19. *Natural Equations of Curves under Circular Point-Transformation Groups and their Duals, II.*

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In Part I, the natural equations of curves under the groups spoken of in the title have been treated in terms of the tetracyclic circle-(pentaspherical sphere-) coordinates (or of their natural duals). In the present part (II), they will be treated in terms of the tetracyclic (pentaspherical) point-coordinates (or of their natural duals). The results are new for the three-dimensional case. The present theory seems to be more suitable for the study of surface curves than the former.

I. Non-Euclidean Plane Curves.

1. If \((\xi)\) be the tetracyclic point-coordinates of a doubly oriented point of a plane curve, then the expression

\[
\varepsilon_{\alpha} d\rho^2 = \frac{|\xi d\xi\frac{d^2\xi}{d\xi^2}|}{(d\xi\frac{d^2\xi}{d\xi^2})^2},
\]

is an absolute invariant under direct quaternary orthogonal transformations as well as under the following transformations: \(\xi = h\xi\). It changes sign by an indirect quaternary orthogonal transformation.

\[
d\rho^2 = \frac{dR}{R^2},
\]

where \(R^{-1} = k^{-1} \cot \frac{v}{k}\) is the curvature. We call such \(\rho\) the *inversion-length.* It agrees with \(dp = (d\eta d\eta)\) in Part I.

2. We normalize \((\xi)\) as follows:

\[
\tilde{\xi} = \frac{\xi}{\varepsilon_{\alpha} (d\xi d\xi)^{\frac{1}{2}}},
\]

where \(t=1\) or \(0\) according as \((\xi)\) has or has not undergone an indirect quaternary orthogonal transformation. In the subsequent lines the factor \(\tilde{v}\) will be dropped. But it should be understood.
3. Then

\[ dp^2 = (d\xi d\bar{\xi})^4, \]

\[ dp = (-\varepsilon_0)^{\frac{4}{3}} \cdot \xi d\xi d\bar{\xi} d\bar{\xi} |^{\frac{1}{6}}. \]

We call such curves as \( dp = 0 \) *inversion minimal curves*. They are doubly oriented circles \((a \xi) = 0\).

4. The expression

\[(4) \]

\[ \phi = -\left( \frac{d^2 \xi}{dp^2} \right)^4 = -\frac{5}{4} R^2 \left( \frac{d^2 R}{ds^2} \right)^2 - \frac{R^2 d^2 R}{ds^2} - \frac{1}{ds} + \frac{dR}{ds} + \frac{R^2 d^2 R}{ds^3} - \frac{R^2}{\kappa d^2 R ds} \]

is an absolute invariant under direct quaternary orthogonal transformations as well as under the transformations \( \xi = b \xi \) and changes its sign by an indirect quaternary orthogonal transformation.

5. **Theorem.** In order that two plane curves may be transformable into each other by direct circular point-transformations in the plane, it is necessary and sufficient that we can establish the correspondence in such a way that at the corresponding doubly oriented points, where the relations \( dp_1^2 = dp^2 \) holds, the relation \( \phi_1 = \phi \) holds also. In order that two plane curves may be transformable into each other by *indirect* circular point-transformations, it is necessary and sufficient that we can establish the correspondence in such a way that at the corresponding doubly oriented points, where the relation \( dp_1^2 = -dp^2 \) holds, the relation \( \phi_1 = -\phi \) also holds \((1)\).

6. The theory of the present § may be dualized.

II. **Euclidean Plane Curves.**

7. As long as the *Euclidean circular point-transformations* are concerned, it is perfectly analogous to the theory in § 1 except the forms of the expressions \((2)\) and \((4)\). They should be replaced by

\[(2') \]

\[ dp^2 = \frac{dr ds}{r^2} \]

and

\[(1) \text{ This theorem is new only in the form of the expression } \phi. \text{ See Part I.} \]
respectively.

8. When the Laguerre group in plane is concerned, the theory remains again almost parallel to that in § 1. Peculiarities take place only in the corresponding expressions:

\((\xi)\): Laguerre’s doubly oriented line coordinates,

\[(1'') \quad \varepsilon_{\phi} dp^2 = \frac{\xi d\xi d^2\xi d^3\xi}{(\xi^2 d^3\xi)^{\frac{3}{2}}} ,\]

\[(2'') \quad dp^3 = \frac{dP d\theta}{P^2} ,\]

\[(3'') \quad i\tilde{\xi} = \frac{\varepsilon \xi d\xi d^2\xi d^3\xi}{\sqrt{\varepsilon_{\phi} (d^2 \xi d^3 \xi)^{\frac{3}{2}}}} , \quad \varepsilon = \pm 1 ,\]

\[(4'') \quad \phi = -\frac{\xi d^3\xi d^2\xi d\xi}{(\xi^2 d^3\xi)^{\frac{3}{2}}} = -\frac{r d^2 r}{d\xi^2} + \frac{3}{2} \left(\frac{d^2 r}{d\xi^2}\right)^2 + 5 \left(\frac{d^3 r}{d\xi d^2\xi}\right)^2 + \frac{1}{r} \left(\frac{d^2 r}{d\xi^2}\right)^3 + \frac{1}{r^2} \frac{d^2 r}{d\xi^2} .\]

III. Non-Euclidean Space-Curves.

9. If \((\xi)\) be the pentaspherical point-coordinates of a doubly oriented point of a space curve, then the expression

\[(5) \quad \varepsilon_{\phi} dp^2 = \frac{\xi d\xi d^2\xi d^3\xi d^4\xi}{(d^2 \xi d^3 \xi d^4 \xi)^{\frac{3}{2}}} ,\]

is an absolute invariant under direct quinary orthogonal transformations as well as under the transformations \(\xi = h\xi\). It changes sign by an indirect quinary orthogonal transformation.

\[(6) \quad dp^3 = -\frac{ds^5 S ds}{R^5 T^3 dR ds} , \quad S^2 = R^5 + T^3 \left(\frac{dR}{ds}\right)^2 ,\]

where \(T^{-1}\) is the torsion. We call such \(p\) the inversion-length of the first kind. It is historically new.
10. The expression

\[ \frac{dt^i}{(d\xi^i d\xi^j)^{\alpha}} = \frac{S^2}{R^2 T^2} \]

is an absolute invariant under all quinary orthogonal transformations as well as under \( \xi = h\xi \). Such \( t \) is the inversion-length in Liebman's sense. But let us refer to it as the inversion-length of the second kind.

11. We normalize \( (\xi) \) as follows:

\[ (-1)^{\xi} = \frac{|\xi d\xi d\xi^i d\xi^j|^{\frac{1}{2}}}{(d\xi^i d\xi^j)^{\alpha}} \xi^i, \]

where \( t = 1 \) or 0 according as \( (\xi) \) has or has not undergone an indirect quinary orthogonal transformation.

In the subsequent lines, we will drop the factors \((-1)^{\xi}\). But it should always be understood.

12. Then

\[ dp^2 = (d\xi^i d\xi^j), \quad dp = (-\xi^i, j)^{\frac{1}{2}} |\xi d\xi d\xi^i d\xi^j|^{\frac{1}{2}}. \]

We call such curves as \( dp = 0 \) inversion minimal curves of the first kind. They are spherical curves: \((a\xi^i)^{\alpha} = 0\).

13. We call such curves as \( dt = 0 \) inversion minimal curves of the second kind. Their osculating spheres are everywhere null.

14. The expressions

\[ \phi = -\left(\frac{d^2\xi^i}{d\xi^j d\xi^j}\right) = \frac{2}{h^2} \frac{d^2h}{ds^2} - \frac{3}{h^3} \left(\frac{dh}{ds}\right)^2 - \frac{1}{h^4} \left(\frac{1}{R^2} + \frac{1}{k^2}\right), \]

\[ f = \left(\frac{d^2\xi^i}{d\xi^j d\xi^j}\right) = \phi^2 + \frac{1}{h^4} R^2 \left(\frac{dR}{ds}\right)^2 + \frac{1}{h^6 R^2 T^2} (\equiv \phi^2 + \psi, \text{say}), \]

where \( h \equiv -\left(\frac{R + \frac{dR}{ds}}{R^2 T^2}\right)^{\frac{1}{2}}, \quad \phi = -\frac{dp}{d\Theta}(^z), \)

\( \Theta \) the inversion twisting\(^(\text{a}) \),

are absolute invariants under all quinary orthogonal transformations as well as under \( \xi = h\xi \).

15. **Theorem.** In order that two space curves may be transformable into each other by a direct spherical point-transformation, it is necessary

\(^{(\text{a})} - \phi \) may be named the inversion-dualtorsion.

\(^{(\text{b})} \) See Part I.
and sufficient that we can establish the correspondence in such a way that at the corresponding doubly oriented points, where the relation \( dp_1 = dp \) holds, the following relations also hold: \( \phi_1 = \phi, \varphi_1 = \varphi \). In order that two space curves may be transformable into each other by an indirect spherical point-transformation, it is necessary and sufficient that we can establish the correspondence in such a way that at the corresponding doubly oriented points, where the relation \( dp_1 = -dp \) holds, the following relations hold also:

\[ \phi_1 = \phi, \quad \varphi_1 = \varphi. \]

16. The theory of the present § may be dualized.

IV. Euclidean Space-Curves.

17. As long as the Euclidean spherical point-transformations are concerned, it is perfectly analogous to the theory in § 3 except the forms of the expressions (6), (9) and (10). They should be replaced by

\[
(6') \quad dp^2 = - \frac{ds^3 S dS}{r^3 T^3 dS/ds}, \quad S^2 = r^2 + T^2 \left( \frac{dr}{ds} \right)^2,
\]

\[
(9') \quad \varphi = -\left( \frac{d^2 \tilde{\xi}}{dp^2} \frac{d^2 \tilde{\xi}}{dp^2} \right) \approx \frac{2}{k^3} \frac{dh}{ds} - \frac{2}{k^2} \left( \frac{dh}{ds} \right)^2 - \frac{1}{k^2 r^2},
\]

and

\[
(10') \quad f = \left( \frac{d^2 \tilde{\xi}}{dp^2} \frac{d^2 \tilde{\xi}}{dp^2} \right) \approx \varphi^2 + \frac{1}{k^4 r^4} \left( \frac{dr}{ds} \right)^2 + \frac{1}{k^4 r^4 T^2}
\]

respectively, where

\[ h \equiv -\left( \frac{r^3 + \frac{d^2 \tilde{r}}{dp^2}}{r^3 T^3} \right)^{\frac{1}{2}}, \quad \varphi \equiv -\frac{dp}{d\Theta}^{(*)}, \]

\[ \Theta \text{ the inversion twisting.} \]

18. When the Laguerre group in space is concerned, the theory remains again almost paralleled to that in § 3. Peculiarities take place only in the corresponding expressions:

\[ (\xi) : \text{Laguerre's doubly oriented plane coordinates}, \]

\[ (*) - \varphi \text{ may be named the inversion-dual torsion.} \]
\[ e_\phi \, dp^5 = \frac{|\frac{d\xi}{d\phi} \frac{d^2\xi}{d\phi^2} \frac{d^3\xi}{d\phi^3} \frac{d^4\xi}{d\phi^4}|}{(\frac{d\xi}{d\phi})^2} \],

\[ (-1)^{\frac{1}{2}} \xi = \frac{|\frac{d\xi}{d\phi} \frac{d^2\xi}{d\phi^2} \frac{d^3\xi}{d\phi^3} \frac{d^4\xi}{d\phi^4}|^{\frac{1}{2}}}{\sqrt{e_\phi (\frac{d\xi}{d\phi})}} \xi, \]

\[ q = -\left( \frac{\frac{d^3 \xi}{d \phi^3}}{\frac{d \xi}{d \phi} \frac{d^2 \xi}{d \phi^2}} \right)_i = \frac{2}{h^3} \frac{d^2 h}{d \theta^2} - \frac{2}{h^3} \left( \frac{dh}{d \theta} \right)^2 - \frac{1}{h^3} \left( \frac{1}{P^2} + 1 \right), \]

\[ f = -\left( \frac{\frac{d^3 \xi}{d \phi^3}}{\frac{d \xi}{d \phi} \frac{d^2 \xi}{d \phi^2}} \right)_i = \phi^2 + \frac{1}{h^4 P^4} \left( \frac{dP}{d \theta} \right)^2 + \frac{1}{h^4 P^4}, \]

where

\[ h = -\left( \frac{d^2 P}{d \xi^2} \right)^{\frac{1}{3}}, \quad \varphi = -\frac{dP}{d \Sigma}, \quad \Sigma = \text{Laguerre length}. \]

\((*) - \varphi \text{ may be named the Laguerre torsion}\)