_{n \rightarrow \infty} |x_n| \leq e^{\frac{K}{L}},$$

which proves the theorem.

Remark. When the coefficients a_n of the power series (1) satisfy the condition of Theorem I., then $\overline{\lim}_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = 1$, i. e., the radius of convergence of the power series is 1.

2.

4. We will prove that if $\lim_{n \rightarrow \infty} \frac{a_n}{n^k} = A \neq 0$, then there are no limiting points outside the unit-circle, where k is any real number. First we will prove three lemmas.

Lemma I. In the domain D defined by $1 < r_0 \leq |x| \leq r_1$ there exists a positive constant A which is independent of x and n , such that

$$\left| \frac{x}{1} + \frac{x^2}{2^k} + \cdots + \frac{x^n}{n^k} \right| > A \left| \frac{x^n}{n^k} \right|$$

for $n \geq n_0$, where k is any real number.

We will distinguish the two cases, according as $k > 0$ or $k \leq 0$

5. *Proof of the lemma for $k > 0$.*

Putting
$$\varphi(t) = \frac{x}{1} + \frac{x^2}{2^k} + \cdots + \frac{x^n}{n^k}$$

we have

$$\begin{aligned} \varphi(t) &= \frac{1}{\Gamma(k)} \int_0^1 \frac{1}{t} \left(\log \frac{1}{t} \right)^{k-1} (tx + \cdots + t^n x^n) dt \\ &= \frac{1}{\Gamma(k)} \int_0^1 \frac{1}{t} \left(\log \frac{1}{t} \right)^{k-1} \frac{x^{n+1} t^{n+1} - xt}{xt - 1} dt \dots (2) \end{aligned}$$

Since x lies in the domain D , $\frac{1}{|x|} \leq \frac{1}{r_0} < 1$

When we put $t_0 = \frac{1}{|x|}$ and $\tau_0 = \frac{1}{r_0}$, then $t_0 = \frac{1}{|x|} \leq \frac{1}{r_0} = \tau_0 < 1$.

Denoting by ε a small quantity, to be specified later on, we decompose the integral (2) into two parts:

$$\int_0^1 = \int_0^{t_0 + \varepsilon} + \int_{t_0 + \varepsilon}^1,$$

which we estimate

$$\begin{aligned}
 \text{a)} \quad & \left| \int_0^{t_0+\varepsilon} \left(\log \frac{1}{t} \right)^{k-1} (x + x^2 t + \dots + x^n t^{n-1}) dt \right| \\
 & \leq \int_0^{t_0+\varepsilon} \left(\log \frac{1}{t} \right)^{k-1} (|x| + |x|^2 t + \dots + |x|^n t^{n-1}) dt \\
 & < (|x| + |x|^2 (t_0 + \varepsilon) + \dots + |x|^n (t_0 + \varepsilon)) \int_0^{t_0+\varepsilon} \left(\log \frac{1}{t} \right)^{k-1} dt \\
 & = |x| \left(1 + (1 + \varepsilon |x|) + \dots + (1 + \varepsilon |x|)^{n-1} \right) \int_0^{t_0+\varepsilon} \left(\log \frac{1}{t} \right)^{k-1} dt \\
 & < n |x| (1 + \varepsilon |x|)^{n-1} \int_0^1 \left(\log \frac{1}{t} \right)^{k-1} dt = n |x| (1 + \varepsilon |x|)^{n-1} \Gamma(k)
 \end{aligned}$$

Hence

$$\begin{aligned}
 & \left| \int_0^{t_0+\varepsilon} \frac{1}{t} \left(\log \frac{1}{t} \right)^{k-1} \frac{x^{n+1} t^{n+1} - xt}{xt - 1} dt \right| < n |x| (1 + \varepsilon |x|)^{n-1} \Gamma(k) \quad (3) \\
 \text{b)} \quad & \left| \int_{t_0+\varepsilon}^1 \frac{1}{t} \left(\log \frac{1}{t} \right)^{k-1} \frac{x^{n+1} t^{n+1} - xt}{xt - 1} dt \right| \geq \left| \int_{t_0+\varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} \frac{x^{n+1} t^n}{xt - 1} dt \right| \\
 & \quad - \left| \int_{t_0+\varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} \frac{x}{xt - 1} dt \right| \quad (4) \\
 & \left| \int_{t_0+\varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} \frac{x}{xt - 1} dt \right| \leq |x| \int_{t_0+\varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} \frac{dt}{|xt - 1|} \\
 & < \frac{|x|}{|x|(t_0 + \varepsilon) - 1} \int_{t_0+\varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} dt < \frac{|x|}{|x|(t_0 + \varepsilon) - 1} \int_0^1 \left(\log \frac{1}{t} \right)^{k-1} dt \\
 & = \frac{1}{\varepsilon} \int_0^1 \left(\log \frac{1}{t} \right)^{k-1} dt = \frac{\Gamma(k)}{\varepsilon}
 \end{aligned}$$

Hence

$$\left| \int_{t_0+\varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} \frac{x}{xt - 1} dt \right| < \frac{\Gamma(k)}{\varepsilon} \quad (5)$$

And

$$\begin{aligned}
 & \left| \int_{t_0+\varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} \frac{x^{n+1} t^n}{xt - 1} dt \right| = |x|^{n+1} \left| \int_{t_0+\varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} \frac{t^n}{xt - 1} dt \right| = \\
 & |x|^{n+1} \sqrt{\left[\int_{t_0+\varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} \frac{t^n (\xi t - 1)}{(\xi t - 1)^2 + \eta^2 t^2} dt \right]^2 + \left[\int_{t_0+\varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} \frac{\eta t^{n+1}}{(\xi t - 1)^2 + \eta^2 t^2} dt \right]^2} \quad (6)
 \end{aligned}$$

where we have put $x = \xi + i\eta$

We will divide the domain D into two parts, D_1 and D_2 defined respectively by

$$|\eta| \geq \alpha > 0 \quad \text{and} \quad |\eta| \leq \alpha:$$

$$D = D_1 + D_2$$

Before we begin the evaluation of the integral (6) we will prove that we can determine n_1 so that for $n \geq n_1$

$$\int_{t_0+\epsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} t^{n+1} dt \geq \frac{\Gamma(k)(1-\delta)}{(n+2)^k} \quad (7)$$

uniformly for $1 < r_0 \leq |x| \leq r_1$, where δ is a small quantity less than 1.

To prove this, we start from the relation;

$$\begin{aligned} \int_0^1 \left(\log \frac{1}{t} \right)^{k-1} t^{n+1} dt &= \frac{\Gamma(k)}{(n+2)^k} = \int_0^{t_0+\epsilon} + \int_{t_0+\epsilon}^1 \\ \int_0^{t_0+\epsilon} \left(\log \frac{1}{t} \right)^{k-1} t^{n+1} dt &\leq (t_0 + \epsilon)^{n+1} \int_0^1 \left(\log \frac{1}{t} \right)^{k-1} dt = \Gamma'(k) (t_0 + \epsilon)^{n+1} \end{aligned} \quad (8)$$

Therefore

$$\begin{aligned} (n+2)^k \int_0^{t_0+\epsilon} \left(\log \frac{1}{t} \right)^{k-1} t^{n+1} dt &\leq \Gamma(k) (n+2)^k (t_0 + \epsilon)^{n+1} \\ &\leq \Gamma'(k) (n+2)^k (\tau_0 + \epsilon)^{n+1} \end{aligned}$$

When we choose ε so small that $\tau_0 + \varepsilon < 1$; then

$$\lim_{n \rightarrow \infty} (n+2)^k (\tau_0 + \varepsilon)^{n+1} = 0,$$

hence we can determine n_1 so that for $n \geq n_1$

$$(n+2)^k (\tau_0 + \varepsilon)^{n+1} < \delta < 1$$

Hence

$$\left| \int_0^{\tau_0 + \varepsilon} \left(\log \frac{1}{t} \right)^{k-1} t^{n+1} dt \right| < \frac{\delta}{(n+2)^k} I(k)$$

From (8) we get for $n \geq n_1$

$$\int_{\tau_0 + \varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} t^{n+1} dt = \frac{\Gamma(k)}{(n+2)^k} - \int_0^{\tau_0 + \varepsilon} \geq \frac{\Gamma(k)(1-\delta)}{(n+2)^k}$$

i.e.

$$\int_{\tau_0 + \varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} t^{n+1} dt \geq \frac{\Gamma(k)(1-\delta)}{(n+2)^k}$$

which is the relation to be proved.

Now we begin the evaluation of the integral (6).

(i) When x lies in the domain D_1 , we get, neglecting the first term of the radicand in (6)

$$\begin{aligned} & \left| \int_{\tau_0 + \varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} \frac{x^{n+1} t^n}{xt-1} dt \right| \geq |x|^{n+1} \alpha \int_{\tau_0 + \varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} \frac{t^{n+1} dt}{|x|^2 t + 2|\xi|t + 1} \\ & \geq \frac{|x|^{n+1} \alpha}{(|x|+1)^2} \int_{\tau_0 + \varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} t^{n+1} dt \geq \frac{|x|^{n+1} \alpha}{(r_1+1)^2} \int_{\tau_0 + \varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} t^{n+1} dt \quad (9) \end{aligned}$$

Combining (7) and (9), we get

$$\left| \int_{\tau_0 + \varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} \frac{x^{n+1} t^n dt}{xt-1} \right| \geq \frac{|x|^{n+1} \alpha \Gamma(k)(1-\delta)}{(r_1+1)^2 (n+2)^k} \geq \frac{|x|^n r_0 \alpha \Gamma(k)(1-\delta)}{(r_1+1)^2 (n+2)^k}$$

Hence there exists a constant A_1 , independent of x , such that for $n \geq n_1$

$$\left| \int_{\tau_0 + \varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} \frac{x^{n+1} t^n}{xt-1} dt \right| \geq A_1 \frac{|x|^n}{n^k} \quad (10)$$

(ii) When x lies in the domain D_2 and:

- (a) When $\xi < 0$, we have $\xi t - 1 < 0$ and $|\xi t - 1| > 1$.
- (b) When $\xi > 0$, we will prove that by taking α very small, we can put

$$\xi t - 1 \geq \varepsilon_1 \quad (11)$$

uniformly for x , when t varies from $t_0 + \varepsilon$ to 1,

In fact, we can make $\beta = \sqrt{r_0^2 - \alpha^2} > 1$, since $r_0 > 1$, so that

$$r_1 \geq \xi \geq \beta > 1.$$

Hence it is sufficient to prove that

$$\xi(t_0 + \varepsilon) - 1 \geq \varepsilon_1$$

$$\text{or} \quad \frac{\xi}{\sqrt{\xi^2 + \eta^2}} + \varepsilon \xi - 1 \geq \varepsilon_1 \quad (12)$$

$$\text{since} \quad t_0 = \frac{1}{|x|} = \frac{1}{\sqrt{\xi^2 + \eta^2}}$$

The inequality (12) becomes

$$\varepsilon \xi + \frac{\xi - \sqrt{\xi^2 + \eta^2}}{\sqrt{\xi^2 + \eta^2}} \geq \varepsilon_1 \quad \text{or} \quad \varepsilon \xi - \frac{\eta^2}{|x|(\xi + |x|)} \geq \varepsilon_1 \quad (13)$$

which is satisfied, if we take

$$\varepsilon \beta - \frac{\alpha^2}{r_0(1 + r_0)} \geq \varepsilon_1 \quad (14)$$

Hence we can take α so small, that (14) holds, if $\varepsilon_1 < \varepsilon r_0$, since

$$\lim_{\alpha \rightarrow 0} \varepsilon \beta = r_0 \varepsilon,$$

For such α and ε_1 (11) holds evidently. Now we see that $\xi t - 1$ is positive, when ξ is positive and is greater than a positive quantity ε_1 and that when ξ is negative, $\xi t - 1$ is negative and its absolute value is greater than 1. Hence by taking ε_1 smaller than 1, we can make $\xi t - 1$ retain the same sign in the domain D_2 and have an absolute value greater than ε_1 .

Now by neglecting the second term of the radicand in (6), we get

$$\left| \int_{t_0 + \varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} \frac{x^{n+1} t^n}{xt - 1} dt \right| \geq |x|^{n+1} \left| \int_{t_0 + \varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} \frac{t^n (\xi t - 1) dt}{(\xi t - 1)^2 + \eta^2 t^2} \right| \quad (15)$$

Since in the integral (15) $\xi t - 1$ does not change sign, we do not augment the absolute value of the integral when we replace the integrand by its absolute value, we get

$$|x|^{n+1} \left| \int_{t_0 + \varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} \frac{t^n |\xi t - 1|}{(\xi t - 1)^2 + \eta^2 t^2} dt \right| = |x|^{n+1} \int_{t_0 + \varepsilon}^1 \left(\log \frac{1}{t} \right)^{k-1} \frac{t^n |\xi t - 1|}{(\xi t - 1)^2 + \eta^2 t^2} dt$$

$$\begin{aligned}
&\geq |x|^{n+1}\varepsilon_1 \int_{t_0+\varepsilon}^1 \left(\log \frac{1}{t}\right)^{k-1} \frac{t^n}{(\xi^2 + \eta^2)t^2 + 2|\xi|t + 1} dt \\
&\quad \geq |x|^{n+1}\varepsilon_1 \int_{t_0+\varepsilon}^1 \left(\log \frac{1}{t}\right)^{k-1} \frac{t^n dt}{|x|^2 t^2 + 2|x|t + 1} \\
&\geq \frac{|x|^{n+1}\varepsilon_1}{(r_1+1)^2} \int_{t_0+\varepsilon}^1 \left(\log \frac{1}{t}\right)^{k-1} t^n dt \geq \frac{|x|^n r_0 \varepsilon_1}{(r_1+1)^2} \int_{t_0+\varepsilon}^1 \left(\log \frac{1}{t}\right)^{k-1} t^n dt \quad (16)
\end{aligned}$$

From (7) and (16) we see that there exists a constant A_2 such that for $n \geq n_2$

$$\left| \int_{t_0+\varepsilon}^1 \left(\log \frac{1}{t}\right)^{k-1} \frac{x^{n+1} t^n}{xt-1} dt \right| \geq \frac{A_2 |x|^n}{n^k} \quad (17)$$

By taking A_3 smaller than A_1 and A_2 , we see from (10) and (17) that there exists a constant A_3 , such that for $n \geq n_3$

$$\left| \int_{t_0+\varepsilon}^1 \left(\log \frac{1}{t}\right)^{k-1} \frac{x^{n+1} t^n}{xt-1} dt \right| \geq \frac{A_3 |x|^n}{n^k} \quad (18)$$

uniformly in the domain D .

Now from (3), (4), (5) and (18) we get,

$$\begin{aligned}
&\left| \int_0^1 \frac{1}{t} \left(\log \frac{1}{t}\right)^{k-1} \frac{x^{n+1} t^{n+1} - xt}{xt-1} dt \right| \geq \frac{A_3 |x|^n}{n^k} - \frac{\Gamma(k)}{\varepsilon} - n|x|(1+\varepsilon|x|)^{n-1} \Gamma'(k) \\
&= \frac{A_3 |x|^n}{n^k} \left[1 - \Gamma(k) \left(\frac{1}{\varepsilon A_3} \frac{n^k}{|x|^n} + \frac{n}{A_3} \left(\frac{1+\varepsilon|x|}{|x|} \right)^{n-1} \right) \right] \\
&\geq \frac{A_3 |x|^n}{n^k} \left[1 - \Gamma(k) \left(\frac{1}{\varepsilon A_3} \frac{n^n}{r_0^n} + \frac{n}{A_3} \left(\frac{1+\varepsilon r_1}{r_0} \right)^{n-1} \right) \right]
\end{aligned}$$

We will take ε so small that $1+\varepsilon r_1 < r_0$; this is possible since $r_0 > 1$,

Then $\lim_{n \rightarrow \infty} n \left(\frac{1+\varepsilon r_1}{r_0} \right)^n = 0$ and $\lim_{n \rightarrow \infty} \frac{n^k}{r_0^n} = 0$, hence we can find n_4 , independent of x , so that for $n \geq n_4$

$$\left| \int_0^1 \frac{1}{t} \left(\log \frac{1}{t}\right)^{k-1} \frac{x^{n+1} t^{n+1} - xt}{xt-1} dt \right| \geq A_3(1-\delta') \frac{|x|^n}{n^k}. \quad (\delta' < 1)$$

Hence there exists a positive constant A , such that for $n \geq n_4$

$$\left| \frac{1}{\Gamma(k)} \int_0^1 \frac{1}{t} \left(\log \frac{1}{t}\right)^{k-1} \frac{x^{n+1} t^{n+1} - xt}{xt-1} dt \right| \geq A \frac{|x|^n}{n^k} \quad (19)$$

uniformly in the domain D ; which proves the lemma in case k is positive.

6. *Proof of the lemma in case $k \leq 0$.*

When k is zero or negative, we put $\alpha = -k$

Hence $\alpha \geq 0$

Consider $\varphi(x) = x + 2^\alpha x^2 + \dots + n^\alpha x^n = x + \frac{x^2}{2^\alpha} + \dots + \frac{x^n}{n^\alpha}$

Let

$$\alpha = [\alpha] + \zeta \quad 1 > \zeta \geq 0,$$

And

$$\mu = [\alpha] + 1, \quad \lambda = \mu - \alpha;$$

then

$$1 \geq \lambda > 0.$$

Put

$$\varphi_1(x) = x + \frac{x^2}{2^{\mu-\alpha}} + \dots + \frac{x^n}{n^{\mu-\alpha}} = x + \frac{x^2}{2^\lambda} + \dots + \frac{x^n}{n^\lambda}.$$

then

$$\varphi(x) = x \underbrace{\frac{d}{dx} \left(x \left(\frac{d}{dx} \left(\dots x \left(\frac{d}{dx} \varphi_1(x) \right) \right) \right) \right)}_{\mu \text{ times}} \quad (20)$$

Since λ is positive, we can apply (2),

$$\varphi_1(x) = \frac{1}{\Gamma(\lambda)} \int_0^1 \frac{1}{t} \left(\log \frac{1}{t} \right)^{\lambda-1} \frac{x^{n+1} t^{n+1} - xt}{xt-1} dt \quad (21)$$

Now we put

$$\begin{aligned} \psi_0(\rho) &= \rho + \dots + \rho^n = \frac{\rho^{n+1} - \rho}{\rho - 1} \\ \psi_1(\rho) &= \rho \psi_0'(\rho) = \rho + \dots + n\rho^n = \frac{(n+1)\rho^{n+1}}{\rho-1} + \frac{\rho^{n+2} - \rho}{(\rho-1)^2} \\ \psi_2(\rho) &= \rho \psi_1'(\rho) = \rho + \dots + n^2 \rho^n \\ &\dots \dots \dots \\ \psi_r(\rho) &= \rho \psi_{r-1}'(\rho) = \rho + \dots + n^r \rho^n \end{aligned}$$

We can prove by induction that

$$\psi_r(\rho) = \frac{(n+1)^r \rho^{n+1}}{\rho-1} + \frac{\theta_r}{(\rho-1)^{r+1}}, \quad (22)$$

θ_r being a polynomial of the $(n+r+1)$ -th degree at most:

$$\theta_r = a_0 + a_1 \rho + \dots + a_{n+r+1} \rho^{n+r+1}$$

where

$$|a_i| < K (n+1)^{r-1} \quad (23)$$

K being a function of r , but independent of n .

Hence

$$\begin{aligned} \left| \frac{\theta_r}{(n+1)^r \rho^{n+1}} \right| &\leq \frac{K}{n+1} \frac{(1 + |\rho| + \cdots + |\rho|^{n+r+1})}{|\rho|^{n+1}} = \frac{K}{n+1} \frac{|\rho|^{n+r+2} - 1}{(|\rho| - 1)|\rho|^{n+1}} \\ &\leq \frac{K}{n+1} \cdot \frac{|\rho|^{r+1}}{|\rho| - 1} \leq \frac{K \rho_1^{r+1}}{(n+1)(\rho_0 - 1)}, \end{aligned}$$

for $1 < \rho_0 \leq |\rho| \leq \rho_1$.

Therefore

$$\lim_{n \rightarrow \infty} \frac{\theta_r}{(n+1)^r \rho^{n+1}} = 0 \quad (24)$$

uniformly for $1 < \rho_0 \leq |\rho| \leq \rho_1$.

Now

$$x \frac{d}{dx} \left(\frac{x^{n+1} t^{n+1} - xt}{xt - 1} \right) = xt \frac{d}{d(xt)} \left(\frac{x^{n+1} t^{n+1} - xt}{xt - 1} \right) = \rho \frac{d}{d\rho} \left(\frac{\rho^{n+1} - \rho}{\rho - 1} \right) = \psi_1(\rho), \quad (25)$$

where $\rho = xt$.

From (20) (21) (25) we get,

$$\varphi(x) = \frac{1}{\Gamma(\lambda)} \int_0^1 \frac{1}{t} \left(\log \frac{1}{t} \right)^{\lambda-1} \psi_\mu(xt) dt \quad (26)$$

In order to evaluate the integral (26), we divide it into two parts:

$$\int_0^1 = \int_0^{t_0+\varepsilon} + \int_{t_0+\varepsilon}^1$$

where $t_0 = \frac{1}{|x|}$.

$$\begin{aligned} \text{a) } \left| \int_0^{t_0+\varepsilon} \frac{1}{t} \left(\log \frac{1}{t} \right)^{\lambda-1} \psi_\mu(xt) dt \right| &\leq \int_0^{t_0+\varepsilon} \left(\log \frac{1}{t} \right)^{\lambda-1} (|x| + \cdots + n^\mu |x|^n t^{n-1}) dt \\ &< n^{\mu+1} |x|^n (t_0 + \varepsilon)^{n-1} \int_0^{t_0+\varepsilon} \left(\log \frac{1}{t} \right)^{\lambda-1} dt < n^{\mu+1} (1 + \varepsilon |x|)^{n-1} |x| \int_0^1 \left(\log \frac{1}{t} \right)^{\lambda-1} dt \\ &\leq n^{\mu+1} r_1 (1 + \varepsilon r_1)^{n-1} \Gamma(\lambda) \end{aligned}$$

Hence there exists a constant A_4 , such that

$$\left| \int_0^{t_0+\varepsilon} \frac{1}{t} \left(\log \frac{1}{t} \right)^{\lambda-1} \psi_\mu(xt) dt \right| < A_4 n^{\mu+1} (1 + \varepsilon r_1)^n \quad (27)$$

$$\begin{aligned} \text{b) } \left| \int_{t_0+\varepsilon}^1 \frac{1}{t} \left(\log \frac{1}{t} \right)^{\lambda-1} \psi_\mu(xt) dt \right| &\geq \left| \int_{t_0+\varepsilon}^1 \frac{1}{t} \left(\log \frac{1}{t} \right)^{\lambda-1} \frac{(n+1)^\mu x^{n+1} t^{n+1}}{xt-1} dt \right| \\ &- \left| \int_{t_0+\varepsilon}^1 \frac{1}{t} \left(\log \frac{1}{t} \right)^{\lambda-1} \frac{\theta_\mu}{(xt-1)^{\mu+1}} dt \right| \end{aligned} \quad (28)$$

c) When t varies from $t_0 + \varepsilon$ to 1 and x lies in the domain D ,

$$|x| (t_0 + \varepsilon) \leq |xt| \leq r_1 \quad (29)$$

Since

$$|x| (t_0 + \varepsilon) = 1 + \varepsilon |x| \geq 1 + \varepsilon r_0 > 1,$$

we get

$$1 < 1 + \varepsilon r_0 \leq |xt| \leq r_1,$$

Hence from (24) we can determine n_3 so that for $n \geq n_3$

$$\left| \frac{\theta_\mu}{(n+1)^\mu (xt)^{n+1}} \right| \leq \varepsilon_1 \quad (30)$$

uniformly in the domain D , when t varies from $t_0 + \varepsilon$ to 1.

$$\begin{aligned} \text{Now } &\left| \int_{t_0+\varepsilon}^1 \frac{1}{t} \left(\log \frac{1}{t} \right)^{\lambda-1} \frac{\theta_\mu}{(xt-1)^{\mu+1}} dt \right| \\ &\leq \varepsilon_1 \int_{t_0+\varepsilon}^1 \frac{1}{t} \left(\log \frac{1}{t} \right)^{\lambda-1} \frac{(n+1)^\mu |xt|^{n+1}}{(|xt|-1)^{\mu+1}} dt \\ &\leq \frac{\varepsilon_1 (n+1)^\mu |x|^{n+1}}{\varepsilon^{\mu+1} r_0^{\mu+1}} \int_{t_0+\varepsilon}^1 \left(\log \frac{1}{t} \right)^{\lambda-1} t^n dt < \frac{\varepsilon_1 (n+1)^\mu |x|^{n+1}}{\varepsilon^{\mu+1} r_0^{\mu+1}} \int_0^1 \left(\log \frac{1}{t} \right)^{\lambda-1} t^n dt \\ &= \frac{\varepsilon_1 (n+1)^\mu |x|^{n+1}}{\varepsilon^{\mu+1} r_0^{\mu+1}} \frac{\Gamma(\lambda)}{(n+1)^\lambda} = \frac{\varepsilon_1 (n+1)^\mu |x|^{n+1} \Gamma(\lambda)}{\varepsilon^{\mu+1} r_0^{\mu+1}} \leq B n^\alpha |x|^n \varepsilon_1, \end{aligned}$$

where B is a constant.

Hence

$$\left| \int_{t_0+\varepsilon}^1 \frac{1}{t} \left(\log \frac{1}{t} \right)^{\lambda-1} \frac{\theta_\mu}{(xt-1)^{\mu+1}} dt \right| < B n^\alpha |x|^n \varepsilon_1. \quad (31)$$

$$\begin{aligned} \text{d)} \quad & \left| \int_{t_0+\varepsilon}^1 \frac{1}{t} \left(\log \frac{1}{t} \right)^{\lambda-1} \frac{x^{n+1} t^{n+1} (n+1)^\mu}{xt-1} dt \right| \\ & = |x|^{n+1} (n+1)^\mu \left| \int_{t_0+\varepsilon}^1 \left(\log \frac{1}{t} \right)^{\lambda-1} \frac{t^n dt}{xt-1} \right| \end{aligned}$$

By (17) there exists a constant A_3 , such that for $n \geq n_3$.

$$|x|^{n+1} (n+1)^\mu \left| \int_{t_0+\varepsilon}^1 \left(\log \frac{1}{t} \right)^{\lambda-1} \frac{t^n dt}{xt-1} \right| \geq |x|^n \frac{A_3 n^\mu}{n^\lambda}, \quad (32)$$

Therefore

$$\left| \int_{t_0+\varepsilon}^1 \frac{1}{t} \left(\log \frac{1}{t} \right)^{\lambda-1} \frac{x^{n+1} t^{n+1} (n+1)^\mu}{xt-1} dt \right| \geq A_3 |x|^n n^\mu \quad (33)$$

e) From (27) (28) (31) (33) we get

$$\begin{aligned} & \left| \int_0^1 \frac{1}{t} \left(\log \frac{1}{t} \right)^{\lambda-1} \psi_n(xt) dt \right| \geq A_3 |x|^n n^\mu - B n^\mu |x|^n \varepsilon_1 - A_4 n^{\mu+1} (1 + \varepsilon r_1)^\mu \\ & \geq A_3 |x|^n n^\mu \left(1 - \frac{B}{A_3} \varepsilon_1 - \frac{A_4}{A_3} n^{\lambda+1} \left(\frac{1 + \varepsilon r_1}{r_0} \right)^n \right). \end{aligned}$$

Now we choose ε and ε_1 so that $4 \frac{B}{A_3} \varepsilon_1 \leq 1$; $1 + \varepsilon r_1 < r_0$, then

$\lim_{n \rightarrow \infty} n^{\lambda+1} \left(\frac{1 + \varepsilon r_1}{r_0} \right)^n = 0$, hence we can find n_6 , such that for $n \geq n_6$

$$\left| \frac{A_4}{A_3} n^{\lambda+1} \left(\frac{1 + \varepsilon r_1}{r_0} \right)^n \right| \leq \frac{1}{4}, \quad (34)$$

Therefore

$$\left| \int_0^1 \frac{1}{t} \left(\log \frac{1}{t} \right)^{\lambda-1} \psi_n(xt) dt \right| \geq A_3 |x|^n n^\mu \left(1 - \frac{1}{4} - \frac{1}{4} \right) = \frac{A_3}{2} |x|^n n^\mu.$$

Hence there exists a constant A so that for $n \geq n_7$

$$|\varphi(x)| = \left| \frac{1}{\Gamma(\lambda)} \int_0^1 \frac{1}{t} \left(\log \frac{1}{t} \right)^{\lambda-1} \psi_n(xt) dt \right| \geq A |x|^n n^\mu = \frac{A |x|^n}{n^k},$$

uniformly in the domain D , which proves the lemma.

7. Lemma II. In the domain D defined by $1 < r_0 \leq |x| \leq r_1$ there exists a constant B , such that for $n \geq n_0$

$$1 + \frac{|x|}{1} + \dots + \frac{|x|^n}{n^k} \leq B \frac{|x|^n}{n^k},$$

where k is any real number.

Proof. We put

$$|x| = r$$

and

$$f(t) = \frac{r^t}{t^k}$$

then

$$f'(t) = \frac{r^t}{t^k} \left(\log r - \frac{k}{t} \right) \quad (35)$$

$$\text{Hence } f'(t) > 0 \text{ for } \log r > \frac{|k|}{t} \text{ or } t > \frac{|k|}{\log r}.$$

Therefore in the domain D

$$f'(t) > 0,$$

if we take

$$t > \frac{|k|}{\log r_0}. \quad (36)$$

Similarly putting

$$\psi(t) = \frac{r^t}{t^{k+1}},$$

we see that

$$\psi'(t) > 0$$

uniformly in the domain D , if we take

$$t > \frac{|k| + 1}{\log r_0} \quad (37)$$

Hence when we denote by n_0 any natural number greater than $\frac{|k| + 1}{\log r_0}$, $f(t)$

and $\psi(t)$ are monotonously increasing for $t \geq n_0$, uniformly in the domain D .

Now we put

$$B_1 = 1 + \frac{r_1}{1} + \dots + \frac{r_1^{n_0-1}}{(n_0-1)^k}$$

Then, since $f(t)$ is increasing, we get,

$$1 + \frac{|x|}{1} + \dots + \frac{|x|^n}{n^k} \leq B_1 + \frac{|x|^{n_0}}{n_0^k} + \dots + \frac{|x|^n}{n^k} < B_1 + \int_{n_0}^n \frac{|x|^t}{t^k} dt + \frac{|x|^n}{n^k}$$

$$\leq B_1 + \left[\frac{|x|^t}{\log|x|t^k} \right]_{n_0}^n + \frac{|k|}{\log|x|} \int_{n_0}^n \frac{|x|^t}{t^{k+1}} dt + \frac{|x|^n}{n^k}.$$

And since $\frac{|x|^t}{t^{k+1}}$ is also increasing, we get

$$\int_{n_0}^n \frac{|x|^t}{t^{k+1}} dt < n \frac{|x|^n}{n^{k+1}} = \frac{|x|^n}{n^k}.$$

Hence

$$\begin{aligned} 1 + \frac{|x|}{1} + \cdots + \frac{|x|^n}{n^k} &< B_1 + \frac{|x|^n}{\log|x|n^k} + \frac{|k|}{\log|x|} \frac{|x|^n}{n^k} + \frac{|x|^n}{n^k} \\ &\leq \frac{|x|^n}{n^k} \left(1 + \frac{|k|+1}{\log r_0} + B_1 \frac{n^k}{r_0^n} \right). \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \frac{n^k}{r_0^n} = 0$, there exists a constant B , such that for $n \geq n_0$

$$1 + \frac{|x|}{1} + \cdots + \frac{|x|^n}{n^k} \leq B \frac{|x|^n}{n^k}$$

which proves the lemma.

8. Lemma III. In the domain D defined by $1 < r_0 \leq |x| \leq r_1$ there exists a positive constant M , such that for $n \geq n_0$

$$\left| 1 + \frac{x}{1} + \frac{x^2}{n^k} + \cdots + \frac{x^n}{n^k} \right| \geq M \left(1 + \frac{|x|}{1} + \cdots + \frac{|x|^n}{n^k} \right)$$

where k is any real number.

Proof. We have

$$\left| 1 + \frac{x}{1} + \cdots + \frac{x^n}{n^k} \right| \geq \left| \frac{x}{1} + \cdots + \frac{x^n}{n^k} \right| - 1$$

By Lemma I

$$\left| \frac{x}{1} + \cdots + \frac{x^n}{n^k} \right| \geq A \left| \frac{x^n}{n^k} \right|,$$

for $n \geq n_0$

Hence there exists a constant C such that for $n \geq n_0$

$$\left| 1 + \frac{x}{1} + \cdots + \frac{x^n}{n^k} \right| \geq A \frac{|x|^n}{n^k} - 1 \geq A \frac{|x|^n}{n^k} \left(1 - \frac{n^k}{r_0^n} \right) > C \frac{|x|^n}{n^k}, \quad (38)$$

uniformly in the domain D .

By Lemma II and (38) we get

$$\left| 1 + \frac{x}{1} + \dots + \frac{x^n}{n^k} \right| \geq \frac{C}{B} \left(1 + \frac{|x|}{1} + \dots + \frac{|x|^n}{n^k} \right) = M \left(1 + \frac{|x|}{1} + \dots + \frac{|x|^n}{n^k} \right)$$

for $n \geq n_0$, uniformly in the domain D , M being a positive constant.

3.

9. Now we begin to prove our fundamental theorem II:

Theorem II. *If in a power series*

$$F(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

$\lim_{n \rightarrow \infty} \frac{a_n}{n^k} = A (\neq 0)$, where k is any real number, then there are no limiting points of the zero points of sections outside the unit-circle.

Proof.

Since $\lim_{n \rightarrow \infty} \frac{a_n}{n^k} = A$, we can put

$$a_n = An^k + \varepsilon_n n^k; \quad \lim_{n \rightarrow \infty} \varepsilon_n = 0 \quad (39)$$

Suppose that the zero points of sections

$$a_0 + a_1 x + \dots + a_n x^n$$

have a limiting point z_0 outside the unit circle, which is now the convergence circle of $F(x)$, then we can choose a set $n_0, n_1, n_2, \dots$ such that $x_0, x_1, x_2, \dots$ where x_i is a root of the n_i -th section, converge to z_0 . We now take two circles, whose radii are $r_0 = |z_0| - \varepsilon$, and $r_1 = |z_0| + \varepsilon$, and whose centres are at the origin. When ε is very small, the two circles lie outside the unit circle. We may suppose that all the points $x_0, x_1, \dots$ lie within the ring bounded by the two circles, so that

$$r_0 \leq |x_i| \leq r_1 \quad (i=0, 1, 2, \dots)$$

By Lemma III, there exists a constant M , such that for $n \geq n_0$

$$|1 + 1^k x_i + 2^k x_i^2 + \dots + n_i^k x_i^{n_i}| \geq M(1 + 1^k |x_i| + \dots + n_i^k |x_i|^{n_i})$$

Hence

$$|1 + 1^k x_i + 2^k x_i^2 + \dots + n_i^k x_i^{n_i}| \neq 0. \quad (40)$$

Now by the hypothesis

$$a_0 + a_1 x_i + \dots + a_{n_i} x_i^{n_i} = 0. \quad (i=0, 1, 2, \dots) \quad (41)$$

or from (39)

$$A(1 + 1^k x_i + \dots + n_i^k x_i^{n_i}) + (\varepsilon_0 + \varepsilon_1 x_i + \dots + \varepsilon_{n_i} n_i^k x_i^{n_i}) = 0 \quad (42)$$

By putting

$$b_{i0} = \frac{1}{1 + 1^k x_i + \dots + n_i^k x_i^{n_i}}$$

$$b_{ir} = \frac{r^k x_i^r}{1 + 1^k x_i + \dots + n_i^k x_i^{n_i}} \quad (r=1, 2, \dots)$$

Equations (42) can be put in the following form in virtue of (40).

$$A + b_{i0} \varepsilon_0 + b_{i1} \varepsilon_1 + \dots + b_{in_i} \varepsilon_{n_i} = 0 \quad (43)$$

Consider the sequence $\{x_i^1\}$:

$$x_i^1 = b_{i0} \varepsilon_0 + b_{i1} \varepsilon_1 + \dots + b_{in_i} \varepsilon_{n_i} \quad (i=0, 1, 2, \dots)$$

We have,

$$(i) \quad |b_{i0}| + |b_{i1}| + \dots + |b_{in_i}| = \frac{1 + 1^k |x_i| + \dots + n_i^k |x_i|^{n_i}}{|1 + 1^k x_i + \dots + n_i^k x_i^{n_i}|} \leq \frac{1}{M}.$$

$$(ii) \quad |b_{ir}| = \frac{r^k |x_i|^r}{|1 + 1^k x_i + \dots + n_i^k x_i^{n_i}|} < \frac{1}{M} \frac{r^k |x_i|^r}{1 + 1^k |x_i| + \dots + n_i^k |x_i|^{n_i}}$$

$$< \frac{1}{M} \frac{r^k}{n_i^k |x_i|^{n_i-r}} \leq \frac{1}{M} \frac{r^k}{n_i^k r_o^{n_i-r}},$$

$$|b_{i0}| \leq \frac{1}{M} \frac{1}{1 + 1^k |x_i| + \dots + n_i^k |x_i|^{n_i}} < \frac{1}{M} \frac{1}{n_i^k |x_i|^{n_i}} \leq \frac{1}{M} \frac{1}{n_i^k r_o^{n_i}}$$

Hence

$$\lim_{i \rightarrow \infty} |b_{i0}| = 0, \quad \lim_{i \rightarrow \infty} |b_{ir}| = 0 \quad (\text{for a fixed } r) \quad (45)$$

Now by *Toeplitz's Theorem*(¹) the sequence $\{x_i^1\}$ tends to zero, when ε_i tends to zero. But from (43), this contradicts the hypothesis that A is not zero. Hence there is no limiting point outside the unit circle, which proves the theorem.

(¹) *Toeplitz*: Über lineare Mittelbildungen, *Prace matematyczno-fizyczne*. **22**, or *Knopp*. *Theorie und Anwendung der unendlichen Reihen*, 2. Aufl. 392.

Remark (i) The above result is not true, when we only suppose that

$\lim_{n \rightarrow \infty} \frac{|a_n|}{n^k} = A (\neq 0)$, as the following example shows. In the power series

$$F(x) = (1-x-x^2) + (x^3-x^4-x^5) + (x^6-x^7-x^8) + \dots; \quad |a_n| = 1$$

the sections;

$$1-x-x^2, \quad 1-x-x^2+(x^3-x^4-x^5), \text{ etc.}$$

have the roots of $1-x-x^2=0$ in common, one of which

$$x_1 = \frac{-1-\sqrt{5}}{2} \text{ has an absolute value greater than 1.}$$

Remark (ii) When the coefficients a_n satisfy the condition of Theorem II, the radius of convergence is 1 and from Theorem I, the zero points of sections are bounded.

4

10. We will give some examples, where the condition of Theorem II is satisfied.

$$(1) \quad \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots \quad (46)$$

or

$$\frac{1}{i\sqrt{z}} \tan^{-1}(i\sqrt{z}) = 1 + \frac{z}{3} + \dots + \frac{2^n}{2n+1} + \dots \quad (47)$$

$$\text{Here } a_n = \frac{1}{2n+1}, \quad \lim_{n \rightarrow \infty} a_n n = \frac{1}{2}.$$

$$(2) \quad \sin^{-1}x = x + \frac{1}{2} \frac{x^3}{3} + \frac{1.3}{2.4} \frac{x^5}{5} + \dots + \frac{1.3 \dots (2n-1)}{2.4 \dots 2n} \frac{x^{2n+1}}{2n+1} + \dots \quad (48)$$

or

$$\frac{\sin^{-1}\sqrt{z}}{\sqrt{z}} = 1 + \frac{1}{2} \frac{z}{3} + \dots + a_n z^n + \dots \quad (49)$$

$$\lim_{n \rightarrow \infty} a_n n^{\frac{1}{2}} = \frac{1}{2\sqrt{\pi}}. \quad (50)$$

$$(3) \quad \log \frac{1}{1-x} = x + \frac{x^2}{2} + \dots + \frac{x^n}{n} + \dots \quad (51)$$

$$\text{Here } \lim_{n \rightarrow \infty} a_n n = 1.$$

(4) The binomial series

$$(1-x)^r = \sum a_n x^n, \quad (52)$$

where r is any real number which is not a positive integer.

$$\text{Then } a_n = \frac{-r\left(1-\frac{r}{1}\right)\dots\left(1-\frac{r}{n-1}\right)}{n} = -\frac{r\left(1-\frac{r}{1}\right)\dots\left(1-\frac{r}{n}\right)}{n-r}$$

$$\text{Take } m \text{ so large that for } n > m, \quad \left| \frac{r}{n} \right| < 1,$$

then

$$a_n = \frac{A p_n}{n-r},$$

where

$$A = -r\left(1-\frac{r}{1}\right)\dots\left(1-\frac{r}{m}\right); \quad p_n = \left(1-\frac{r}{m+1}\right)\dots\left(1-\frac{r}{n}\right).$$

Now, if in the product $p_n = (1+\alpha_1)(1+\alpha_2)\dots(1+\alpha_n)$, $|\alpha_k| < 1$ and $\sum_{k=1}^{\infty} |\alpha_k|^2$ converges, then⁽²⁾

$$\lim_{n \rightarrow \infty} \frac{p_n}{e^{s_n}} = B \quad (B \neq 0),$$

where

$$s_n = \alpha_1 + \dots + \alpha_n.$$

Hence

$$\begin{aligned} p_n &= (B + \varepsilon_n) e^{s_n}; & \lim_{n \rightarrow \infty} \varepsilon_n &= 0, \\ a_n &= \frac{A}{n-r} (B + \varepsilon_n) e^{s_n} = \frac{A}{n-r} (B + \varepsilon_n) e^{-\left(\frac{1}{m+1} + \dots + \frac{1}{n}\right)r} = \\ &= \frac{A}{n-r} (B + \varepsilon_n) e^{-\left(\frac{1}{1} + \dots + \frac{1}{n}\right)r + \left(1 + \frac{1}{2} + \dots + \frac{1}{m}\right)r} = \\ &= \frac{A}{n-r} (B + \varepsilon_n) e^{-(\log n + \gamma_n)r + \left(1 + \frac{1}{2} + \dots + \frac{1}{m}\right)r} = \frac{A(B + \varepsilon_n)}{(n-r)n^r} e^{r\left(1 + \dots + \frac{1}{m} - \gamma_n\right)}, \end{aligned}$$

where

$$\lim_{n \rightarrow \infty} \gamma_n = \gamma \quad (\text{Euler's constant}).$$

(²) Knopp, Theorie und Anwendung der unendlichen Reihen, 2. Aufl. 226.

Consequently

$$\lim_{n \rightarrow \infty} a_n n^{r+1} = A B e^{r \left(1 + \dots + \frac{1}{n} - \gamma \right)} \neq 0$$

Specially, when r is negative, $r = -\lambda$ ($\lambda > 0$), we know from the theory of Γ -function that

$$\lim_{n \rightarrow \infty} \frac{B_n^{(\lambda)}}{n^{\lambda-1}} = \frac{1}{\Gamma(\lambda)},$$

where we have put

$$\frac{1}{(1-x)^\lambda} = \sum_{n=0}^{\infty} B_n^{(\lambda)} x^n$$

This case admits also simple verification. In fact

$$B_n^{(\lambda)} = \frac{\lambda(\lambda+1) \dots (\lambda+n-1)}{n!},$$

so that

$$\begin{aligned} B_{n+1}^{(\lambda)} &> B_n^{(\lambda)}, & \text{if } \lambda > 1; \\ B_{n+1}^{(\lambda)} &< B_n^{(\lambda)}, & \text{if } \lambda < 1. \end{aligned}$$

Hence by *Kakeya's* theorem we see that if $\lambda < 1$, all roots of sections lie outside the unit circle, while if $\lambda > 1$, they all lie inside it, however in each case as we have seen above, they converge to the unit circle; the case $\lambda = 1$ is trivial; all roots lie on the unit circle.

(5) The hypergeometric series.

$$F(\alpha, \beta, \gamma; x) = 1 + \frac{\alpha, \beta}{1, \gamma} x + \frac{\alpha(\alpha+1) \beta(\beta+1)}{2! \gamma(\gamma+1)} x^2 + \dots \quad (54)$$

where α, β, γ , are real numbers.

$$\text{Here } \alpha_n = \frac{\alpha(\alpha+1) \dots (\alpha+n-1) \beta(\beta+1) \dots (\beta+n-1)}{n! \gamma(\gamma+1) \dots (\gamma+n-1)}.$$

Now as have shewn under (4) we have

$$\frac{1}{(1-x)^\alpha} = \sum_0^\infty \frac{\alpha(\alpha+1) \dots (\alpha+n-1)}{n!} x^n = \sum_0^\infty b_n x^n \quad ; \quad \lim_{n \rightarrow \infty} b_n n^{-\alpha+1} = A_1 (\neq 0)$$

$$\frac{1}{(1-x)^\beta} = \sum_0^\infty \frac{\beta(\beta+1) \dots (\beta+n-1)}{n!} x^n = \sum_0^\infty c_n x^n \quad ; \quad \lim_{n \rightarrow \infty} c_n n^{-\beta+1} = A_2 (\neq 0)$$

$$\frac{1}{(1-x)^\gamma} = \sum_0^\infty \frac{\gamma(\gamma+1)\dots(\gamma+n-1)}{n!} x^n = \sum_0^\infty d_n x^n \quad ; \quad \lim_{n \rightarrow \infty} d_n n^{-\gamma+1} = A_3 (\neq 0)$$

Consequently

$$\lim_{n \rightarrow \infty} a_n n^{1+\gamma-\alpha-\beta} = \frac{A_1 A_3}{A_3} \neq 0 \quad \text{since} \quad a_n = \frac{b_n c_n}{d_n}.$$

(6) The rational function

$$R(x) = \frac{P(x)}{Q(x)} = \frac{\alpha_0 + \alpha_1 x + \dots + \alpha_m x^m}{\beta_0 + \beta_1 x + \dots + \beta_p x^p} = a_0 + a_1 x + \dots + a_n x^n + \dots \quad (55)$$

where we suppose $Q(x)$ to have been brought, by a suitable linear transformation, in the following form:

$$Q(x) = (1-x)^{n_0} (x_1-x)^{n_1} (x_2-x)^{n_2} \dots \\ 1 \leq |x_1| \leq |x_2| \leq \dots ;$$

Decomposing into partial fractions, let

$$R(x) = \frac{b_{00}}{(1-x)^{n_0}} + \frac{b_{01}}{(1-x)^{n_0-1}} + \dots + \frac{b_{0, n_0-1}}{1-x} + \\ \frac{b_{10}}{\left(1 - \frac{x}{x_1}\right)^{n_1}} + \frac{b_{11}}{\left(1 - \frac{x}{x_1}\right)^{n_1-1}} + \dots + \frac{b_{1, n_1-1}}{1 - \frac{x}{x_1}} + \dots \\ + c_0 + c_1 x + \dots + c_r x^r. \quad (56)$$

Then for $n > r$,

$$a_n = b_{00} B_n^{(n_0)} + b_{01} B_n^{(n_0-1)} + \dots + b_{0, n_0-1} \\ + (b_{10} B_n^{(n_1)} + b_{11} B_n^{(n_1-1)} + \dots + b_{1, n_1-1}) \left(\frac{1}{x_1}\right)^n + \dots \quad (57)$$

and, as we can easily verify,

$$\lim_{n \rightarrow \infty} \frac{B_n^{(n_0-i)}}{B_n^{(n_0)}} = 0, \quad \lim_{n \rightarrow \infty} \frac{B_n^{(n_1)}}{B_n^{(n_0)}} \left(\frac{1}{x_1}\right)^n = 0, \quad \text{if } |x_1| > 1 \quad (58)$$

Suppose now $n_0 = n_1 = \dots = n_k$ and $1 = |x_1| = |x_2| = \dots = |x_k|$; and as to the other roots, let $|x_{k+i}| > 1$ or $n_{k+i} < n_0$ if $|x_{k+i}| = 1$, then we can put, in virtue of (58)

$$\frac{a_n}{n^{n_0-1}} = \left(b_{00} + \frac{b_{10}}{x_1^n} + \dots + \frac{b_{k0}}{x_k^n} + \varepsilon_n \right) \frac{1}{\Gamma(n_0)} \quad (59)$$

where

$$\lim_{n \rightarrow \infty} \varepsilon_n = 0,$$

since

$$\lim_{n \rightarrow \infty} \frac{B_n^{(\lambda)}}{n^{\lambda-1}} = \frac{1}{\Gamma(\lambda)}$$

Hence if $\lim_{n \rightarrow \infty} \left(b_{00} + \frac{b_{10}}{x_1^{n_1}} + \dots + \frac{b_{k0}}{x_k^{n_k}} \right) \neq 0$, we have $\lim_{n \rightarrow \infty} \frac{a_n}{n^{n_0-1}} \neq 0$.

When the condition in the following theorem is satisfied, the above limit certainly exists, hence we get

Theorem III. *If in a rational function $R(x) = \frac{P(x)}{Q(x)} = a_0 + a_1x + \dots$*

$$Q(x) = (x-x_0)^{n_0} (x-x_1)^{n_1} \dots (x-x_\nu)^{n_\nu},$$

and

- i) $|x_0| = |x_1| = \dots = |x_k| < |x_{k+1}| \leq \dots \leq |x_\nu|$
- ii) $n_0, n_1, \dots, n_k$ are all different, then there are no limiting points of the zero points of sections of the expansion of $R(x)$ outside the circle $|x| = |x_k|$.

5.

11. In the following we will investigate the distribution of the zero points of sections of a transcendental integral function.

It has been proved by Prof. Okada⁽³⁾ that a necessary and sufficient condition that

$$F(x) = a_0 + a_1x + \dots + a_nx^n + \dots$$

is a transcendental integral function is that

$$\lim_{n \rightarrow \infty} r_n = \infty, \quad (60)$$

where r_n is the maximum modulus of the zero points of the n -th section.

The original proof of the theorem is very complicated. The following simple proof is due to Prof. Kakeya, who has authorized us to insert it here.

Theorem IV. (Kakeya)

(3) The science reports of the Tôhoku Imperial University, **11** (1922), 43-50.

Let a power series $\sum_{n=0}^{\infty} a_n x^n$ have a convergence-radius r , then by taking suitable sections :

$$S_{m_i}(x) = \sum_{n=0}^{m_i} a_n x^n, \quad (i=1, 2, 3 \dots) \quad (61)$$

we can make $\lim_{i \rightarrow \infty} S_{m_i}(x) = \infty$ uniformly for $|x| > k > 2r$.

Proof. Since $\frac{k}{2} > r$, we get

$$\overline{\lim}_{n \rightarrow \infty} \left| a_n \right| \left(\frac{k}{2} \right)^n = \infty.$$

Hence if we take m_i suitably,

$$\left| a_m \left(\frac{k}{2} \right)^m \right| < \left| a_{m_i} \left(\frac{k}{2} \right)^{m_i} \right|, \quad \text{for } m < m_i.$$

Therefore for $|x| > k$, we have

$$\left| a_m \left(\frac{x}{2} \right)^m \right| < \left| a_{m_i} \left(\frac{x}{2} \right)^{m_i} \right|, \quad \text{for } m < m_i.$$

Now for $|x| > k$,

$$|S_{m_i}(x)| = \left| \sum_{n=0}^{m_i} a_n \left(\frac{x}{2} \right)^n 2^n \right| = \left| a_{m_i} \left(\frac{x}{2} \right)^{m_i} \right| \left| 2^{m_i} + \frac{a_{m_i-1} \left(\frac{x}{2} \right)^{m_i-1}}{a_{m_i} \left(\frac{x}{2} \right)^{m_i}} 2^{m_i-1} + \dots \right|.$$

Since $\left| \frac{a_{m_i-1} \left(\frac{x}{2} \right)^{m_i-1}}{a_{m_i} \left(\frac{x}{2} \right)^{m_i}} \right| < 1$, we get

$$\begin{aligned} |S_{m_i}(x)| &\geq \left| a_{m_i} \left(\frac{x}{2} \right)^{m_i} \right| (2^{m_i} - 2^{m_i-1} - \dots - 2 - 1) \geq \\ &\left| a_{m_i} \left(\frac{k}{2} \right)^{m_i} \right| (2^{m_i} - \dots - 2 - 1) = \left| a_{m_i} \left(\frac{k}{2} \right)^{m_i} \right| \end{aligned}$$

Consequently $\lim_{i \rightarrow \infty} |S_{m_i}(x)| = \infty$ uniformly for $|x| > k$.

Theorem V. (Kakeya)

Let a power series $\sum_{n=0}^{\infty} a_n x^n$ have a convergence circle $|x| = r$ and ρ_n be

the maximum modulus of roots of the n -th section. Then we have

$$r \leq \lim_{n \rightarrow \infty} \rho_n \leq 2r \quad (62)$$

We may suppose that a_0 is not zero; Let the roots of the n -th section be denoted by $\alpha_1, \dots, \alpha_n$; then

$$\left| \frac{a_0}{a_n} \right| = |\alpha_1 \dots \alpha_n| \leq \rho_n^n$$

Hence

$$\frac{\lim_{n \rightarrow \infty} |a_0|^{\frac{1}{n}}}{\lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}} \leq \lim_{n \rightarrow \infty} \rho_n, \quad \therefore r \leq \lim_{n \rightarrow \infty} \rho_n$$

Next suppose if possible that $\lim_{n \rightarrow \infty} \rho_n > 2r$, and let k so taken that $\lim_{n \rightarrow \infty} \rho_n > k > 2r$, then for sufficiently large m , $\sum_{n=0}^m a_n x^n = 0$ would have a root,

whose modulus is greater than k in contradiction to Theorem IV; hence we have $\lim_{n \rightarrow \infty} \rho_n \leq 2r$

Remark. Okada's theorem is a special case of Theorem V, when $r = \infty$.

12. We know that ρ_n tends to infinity by Okada's theorem. In the following lines I will investigate into the order of ρ_n with respect to n .

Theorem VI. Let $F(x) = a_0 + a_1 x + \dots + a_n x^n + \dots$ be a transcendental integral function, and suppose that for non-vanishing a_n

$$\overline{\lim_{n \rightarrow \infty}} |a_n| n^{kn+r} = K < \infty$$

$$\lim_{n \rightarrow \infty} |a_n| n^{kn+r} = L > 0$$

where k is positive and r is any real number.

For any given small quantity ϵ , we have for sufficiently large n

$$1 - \epsilon < \frac{\rho_n}{n^k} < e^{\frac{K}{L} + k} + \epsilon \quad (63)$$

where ρ_n is the maximum modulus of roots of the n -th section.

Proof. By the hypothesis we can find m so that for $n > m$,

$$\frac{L - \epsilon}{n^{kn+r}} \leq |a_n| \leq \frac{K + \epsilon}{n^{kn+r}} \quad (64)$$

for non-vanishing a_n , when ϵ is given.

Now put

$$|a_0| + |a_1| + \dots + |a_{m-1}| = M$$

then for $|x| > 1$, we have

$$|a_0 + a_1x + \dots + a_{m-1}x^{m-1}| \leq M|x|^m \quad (65)$$

From (64) we get

$$|a_mx^m + \dots + a_{n-1}x^{n-1}| \leq (K + \varepsilon) \left[\frac{|x|^m}{m^{mk+r}} + \dots + \frac{|x|^{n-1}}{(n-1)^{(n-1)k+r}} \right] \quad (66)$$

In order to evaluate the right hand side of (66), we put

$$f(t) = \frac{|x|^t}{t^{kt+r}} \quad (67)$$

Then

$$f'(t) = f(t) \left[\log |x| - k \log t - k - \frac{r}{t} \right] \quad (68)$$

Take $|x|$ so large, that when t varies from 1 to n ,

$$\log |x| - k \log t - k - \frac{r}{t} \geq \lambda > 1 \quad (69)$$

or

$$|x| \geq e^{k+\lambda} \cdot t^k e^{\frac{r}{t}} \quad (70)$$

Now we will determine the maximum value of $t^k e^{\frac{r}{t}}$ when t varies from 1 to n . For this purpose we put

$$\varphi(t) = t^k e^{\frac{r}{t}} \quad (71)$$

Then

$$\varphi'(t) = e^{\frac{r}{t}} t^{k-2} (kt - r)$$

Hence $\varphi'(t) = 0$ gives $t = \frac{r}{k}$, which corresponds to the minimum of $\varphi(t)$.

Hence $\varphi(t)$ has no maximum, consequently it attains the greatest value at $t=1$ or $t=n$ in the interval $(1, n)$.

Since $\varphi(1) = e^r$ while $\varphi(n) = n^k e^{\frac{r}{n}}$ tends to infinity, when t tends to infinity, we see that for a sufficiently large n , $\varphi(n) = n^k e^{\frac{r}{n}}$ is the greatest value of $\varphi(t)$ in the interval $(1, n)$.

From (70) we know that if we take

$$|x| \geq e^{k+\lambda+\frac{r}{n}} n^k \quad (72)$$

then $f(t)$ is increasing in the interval $(1, n)$, we get

$$\frac{|x|^m}{n^{mk+r}} + \dots + \frac{|x|^{n-1}}{(n-1)^{(n-1)k+r}} < \int_m^n \frac{|x|^t dt}{t^{tk+r}} = \int_m^n f(t) dt \quad (73)$$

From (68) and (69)

$$f'(t) \geq \lambda f(t) \quad (74)$$

Hence

$$\int_m^n f(t) dt \leq \frac{1}{\lambda} [f(t)]_m^n < \frac{f(n)}{\lambda} = \frac{1}{\lambda} \frac{|x|^n}{n^{nk+r}}.$$

From (73) and (74) we get

$$\frac{|x|^m}{n^{mk+r}} + \dots + \frac{|x|^{n-1}}{(n-1)^{(n-1)k+r}} < \frac{1}{\lambda} \frac{|x|^n}{n^{nk+r}} \quad (75)$$

From (65) and (75) we have,

$$\begin{aligned} |a_0 + a_1 x + \dots + a_{n-1} x^{n-1}| &< M |x|^m + \frac{(K+\varepsilon) |x|^n}{\lambda n^{kn+r}} = \\ &= \frac{(K+\varepsilon) |x|^n}{\lambda n^{kn+r}} \left[1 + \frac{M \lambda}{K+\varepsilon} \frac{n^{kn+r}}{|x|^{n-m}} \right] \end{aligned} \quad (76)$$

We will show that $\frac{n^{kn+r}}{|x|^{n-m}}$ can be made very small by taking n very large.

From (72)

$$|x| \geq e^{k+\lambda+\frac{r}{n}} n^k > e^{k+\frac{r}{n}} n^k.$$

Hence

$$\frac{n^{kn+r}}{|x|^{n-m}} < \frac{n^{km}}{e^{kn-km+\frac{r}{n}(n-m)}} \quad (77)$$

Since $\lim_{n \rightarrow \infty} \frac{n^\alpha}{e^{\beta n}} = 0$, when β is positive, we can determine n_0 so large that

for $n \geq n_0$ we have,

$$\frac{n^{km}}{e^{kn-km+\frac{r}{n}(n-m)}} < \varepsilon_1 \frac{K+\varepsilon}{M} \quad (78)$$

Hence

$$\frac{M n^{kn+r}}{(K+\varepsilon) |x|^{kn}} < \varepsilon_1$$

Therefore for $n \geq n_0$ and $|x| \geq e^{k+\lambda+\frac{r}{n}} n^k$

$$|a_0 + a_1 x + \dots + a_{n-1} x^{n-1}| < \frac{|x|^n (K+\varepsilon)}{\lambda n^{kn+r}} (1 + \varepsilon_1 \lambda) \quad (79)$$

Next we will determine λ , which is not specified till now, so that

$$L - \varepsilon \geq \frac{(K+\varepsilon) (1 + \varepsilon_1 \lambda)}{\lambda}$$

or

$$\lambda \geq \frac{K+\varepsilon}{L - \varepsilon - \varepsilon_1 (K+\varepsilon)} \quad (80)$$

This is always possible by taking ε_1 very small.

Then

$$\begin{aligned} |a_n x^n| &\geq \frac{(L-\varepsilon) |x|^n}{n^{kn+r}} \geq \frac{(K+\varepsilon) (1 + \varepsilon_1 \lambda)}{\lambda} \frac{|x|^n}{n^{kn+r}} \\ &> |a_0 + a_1 x + \dots + a_{n-1} x^{n-1}| \end{aligned}$$

Hence $a_0 + a_1 x + \dots + a_n x^n = 0$ has no root, whose modulus is greater than $n^k e^{k+\lambda+\frac{r}{n}}$, hence we have,

$$\lim_{n \rightarrow \infty} \frac{|x_n|}{n^k} \leq e^{\lambda+k},$$

Since ε and ε_1 are arbitrary, we get from (80)

$$\lim_{n \rightarrow \infty} \frac{|x_n|}{n^k} \leq e^{\frac{\kappa}{L} + k} \quad (81)$$

Next let $\alpha_1 \dots \alpha_n$ be the roots of $a_0 + a_1 x + \dots + a_n x^n = 0$

Then

$$|\alpha_1 \dots \alpha_n| = \left| \frac{a_0}{a_n} \right|$$

Hence if we denote the maximum modulus of the roots of the n -th section by ρ_n we get,

$$\left| \frac{a_0}{a_n} \right| \leq \rho_n^n$$

or

$$\frac{n^{nk+r}}{(K+\varepsilon)} |a_0| < \rho_n^n \quad (n \geq m)$$

Hence

$$\lim_{n \rightarrow \infty} \frac{\rho_n}{n^k} \geq 1 \quad (82)$$

The theorem is proved.

Cor. In a transcendental integral function, $F(x) = \sum_{n=0}^{\infty} a_n x^n$ if for non-vanishing a_n

$$\lim_{n \rightarrow \infty} |a_n| n^{kn+r} = A (\neq 0),$$

where k is positive and r is any real number, then, when ε is given, however small, we can find n_0 such that for $n \geq n_0$

$$1 - \varepsilon < \frac{\rho_n}{n^k} < e^{1+k} + \varepsilon$$

13. In place of the above hypothesis, we suppose that for non-vanishing a_n .

$$\overline{\lim}_{n \rightarrow \infty} |a_n| (n!)^k = K < \infty$$

$$\overline{\lim}_{n \rightarrow \infty} |a_n| (n!)^k = L > 0$$

By Stirling's formula

$$(n!)^k = \left(\frac{n}{e} \right)^{kn} (2\pi n)^{\frac{k}{2}} \left(1 + \frac{k}{n} + \dots \right).$$

Now we put

$$z = e^k x, \quad (83)$$

Then

$$a_0 + a_1 x + \dots + a_n x^n + \dots = b_0 + b_1 z + \dots + b_n z^n + \dots,$$

where

$$a_n = b_n e^{kn}.$$

Hence

$$|a_n| (n!)^k = |b_n| n^{kn + \frac{k}{2}} (2\pi)^{\frac{k}{2}} \left(1 + \frac{k}{n} + \dots \right).$$

Consequently

$$\overline{\lim}_{n=\infty} |b_n| n^{\frac{kn+\frac{k}{2}}{2}} = K_1 = \frac{K}{(2\pi)^{\frac{k}{2}}}$$

$$\lim_{n=\infty} |b_n| n^{\frac{kn+\frac{k}{2}}{2}} = L_1 = \frac{L}{(2\pi)^{\frac{k}{2}}}$$

Hence by Theorem VI we get

$$1 \leq \lim_{n=\infty} \frac{|z_n|}{n^k} \leq \overline{\lim}_{n=\infty} \frac{|z_n|}{n^k} \leq e^{\frac{K_1}{L_1} + k} = e^{\frac{K}{L} + k}$$

where $|z_n|$ is the maximum modulus of roots of the n -th section :

$$b_0 + b_1 z + \dots + b_n z^n = 0.$$

From (83) we get

$$\frac{1}{e^k} \leq \lim_{n=\infty} \frac{|x_n|}{n^k} \leq \overline{\lim}_{n=\infty} \frac{|x_n|}{n^k} \leq e^{\frac{K}{L}}$$

Hence we have

Theorem VII. In a transcendental integral function $F(x) = \sum_{n=0}^{\infty} a_n x^n$ suppose that for non-vanishing a_n

$$\overline{\lim}_{n=\infty} |a_n| (n!)^k = K < \infty$$

$$\lim_{n=\infty} |a_n| (n!)^k = L > 0$$

then for $n \geq n_0$

$$\frac{1}{e^k} - \varepsilon < \frac{\rho_n}{n^k} < e^{\frac{K}{L}} + \varepsilon \quad (92)$$

when ε is given.

Cor. In a transcendental integral function $F(x) = \sum_{n=0}^{\infty} a_n x^n$

if for non-vanishing a_n ; $\lim_{n=\infty} |a_n| (n!)^k = A (\neq 0)$, then for a sufficiently large n ,

$$\frac{1}{e^k} - \varepsilon < \frac{\rho_n}{n^k} < e + \varepsilon \quad (85)$$

To this category of functions belong e^x , $\sin x$, $\cos x$, $\sinh x$, $\cosh x$.

14. Take as another example the Bessel-functions.

$$J_r(z) = \left(\frac{z}{2}\right)^r \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1) \Gamma(n+r+1)} \left(\frac{z}{2}\right)^{2n} \quad (86)$$

The root of sections of (86) is equivalent to that of the power series;

$$F(x) = \sum_{n=0}^{\infty} \frac{(-)^n}{\Gamma(n+1)\Gamma(n+r+1)} x^n = \sum_{n=0}^{\infty} a_n x^n \quad (87)$$

Hence

$$x^2 = 4x \quad (88)$$

$$|a_n| = \frac{1}{\Gamma(n+1)\Gamma(n+r+1)}$$

By Stirling's formula, we get,

$$\Gamma(n+1) = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} (1 + \varepsilon_n) \quad \lim \varepsilon_n = 0$$

$$\Gamma(n+r+1) = \left(\frac{n+r}{e}\right)^{n+r} \sqrt{2\pi n} (1 + \varepsilon'_n) \quad \lim \varepsilon'_n = 0$$

Hence

$$\Gamma(n+1)\Gamma(n+r+1) = 2\pi \frac{n^{2n+r+1}}{e^{2n+r}} \left(1 + \frac{r}{n}\right)^{n+r} (1 + \varepsilon''_n) =$$

$$\frac{2\pi n^{2n+r+1}}{e^{2n+r}} \left(1 + \frac{r}{n}\right)^{\frac{n}{r}r+r} (1 + \varepsilon''_n)$$

Since

$$\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^{\frac{n}{r}} = e$$

We get

$$\Gamma(n+1)\Gamma(n+r+1) = \frac{2\pi n^{2n+r+1}}{e^{2n}} (1 + \varepsilon''_n) \quad (89)$$

Hence

$$|a_n| = \frac{e^{2n}}{2\pi n^{2n+r+1}} (1 + \varepsilon_0) \quad \lim \varepsilon_0 = 0 \quad (90)$$

Next put

$$y = e^2 x \quad (91)$$

Then

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n y^n$$

where

$$a_n = b_n e^{2n}$$

Hence

$$\lim_{n \rightarrow \infty} |b_n| n^{2n+r+1} = \frac{1}{2\pi}$$

From the above corollary, we get

$$1 \leq \lim_{n \rightarrow \infty} \frac{|y_n|}{n^2} \leq \overline{\lim}_{n \rightarrow \infty} \frac{|y_n|}{n^2} \leq e^3$$

where $|y_n|$ is the maximum modulus of roots of the n -th section.

From (88) and (91) we have, $\sqrt{y} = \frac{ez}{2}$

Hence

$$\frac{2}{e} \leq \lim_{n \rightarrow \infty} \frac{|z_n|}{n} \leq \overline{\lim}_{n \rightarrow \infty} \frac{|z_n|}{n} \leq 2\sqrt{e}$$

Theorem VIII. Let ρ_n be the maximum modulus of roots of the n -th section of Bessel-function $J_\nu(z)$, then for a sufficiently large n we have,

$$\frac{2}{e} - \varepsilon < \frac{\rho_n}{n} < 2\sqrt{e} + \varepsilon$$

.....

I will here express my cordial thanks to Prof. *Kakeya*, who gave me the proofs of his theorems which are unpublished and permitted me to publish them in my paper.

Tokyo, Nov. 1924.